EXAM 1

Math 219, 2024 Spring, Clark Bray.

Name: Solutions ________ Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not remove the staple, tear pages out of the staple, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams (only!), but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)*

(a) Find a parametrization of the curve in the xy-plane with equation $x\sin(y) - y^3 = 4 - 2x$.

$$\Leftrightarrow (x)(2+\sin y) = 4 + y^3$$

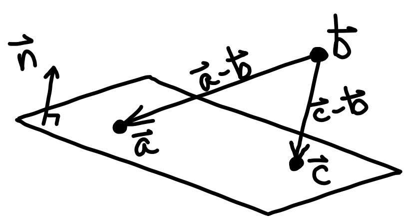
$$\Leftrightarrow x = \frac{4+y^3}{2+\sin y}$$

Using the graph parametrization, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{4+t^3}{2+\sin t} \\ t \end{pmatrix}$$

(b) The plane P contains $\vec{a} = (1,1,1)$ and is perpendicular to $\vec{n} = (1,2,2)$. The point $\vec{b} = (1,5,6)$ is not on P. Find the point $\vec{c}$ on P that is closest to $\vec{b}$. *(Hint: Draw a figure! And, consider how $\vec{c} - \vec{b}$ relates to $\vec{n}$ and $\vec{a} - \vec{b}$?)*

From the figure, we see



$$\vec{c}-\vec{b} = \text{proj}_{\vec{n}}(\vec{a}-\vec{b})$$

$$= \left((\vec{a}-\vec{b}) \cdot \frac{\vec{n}}{\|\vec{n}\|}\right)\frac{\vec{n}}{\|\vec{n}\|}$$

$$= \left(\begin{pmatrix} 0 \\ -4 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}/3\right)\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}/3 = (-2)\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -4 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} -2 \\ -4 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

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2. *(20 pts)* In this question we consider the vectors $\vec{u} = (1,3,2)$, $\vec{v} = (2,0,-1)$, $\vec{w} = (-2,1,-2)$. The parallelepiped P has one vertex at the origin and edges defined by these three vectors. The parallelogram G has one vertex at the origin and edges defined by $\vec{v}$ and $\vec{w}$.

   (a) Compute the area of the parallelogram G.

$$\vec{v}\times\vec{w} = \begin{pmatrix}2\\0\\-1\end{pmatrix}\times\begin{pmatrix}-2\\1\\-2\end{pmatrix} = \begin{pmatrix}1\\6\\2\end{pmatrix}$$

$$\text{area} = \|\vec{v}\times\vec{w}\| = \sqrt{1^2+6^2+2^2} = \sqrt{41}$$

   (b) Compute the volume of P and decide if the list $(\vec{u},\vec{v},\vec{w})$ is in right hand order or left hand order. *(Hint: Some of your computations from (a) might be useful here.)*

$$\det\begin{pmatrix}| & | & |\\ \vec{u} & \vec{v} & \vec{w}\\ | & | & |\end{pmatrix} = \vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{pmatrix}1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\6\\2\end{pmatrix} = 23$$

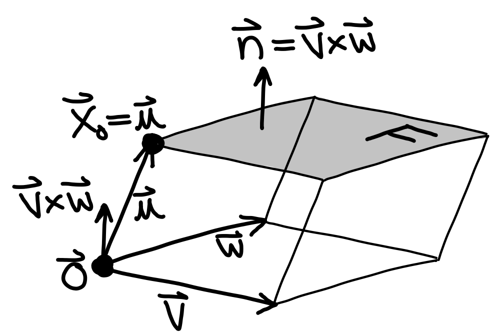
$$\text{Vol} = |\det| = 23. \qquad \text{sgn}(\det) > 0 \Rightarrow \text{RHO}.$$

   (c) Exactly one of the faces of P is (1) not parallel to $\vec{u}$ and (2) does not contain the origin. Find the equation of the plane F that contains this face. *(Hint: Some of your computations from (a) might be useful here.)*

We use

$$\vec{n} = \vec{v}\times\vec{w} = \begin{pmatrix}1\\6\\2\end{pmatrix}$$

$$\vec{x}_0 = \vec{u} = \begin{pmatrix}1\\3\\2\end{pmatrix}$$



Then the equation of F is

$$\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{x}_0 \iff 1x+6y+2z = 23$$

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3. *(20 pts)*

(a) Find the polar equation for the circle in the xy-plane with center at $(3, 0)$ and radius 3.

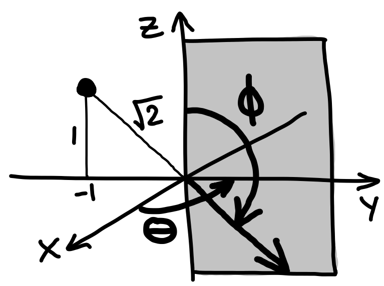
$$(x-3)^2 + y^2 = 3^2$$

$$x^2 + y^2 - 6x + 9 = 9$$

$$r^2 - 6(r\cos\theta) = 0$$

$$r = 6\cos\theta$$

(b) The point $(x, y, z) = (0, -1, 1)$ has exactly one set of spherical coordinates (ρ, ϕ, θ) with $\theta \in [0, \pi]$ and $\phi \in [0, \pi]$. Find these spherical coordinates (ρ, ϕ, θ).

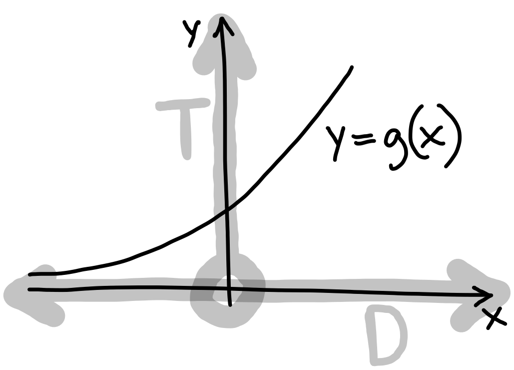


$\theta = \pi/2$

$\phi = 3\pi/4$

$\rho = -\sqrt{2}$

(c) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is not invertible. But, restricting the domain to $D \subset \mathbb{R}$ and the target to $T \subset \mathbb{R}$, the function $g : D \to T$ defined by $g(x) = e^x$ could be made invertible with appropriate choices of D and T. What are the largest choices for D and T that make g invertible?



$D = \mathbb{R}$

$T = (0, \infty)$

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4. *(20 pts)* For this question we consider $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $f(x, y) = x^2 + 4y^2$. The surface S is the graph of f.

(a) Find the equation for the surface S.

$$z = x^2 + 4y^2$$

(b) Find a function g (indicate the domain, the target, and the formula) for which the level set $g^{-1}(0)$ is the surface S.

$$z = x^2 + 4y^2 \iff z - x^2 - 4y^2 = 0$$

This is the level set $g^{-1}(0) = \{g = 0\}$ of

$g : \mathbb{R}^3 \to \mathbb{R}^1$ defined by $g(x, y, z) = z - x^2 - 4y^2$

(c) By what factor must the surface S be stretched in the y-direction to produce a surface that is rotationally symmetric around the z-axis?

$$z = x^2 + 4y^2 \xrightarrow{\text{"}y\text{"} \mapsto \text{"}\frac{y}{2}\text{"}} z = x^2 + y^2$$

This is a stretch by a factor of 2 in the y-direction, and the result has the required symmetry.

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5. *(20 pts)*

(a) Show that $f(x, y) = (x^2 - y^2, x^2 y)$ is differentiable.

All partials are polynomials, which are continuous, so f is continuously differentiable, and therefore is differentiable.

(b) Find the linear approximation for f at the point $(a, b) = (1, 2)$.

$$Df = \begin{pmatrix} 2x & -2y \\ 2xy & x^2 \end{pmatrix} \qquad Df(1,2) = \begin{pmatrix} 2 & -4 \\ 4 & 1 \end{pmatrix}$$

$$L(x,y) = f(1,2) + \left(Df(1,2)\right)\begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 & -4 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

(c) Suppose that at $(1, 2)$ the input velocity $d\vec{x}/dt$ is $(3, 1)$; compute the output velocity df/dt.

$$\frac{df}{dt} = \left(Df(1,2)\right)\frac{d\vec{x}}{dt}$$

$$= \begin{pmatrix} 2 & -4 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 13 \end{pmatrix}$$

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