EXAM 2

Math 219, 2024 Summer Term 1, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not remove the staple, tear pages out of the staple, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams (only!), but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the physical page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

You may use a straight edge to assist in your drawings, but ONLY if there is zero mathematical content on the item.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)* We have differentiable functions f and g, with $f(x, y) = (u, v)$ and $g(u, v) = (p, q)$. At the point where $x = 1$ and $y = 0$ and at the resulting values of u and v, we have

$$Df = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad Dg = \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$$

(a) Compute the partial derivatives $\frac{\partial q}{\partial x}$, $\frac{\partial q}{\partial y}$, and $\frac{\partial p}{\partial y}$ by the most efficient possible method.

$h = g \circ f$

$\begin{matrix} x \\ y \end{matrix} \xrightarrow{f} \begin{matrix} u \\ v \end{matrix} \xrightarrow{g} \begin{matrix} p \\ q \end{matrix}$

$$Dh = D(g \circ f) = (Dg)(Df)$$

$$= \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}$$

$43 = \frac{\partial p}{\partial y}$, $22 = \frac{\partial q}{\partial x}$, $50 = \frac{\partial q}{\partial y}$

(b) Find $\nabla u(1, 0)$.

$$\nabla u(1,0) = \begin{pmatrix} \frac{\partial u}{\partial x}(1,0) \\ \frac{\partial u}{\partial y}(1,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(c) From the given point compute the directional derivative $D_{\vec{c}}\, u(1, 0)$ where $\vec{c} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$$D_{\vec{c}}\, u(1,0) = \nabla u(1,0) \cdot \vec{c} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} = \frac{1 + 3\sqrt{3}}{2}$$

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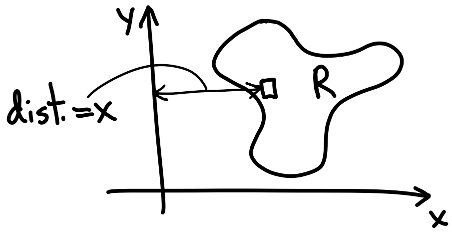
2. *(20 pts)* (The three parts of this question are not related to each other.)

(a) In this question, the amount of work that it takes to move a piece of mass by some distance is the product of the mass and the distance. Mass is distributed over the region R in the xy-plane with density $\delta(x, y) = xy$. All of this mass needs to be moved to the y-axis. Write a double integral (do not rewrite it as an iterated integral) that represents the amount of work it takes to accomplish this task.

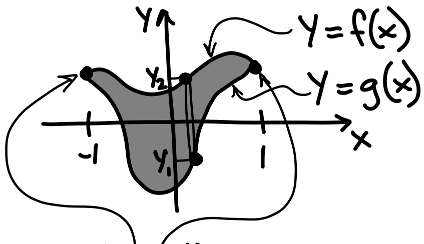


$$W = \iint_R dW$$

$$= \iint_R (\text{dist.})\, dm$$

$$= \iint_R (\text{dist.})(\delta\, dA) = \iint_R (x)(xy)\, dA$$

(b) The region D in the xy-plane is bounded by the curves with equations $y = f(x) = x^3 - x^2 + 1$ and $y = g(x) = x^4 + x^3 - x^2$. Express the double integral $\iint_D h(x, y)\, dA$ as an iterated integral.



$f(x) = g(x)$

$\Rightarrow 1 = x^4$

$\Rightarrow x = \pm 1$

$$\iint_D h(x,y)\, dA$$

$$= \int_{-1}^{1} \int_{x^4+x^3-x^2}^{x^3-x^2+1} h(x,y)\, dy\, dx$$

(c) Compute $\int_0^1 \int_y^{y^2} 30x\, dx\, dy$

$$= \int_0^1 \left(15x^2 \Big]_{x=y}^{x=y^2} \right) dy$$

$$= \int_0^1 15y^4 - 15y^2\, dy$$

$$= 3y^5 - 5y^3 \Big]_0^1 = (-2) - (0) = -2$$

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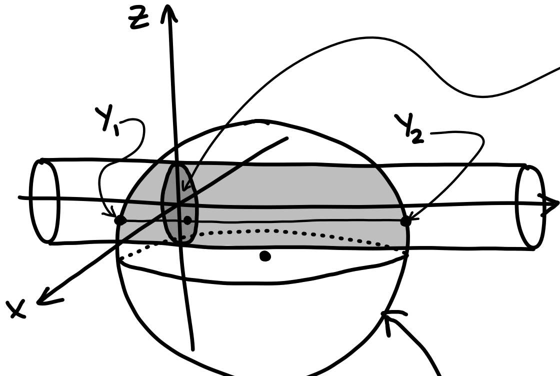
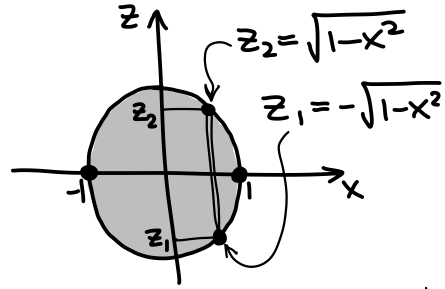
3. *(20 pts)* The solid R in xyz-space is the region inside of both (a) the cylinder centered on the y-axis of radius 1, and (b) the sphere centered at $(2, 3, 1)$ of radius 4. Set up (but do not evaluate) an explicit iterated integral representing $\iiint_R f(x, y, z)\, dV$.

Proj. to xz-plane!





$$(x-2)^2 + (y-3)^2 + (z-1)^2 = 4^2$$

$$y = 3 \pm \sqrt{16 - (x-2)^2 - (z-1)^2}$$

$$\iiint_R f\, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{3-\sqrt{16-(x-2)^2-(z-1)^2}}^{3+\sqrt{16-(x-2)^2-(z-1)^2}} f \, dy\, dz\, dx$$

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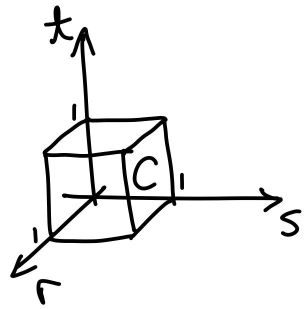
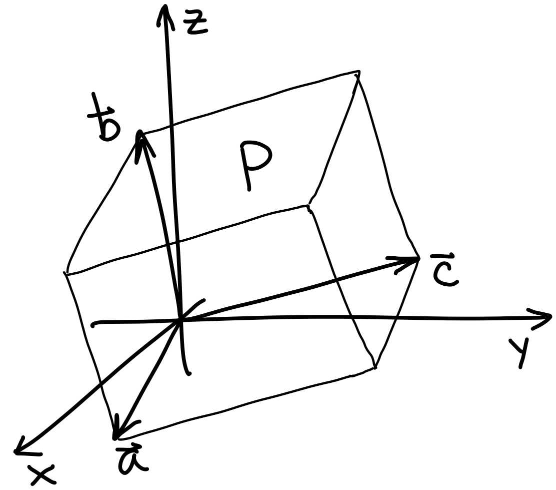
4. *(20 pts)* The parallelepiped P is defined by $P = \{r\vec{a} + s\vec{b} + t\vec{c} \mid 0 \le r, s, t \le 1\}$ where $\vec{a} = (3, 0, 1)$, $\vec{b} = (1, 0, 4)$, $\vec{c} = (0, 5, 2)$. Compute the integral $\iiint_P y\, dV$.



$$\xrightarrow{T} \quad A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 5 \\ 1 & 4 & 2 \end{pmatrix}$$



$$T\begin{pmatrix} r \\ s \\ t \end{pmatrix} = A\begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(r,s,t)} = \det DT = \det A = (-)(5)(3\cdot 4 - 1\cdot 1) = -55$$

$$\iiint_P y \, dx\,dy\,dz = \iiint_C (5t)\left(|-55|\right) dr\,ds\,dt$$

$$= \int_0^1 \int_0^1 \int_0^1 275\,t \; dr\,ds\,dt$$

$$= \int_0^1 \underbrace{\int_0^1 \int_0^1 dr\,ds}_{=1} \; 275\,t\,dt$$

$$= \int_0^1 275\,t\,dt = \frac{275}{2}$$

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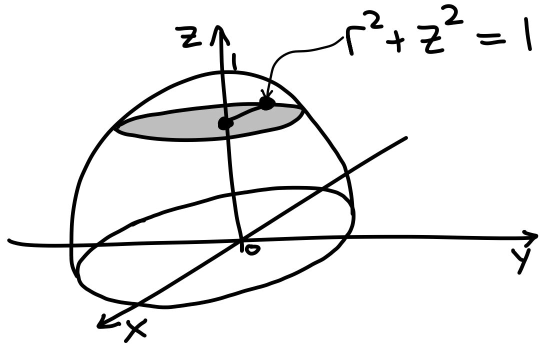
5. *(20 pts)* Compute the integral below (over part of the unit ball) by rewriting it in cylindrical coordinates.

$$\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 4x^2 + 4y^2 \, dx \, dy \, dz$$



$$= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-z^2}} (4r^2)(r \, dr \, d\theta \, dz)$$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-z^2}} 4r^3 \, dr \, d\theta \, dz)$$

$$= \int_0^1 \int_0^{2\pi} r^4 \Big]_{r=0}^{r=\sqrt{1-z^2}} d\theta \, dz)$$

$$= \int_0^1 \int_0^{2\pi} (1-2z^2+z^4) \, d\theta \, dz)$$

$$= 2\pi \int_0^1 1-2z^2+z^4 \, dz$$

$$= 2\pi \left(z - \frac{2}{3}z^3 + \frac{1}{5}z^5 \right]_0^1$$

$$= (2\pi)\left(\frac{8}{15}\right) = \frac{16\pi}{15}$$

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