EXAM 1

Math 219-01, 2025 Spring, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class and course websites are in effect on this exam.

WRITING RULES

Do not remove the staple, tear pages out of the staple, fold or "dog-ear" pages, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams (only!), but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the physical page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

You may use a straight edge to assist in your drawings, but ONLY if there is zero mathematical content on the item.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: _________________________

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1. *(20 pts)*

(a) The curve C in the xy-plane has equation $y = x^2 \sin x$. Find a parametrization of C.

Let $x = t \Rightarrow y = t^2 \sin t$

$$\Rightarrow \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \sin t \end{pmatrix}$$

(b) We have vectors $\vec{v} = (6, 2, 3)$ and $\vec{w} = (3, 6, 2)$. Find the unit vector that points in the same direction as $\vec{w}$, the component of $\vec{v}$ in the direction of $\vec{w}$, and then use these to compute the projection of $\vec{v}$ in the direction of $\vec{w}$.

$$\vec{a} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{(3,6,2)}{7}$$

$$\text{comp}_{\vec{w}}(\vec{v}) = \vec{v} \cdot \vec{a} = \frac{36}{7}$$

$$\text{proj}_{\vec{w}}(\vec{v}) = \left(\text{comp}_{\vec{w}}(\vec{v})\right) \vec{a} = \frac{36}{49}(3,6,2)$$

(c) For $\vec{v}$ and $\vec{w}$ as above, find the area of the parallelogram $\|(\vec{v}, \vec{w})$, and the unique unit vector perpendicular to both so that $(\vec{v}, \vec{w}, \vec{u})$ is in left hand order.

$$\vec{c} = \vec{v} \times \vec{w} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -14 \\ -3 \\ 30 \end{pmatrix}$$

$$\text{area} = \|\vec{c}\| = \sqrt{1105}$$

$(\vec{v}, \vec{w}, \vec{c})$ is always RHO, so $\vec{u} = \frac{-\vec{c}}{\|\vec{c}\|}$ gives the desired result.

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2. *(20 pts)*

(a) Find the equation of the unique plane in $\mathbb{R}^3$ that contains the line parametrized by $(3-t, 4+2t, t-5)$ and is parallel to the line parametrized by $(1+3t, 2+2t, 4-t)$.

$$L_1: \vec{r}_1(t) = \underbrace{\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}}_{\vec{a}_1} + t \underbrace{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}_{\vec{v}_1} \qquad L_2: \vec{r}_2(t) = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_{\vec{a}_2} + t \underbrace{\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}_{\vec{v}_2}$$

$\vec{n} = \vec{v}_1 \times \vec{v}_2 = (-4, 2, -8)$; choose $\vec{r}_0 = \vec{a}_1$.

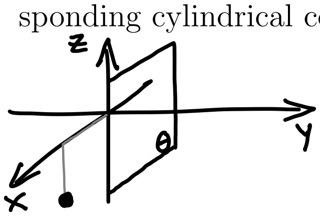
Then $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \iff -4x + 2y - 8z = 36$

(b) The surface S has cylindrical (r, θ, z) equation $rz = 1$.

i. Does S have any rotational symmetry, and if so what is the axis of symmetry?

No θ's in cyl. equation $\Longrightarrow$ rot. symm. around z-axis

ii. Is the point with rectangular coordinates $(1, 0, -1)$ on S, and if so what are the corresponding cylindrical coordinates (r, θ, z) satisfying the above equation?



$(-1, \pi, -1)$ are valid cyl. coords, and these do satisfy the eqn.

iii. Does the surface H with equation $x^2 + y^2 - z^2 = 1$? "look like" S, in the sense that some rotation would turn one into the other?

S surrounds and approaches the z-axis, while H does not do this for any line.

So S and H are not rotations of each other.

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3. *(20 pts)*

(a) S is the surface with equation $e^x z - xy^2 = -z$. Find a function (domain, target, and formula) f whose graph is S, and another function (domain, target, and formula) g whose level set $g^{-1}(0)$ is S.

$$\Leftrightarrow \quad z(1+e^x) = xy^2 \quad \Leftrightarrow \quad z = \frac{xy^2}{1+e^x}$$

This is the graph $z = f(x,y)$ of

$f: \mathbb{R}^2 \to \mathbb{R}^1$ given by $f(x,y) = \dfrac{xy^2}{1+e^x}$

$$\Leftrightarrow \quad e^x z - xy^2 + z = 0$$

This is the level set $g^{-1}(0)$ of

$g: \mathbb{R}^3 \to \mathbb{R}^1$ given by $g(x,y,z) = e^x z - xy^2 + z$

(b) Find the linear approximation L of $h(x,y) = (x^2 y^4)$ at the point $(3,2)$.

$$Dh = \begin{pmatrix} 2xy^4 & 4x^2y^3 \end{pmatrix} \qquad Dh(3,2) = \begin{pmatrix} 96 & 288 \end{pmatrix}$$

$$L(x,y) = h(3,2) + Dh(3,2)(\vec{r} - \vec{a})$$

$$= 144 + \begin{pmatrix} 96 & 288 \end{pmatrix} \begin{pmatrix} x-3 \\ y-2 \end{pmatrix}$$

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4. *(20 pts)*

(a) We have three differentiable functions $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$, with $h = c_1 f + c_2 g$ for some constants c_1 and c_2. The derivative matrices at $\vec{a}$ are given by

$$Df(\vec{a}) = \begin{pmatrix} 1 & 21 \\ -13 & 0 \end{pmatrix} \qquad Dg(\vec{a}) = \begin{pmatrix} 0 & -16 \\ 11 & 1 \end{pmatrix} \qquad Dh(\vec{a}) = \begin{pmatrix} 3 & -1 \\ 5 & 4 \end{pmatrix}$$

Find c_1 and c_2.

By linearity of D, we have $Dh = c_1 Df + c_2 Dg$.

top left $\Rightarrow 3 = c_1(1) + c_2(0) \Rightarrow c_1 = 3$

bottom right $\Rightarrow 4 = c_1(0) + c_2(1) \Rightarrow c_2 = 4$

(b) The velocity of a parametric curve is given by $\vec{v}(t) = (\sin(3t), e^t)$, and it is known also that the initial position is $\vec{r}(0) = (3, 4)$. Find the position function $\vec{r}(t)$.

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \begin{pmatrix} \int \sin 3t\,dt \\ \int e^t\,dt \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}\cos(3t) \\ e^t \end{pmatrix} + \vec{C}$$

At $t=0$: $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{-1}{3} \\ 1 \end{pmatrix} + \vec{C} \Rightarrow \vec{C} = \begin{pmatrix} 10/3 \\ 3 \end{pmatrix}$

$$\Rightarrow \vec{r}(t) = \begin{pmatrix} -\frac{1}{3}\cos(3t) \\ e^t \end{pmatrix} + \begin{pmatrix} 10/3 \\ 3 \end{pmatrix}$$

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5. *(20 pts)* We have two functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$, given by $f(x,y) = (6xy, x^2 - y^2)$ and $g(x,y) = (x - 2y, 3x + y)$.

(a) Find the derivative matrices for f and g.

$$Df = \begin{pmatrix} 6y & 6x \\ 2x & -2y \end{pmatrix} \qquad Dg = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$$

(b) Use the multivariable chain rule to find the derivative matrix for $h(x,y) = (f \circ g)(x,y)$. <u>*DO NOT*</u> explicitly compose f and g to compute h itself.

$$\begin{matrix} x \\ y \end{matrix} \xrightarrow{g} \begin{matrix} u \\ v \end{matrix} \xrightarrow{f}$$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-2y \\ 3x+y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 6uv \\ u^2 - v^2 \end{pmatrix}$$

$$Dh = Df\, Dg = \begin{pmatrix} 6v & 6u \\ 2u & -2v \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 18u + 6v & 6u - 12v \\ 2u - 6v & -4u - 2v \end{pmatrix}$$

$$= \begin{pmatrix} 36x - 30y & -30x - 24y \\ -16x - 10y & -10x + 6y \end{pmatrix}$$

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