EXAM 1

Math 219-07, 2025 Spring, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class and course websites are in effect on this exam.

WRITING RULES

Do not remove the staple, tear pages out of the staple, fold or "dog-ear" pages, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams (only!), but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the physical page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

You may use a straight edge to assist in your drawings, but ONLY if there is zero mathematical content on the item.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)*

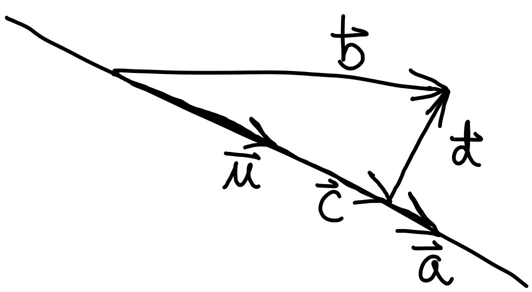
(a) The curve C is parametrized by $\vec{r}(t) = (t-2, t^3)$. Find an equation for C.

$$x = t-2 \iff t = x+2$$

$$y = t^3 \iff \boxed{y = (x+2)^3}$$

(b) We have vectors $\vec{a} = (1,1,0)$ and $\vec{b} = (0,1,1)$, and $\vec{b} = \vec{c} + \vec{d}$ where $\vec{c} \| \vec{a}$ and $\vec{d} \perp \vec{a}$. Find the vectors $\vec{c}$ and $\vec{d}$.

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{(1,1,0)}{\sqrt{2}}$$



$$\vec{c} = \text{proj}_{\vec{a}}(\vec{b}) = (\text{comp}_{\vec{a}}(\vec{b}))\,\vec{u}$$
$$= (\vec{b} \cdot \vec{u})\,\vec{u}$$
$$= \frac{1}{\sqrt{2}}\,\vec{u} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\vec{d} = \vec{b} - \vec{c} = \left(\frac{-1}{2}, \frac{1}{2}, 1\right)$$

(c) For $\vec{a}$ and $\vec{b}$ above and $\vec{e} = (1,2,3)$, decide if the list $(\vec{a}, \vec{b}, \vec{e})$ is in right hand order, left hand order, or neither, and compute the volume of the parallelepiped that they define.

$$\det\begin{pmatrix} \text{---} \vec{a} \text{---} \\ \text{---} \vec{b} \text{---} \\ \text{---} \vec{e} \text{---} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = 2$$

$$\det > 0 \Rightarrow \text{RHO}$$

$$\text{Vol} = |\det| = 2$$

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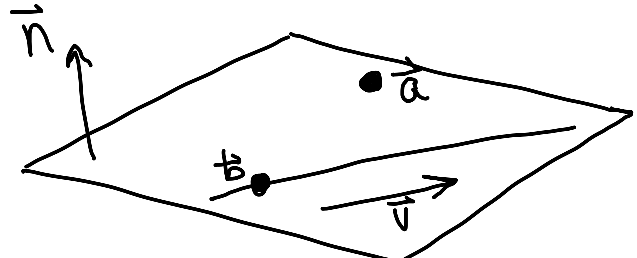
2. *(20 pts)*

(a) Find the equation of the plane in $\mathbb{R}^3$ that contains both the point $\vec{a} = (1, 2, 3)$ and the line parametrized by $\vec{r} = (1 - t, 2 + 3t, 5 - t)$.

$$= \underbrace{\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}}_{\vec{b}} + t \underbrace{\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}}_{\vec{v}}$$



Choose $\vec{n} = \vec{v} \times (\vec{b} - \vec{a}) = (-1, 3, -1) \times (0, 0, 2)$

$= (6, 2, 0)$

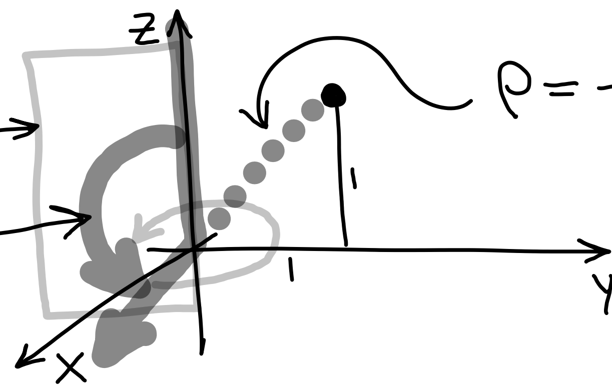
$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{a}$

$$\boxed{6x + 2y = 10}$$

(b) Find the spherical (ρ, ϕ, θ) coordinates with $\phi \in [0, \pi]$ and $\theta \in [\pi, 2\pi]$ for the point with rectangular coordinates $(0, 1, 1)$.

$\theta = \frac{3\pi}{2}$

$\phi = \frac{3\pi}{4}$



$\rho = -\sqrt{2}$

(c) The surface S has equation $x^2 - y^2 + z^2 = 1$ $\Leftrightarrow$ $(x^2 + z^2) - y^2 = 1$

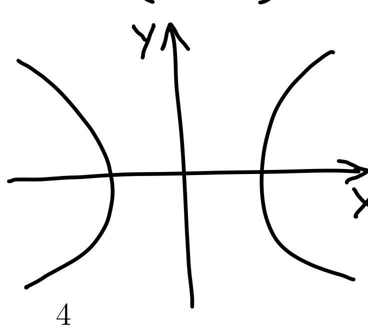
i. Identify the axis of rotational symmetry for S.

S has rotational symmetry around y-axis.

ii. Choose a coordinate plane containing the above axis, and draw (just in 2-d) the cross section of S in that coordinate plane.

In the xy-plane $(z=0)$ the eqn becomes

$x^2 - y^2 = 1$



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3. *(20 pts)*

(a) We have $f : \mathbb{R}^3 \to \mathbb{R}^1$ with $f(x, y, z) = 2x - 4y + 8z$. Find the equation for the level set $f^{-1}(16)$, and find a function g (domain, target, and formula) whose graph is that level set.

$$f^{-1}(16) : f = 16 \iff \boxed{2x - 4y + 8z = 16}$$

$$\iff z = \frac{16 - 2x + 4y}{8}$$

This is the graph $z = g(x,y)$ of

$$\boxed{g : \mathbb{R}^2 \to \mathbb{R}^1, \quad g(x,y) = \frac{16 - 2x + 4y}{8}}$$

(b) The linear approximation of the differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^1$ at the point $\vec{a} = (2, 1)$ is given by $L(x, y) = 1 + 2x + 3y$. Compute $\partial f/\partial x(\vec{a})$, $\partial f/\partial y(\vec{a})$, and $f(\vec{a})$.

$$f(\vec{a}) = L(\vec{a}) = 8$$

$$\frac{\partial f}{\partial x}(\vec{a}) = \frac{\partial L}{\partial x}(\vec{a}) = 2$$

$$\frac{\partial f}{\partial y}(\vec{a}) = \frac{\partial L}{\partial y}(\vec{a}) = 3$$

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4. *(20 pts)*

(a) The "Hessian matrix" for a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}^1$ is

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Suppose we know that at the point $\vec{a}$ we have $\det Hf > 0$. Show that at this point, the concavities of the vertical cross sections of the graph in the x-direction and the y-direction must be the same (either both upward or both downward).

$$\det Hf = f_{xx} f_{yy} - f_{xy} f_{yx} \quad \text{b/c } f \text{ is } C^2$$

$$= f_{xx} f_{yy} - f_{xy}^2 > 0$$

$$f_{xx} f_{yy} > f_{xy}^2 > 0 \Rightarrow f_{xx}, f_{yy} \text{ same sign}$$

$\Rightarrow$ concavities are the same

(b) The differentiable function $f : \mathbb{R}^3 \to \mathbb{R}^2$ has derivative matrix $Df(\vec{a})$ below. In the domain there is a particle with position given by $\vec{r}(t)$, at the point $\vec{a}$ moving with velocity $\vec{v}$ below. What is the velocity of the image $f(\vec{r}(t))$?

$$Df(\vec{a}) = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \qquad \vec{v} = (4, 5, 6) = \frac{d\vec{x}}{dt}$$

$$\frac{df}{dt} = Df(\vec{a}) \frac{d\vec{x}}{dt}$$

$$= \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 31 \end{pmatrix}$$

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5. *(20 pts)* Suppose $z = f(x,y)$ is a C^2 function, and (r,θ) are the usual polar coordinates.

(a) Find the expression for $\partial z/\partial\theta$ in terms of only the above variables and the rectangular partials of z.

$$\begin{tikzcd} r & x \arrow[r] & z \\ \theta \arrow[ur] \arrow[r] & y \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= z_x(-r\sin\theta) + z_y(r\cos\theta)$$

(b) Find an expression for $\dfrac{\partial}{\partial\theta}(y\,z_x)$

$$\begin{tikzcd} r & x \arrow[dr] & z \\ \theta \arrow[ur] \arrow[r] & y \arrow[r] & z_x \end{tikzcd}$$

$$= \frac{\partial y}{\partial \theta} z_x + y\frac{\partial z_x}{\partial \theta}$$

$$= (r\cos\theta) z_x + y\left(\frac{\partial z_x}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z_x}{\partial y}\frac{\partial y}{\partial \theta}\right)$$

$$= (r\cos\theta) z_x + y\left(z_{xx}(-r\sin\theta) + z_{xy}(r\cos\theta)\right)$$

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