

The Brumer–Stark Conjecture over \mathbf{Z}

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Abstract. In this paper we give a complete proof of the Brumer–Stark conjecture over \mathbf{Z} . In previous work, the first two named authors proved the result over $\mathbf{Z}[1/2]$. There are two conceptual difficulties that occur when attempting to generalize to the localization at $p = 2$. The first of these is the need for a general version of Ribet’s Lemma that applies in the residually indistinguishable case, which was handled in a companion paper. The second issue at $p = 2$ concerns the Ritter–Weiss module $\nabla_{\Sigma}^{\Sigma'}(H)$ studied in our previous work. The main technical contribution of this paper is the introduction of a new generalized Ritter–Weiss module $\nabla_{\Sigma}^{\Sigma'}(H)$ in which the depletion set Σ need not contain the archimedean primes, as well as a delicate handling of the constructions with modular forms and Galois representations that ensue arising from the presence of Hecke algebras that are not reduced. The connection between our Ritter–Weiss modules and Galois cohomology is handled in a more conceptual way than in our previous work, and requires new results on class formations.

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1 Introduction

Let F be a totally real field and let H be a finite abelian CM extension of F . Let \mathfrak{p} denote a prime ideal of F that splits completely in H . The Brumer–Stark conjecture states the existence of a \mathfrak{p} -unit in H whose valuations at the primes above \mathfrak{p} are related to the values $L(\chi, 0)$ as χ ranges over the characters of the extension H/F . See Theorem 1.1 below for a precise statement. In previous work, the first two named authors proved the Brumer–Stark conjecture over $\mathbf{Z}[1/2]$, i.e. we proved the existence of an element in the \mathfrak{p} -unit group tensored with $\mathbf{Z}[1/2]$ with the desired properties.

In this paper, we give a complete proof of the Brumer–Stark conjecture over \mathbf{Z} . In fact we obtain a strong refinement of the Brumer–Stark conjecture that yields the Fitting ideal of certain Ritter–Weiss modules. In a companion paper to this one, we show how to use this result to deduce the minus part of the Equivariant Tamagawa Number Conjecture (ETNC) for the Tate motive associated to H/F . Again, we obtain this integrally over \mathbf{Z} and not just over $\mathbf{Z}[1/2]$. The proof of ETNC, which is obtained by applying an idea of Bullach–Burns–Daoud–Seo to the results on Brumer–Stark proven here, yields many important new corollaries. These include Rubin’s higher rank Brumer–Stark conjecture, the integral Gross–Stark conjecture, and the higher rank version due to Popescu. One should also obtain the classical Main Conjecture of Iwasawa Theory over totally real fields at the prime $p = 2$, though there does not appear to be a precise statement of the classical Main Conjecture at $p = 2$ in the literature (but see [5, Conjecture 3.1]). Our proof of ETNC is rather formal, with the main arithmetic input being Theorem 1.2 from this paper.

There are two conceptual difficulties that occur when attempting to generalize our previous work on the Brumer–Stark conjecture to the prime $p = 2$. The first of these is in the

application of Ribet’s method, which provides a strategy for constructing nontrivial extensions of a p -adic Galois representation ρ_1 by another such representation ρ_2 . Our previous proof of Brumer–Stark over \mathbf{Z}_p for p odd is an application of a very general version of Ribet’s method. However, an important assumption that occurs throughout the literature is that the representations ρ_i are residually distinguishable, i.e. that

$$\bar{\rho}_1 \not\equiv \bar{\rho}_2 \pmod{p}.$$

In the setting of Brumer–Stark, ρ_1 is the trivial character and ρ_2 is a totally odd character χ . When p is odd, $\chi(c) = -1 \not\equiv 1 \pmod{p}$, where c denotes complex conjugation. Ribet’s method therefore carries through. However when $p = 2$, we may have $\chi \equiv 1 \pmod{2}$, and a new construction is necessary. This issue was handled in our paper [13], whose main theorem is a general version of Ribet’s Lemma that applies in the residually indistinguishable case.

The next issue at $p = 2$ concerns the Ritter–Weiss module $\nabla_{\Sigma}^{\Sigma'}(H)$ appearing in our previous work. It turns out that this module is not what appears in the Galois representations that we study locally at 2. We must alter the module $\nabla_{\Sigma}^{\Sigma'}(H)$ by modifying the local conditions at the archimedean places. On the analytic side this modification is reflected in the power of 2 in the denominator of the constant terms of Eisenstein series for F .

These modifications have an important effect on our constructions with group-ring valued Hilbert modular forms. In the case that the extension H/F is not ramified at any primes above 2, the set Σ that one would naturally like to take is empty; however it is an important assumption in our study of Ritter–Weiss modules that Σ is non-empty. One is therefore led to introduce an auxiliary prime \mathfrak{l} that does not divide the conductor of the extension H/F into the set Σ . The inclusion of this prime leads to the presence of oldforms in our constructions with Hilbert modular forms, which in turn yields a Hecke algebra \mathbf{T} that is not reduced. Working with a non-reduced Hecke algebra involves new ideas to show that the Galois representations we construct have the desired properties. Therefore, the main technical contribution of this paper is the introduction of our new *generalized Ritter–Weiss module* $\nabla_{\Sigma}^{\Sigma'}(H)$ in which the depletion set Σ need not contain the archimedean primes, as well as a delicate handling of the constructions with modular forms and Galois representations that ensue; the connection between our Ritter–Weiss modules and Galois cohomology is handled in a more conceptual way than in [11], and requires new results on class formations.

We conclude the introduction by stating precisely our main arithmetic results. First we recall the statement of the Brumer–Stark conjecture, and next we describe our stronger results involving Ritter–Weiss modules.

1.1 The Brumer–Stark Conjecture

Recall that we have fixed a totally real field F of degree n and a finite abelian extension H/F that is a CM field. Write $G = \text{Gal}(H/F)$. Let S and T denote finite disjoint sets of

places of F . Associated to any character $\chi: G \rightarrow \mathbf{C}^*$ one has the Artin L -function

$$L_S(\chi, s) = \prod_{\substack{\mathfrak{p} \notin S \\ \mathfrak{p} \text{ finite}}} \frac{1}{1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}}, \quad \operatorname{Re}(s) > 1, \quad (1)$$

and its “ T -smoothed” version

$$L_{S,T}(\chi, s) = L_S(\chi, s) \prod_{\substack{\mathfrak{p} \in T \\ \mathfrak{p} \text{ finite}}} (1 - \chi(\mathfrak{p})N\mathfrak{p}^{1-s}). \quad (2)$$

Assume now that S contains the set S_∞ of real places and the set S_{ram} of finite primes ramifying in H . Assume that T satisfies the Deligne–Ribet condition ensuring the integrality of $L_{S,T}(\chi, 0)$, namely that T contains two primes of different residue characteristic, or one prime of residue characteristic larger than $n + 1$.

Theorem 1.1 (Brumer–Stark Conjecture). *Let $\mathfrak{p} \notin S \cup T$ be a prime of F that splits completely in H . Fix a prime \mathfrak{P} of H above \mathfrak{p} . There exists an element $u \in H^*$ satisfying the following.*

- We have $|u|_w = 1$ for all places w of H not lying above \mathfrak{p} , including the complex places.

- We have

$$\sum_{\sigma \in G} \chi(\sigma) \operatorname{ord}_{\sigma^{-1}(\mathfrak{P})}(u) = L_{S,T}(\chi, 0) \quad (3)$$

for all characters $\chi: G \rightarrow \mathbf{C}^*$.

- We have $u \equiv 1 \pmod{\mathfrak{q}\mathcal{O}_H}$ for all $\mathfrak{q} \in T$.

1.2 Ritter–Weiss Modules

For the moment, let F be an arbitrary number field, and H a finite abelian extension of F with $G = \operatorname{Gal}(H/F)$. Denote by S_∞ the set of archimedean places of F and S_{ram} the set of finite places of F that ramify in H . Let Σ, Σ' denote disjoint finite sets of places of F satisfying the following conditions:

(A1) $\Sigma \cup \Sigma' \supset S_\infty \cup S_{\text{ram}}$.

(A2) The torsion subgroup of

$$\mathcal{O}_{H, S_\infty, \Sigma'}^* = \{u \in \mathcal{O}_H^* : u \equiv 1 \pmod{\mathfrak{q}} \text{ for all primes } \mathfrak{q} \text{ above primes in } \Sigma'\}$$

is trivial.

We will define a $\mathbf{Z}[G]$ -module $\nabla_{\Sigma}^{\Sigma'}(H)$ that generalizes the construction of Ritter–Weiss [24] (which did not consider the smoothing set Σ') and our previous work [11] (which imposed an additional condition $\Sigma \supset S_{\infty}$). The module $\nabla_{\Sigma}^{\Sigma'}(H)$ sits in a short exact sequence of $\mathbf{Z}[G]$ -modules

$$0 \longrightarrow \mathrm{Cl}_{\Sigma}^{\Sigma'}(H) \longrightarrow \nabla_{\Sigma}^{\Sigma'}(H) \longrightarrow X_{H,\Sigma} \longrightarrow 0.$$

Here $X_{H,\Sigma}$ denotes the group of degree zero divisors on the set of places of H above those in Σ , and $\mathrm{Cl}_{\Sigma}^{\Sigma'}(H)$ is the Σ -depleted, Σ' -smoothed class group of H (for a definition see [11, page 292]).

To state our main result, we specialize the situation and impose further conditions on Σ, Σ' . Assume that F is totally real and that H is a CM field. Denote by $c \in G$ the complex conjugation, and define

$$\mathbf{Z}[G]_{-} = \mathbf{Z}[G]/(1+c).$$

Note that $\mathbf{Z}[G]_{-}$ is finite free as a \mathbf{Z} -module and in particular has no torsion. For any $\mathbf{Z}[G]$ -module M , we define

$$M_{-} = M \otimes_{\mathbf{Z}[G]} \mathbf{Z}[G]_{-} = M/(1+c)M.$$

Fix a prime p . We impose the further conditions

$$(A3) \quad \Sigma \neq \emptyset.$$

$$(A4) \quad \Sigma \supset \{v \in S_{\mathrm{ram}} : v \mid p\}.$$

$$(A5) \quad \Sigma' \supset \{v \in S_{\mathrm{ram}} : v \nmid p\}.$$

We write

$$\nabla_{\Sigma}^{\Sigma'}(H)_{p,-} = \nabla_{\Sigma}^{\Sigma'}(H) \otimes_{\mathbf{Z}[G]} \mathbf{Z}_p[G]_{-}.$$

Under conditions (A1)–(A5), we prove that $\nabla_{\Sigma}^{\Sigma'}(H)_{p,-}$ is quadratically presented over $\mathbf{Z}_p[G]_{-}$, and hence that its Fitting ideal is principal. The following is our main result.

Theorem 1.2. *Under conditions (A1)–(A5), we have*

$$\mathrm{Fitt}_{\mathbf{Z}_p[G]_{-}}(\nabla_{\Sigma}^{\Sigma'}(H)_{p,-}) = (\Theta_{\Sigma,\Sigma'}/2^t),$$

where $t = \#(S_{\infty} \cap \Sigma')$.

Here $\Theta_{\Sigma}^{\Sigma'}/2^t$ denotes the unique element of $\mathbf{Z}_p[G]_{-}$ such that

$$\chi(\Theta_{\Sigma}^{\Sigma'}/2^t) = L_{\Sigma,\Sigma'}(\chi^{-1}, 0)/2^t \tag{4}$$

for all odd characters $\chi \in \hat{G}$. The congruence of Deligne–Ribet [16, Corollary 8.9] together with the argument of Kurihara recalled in [11, Lemma 3.4] implies that there exists an

element satisfying (4) in $\frac{c-1}{2}\mathbf{Z}_p[G]$. The existence of $\Theta_{\Sigma'}^{\Sigma'}/2^t \in \mathbf{Z}_p[G]_-$ follows. We thank Kurihara for pointing this out to us.

Theorem 1.2 was proven for p odd in [11], so it suffices to prove the result for $p = 2$. However, since most of the arguments presented here work for all p , we do not in general set $p = 2$ in what follows (though we do provide certain arguments only for $p = 2$ when the analogous results for p odd were already proven in [11]).

The paper is organized as follows. In §2, we state the main properties of the Ritter–Weiss modules $\nabla_{\Sigma'}^{\Sigma'}(H)$ that we construct later. We explain how to deduce the Brumer–Stark conjecture from Theorem 1.2 stated above. In §2.3–§2.4 we establish some notation. In §2.5 we prove the important and surprisingly delicate result that an inclusion in Theorem 1.2 is sufficient to imply an equality. In §2.7 we state precisely the main theorem from [13] on Ribet’s Lemma in the residually indistinguishable case and thereby reduce the proof of Theorem 1.2 to the construction of a triple $(\tilde{\mathbf{T}}, \varphi, \rho)$ satisfying certain properties, where $\tilde{\mathbf{T}}$ is a Hecke algebra, φ is certain homomorphism on $\tilde{\mathbf{T}}$, and ρ is a Galois representation of G_F valued in $\text{Frac}(\tilde{\mathbf{T}})$.

In §3 we construct these objects and verify their properties. We define $\tilde{\mathbf{T}}$ as the Hecke algebra of a space of *group-ring valued Hilbert modular forms*, as defined in [11]. The homomorphism φ arises through the construction of a group-ring family \mathbf{F} that is congruent to a family of Eisenstein series modulo $\Theta_{\Sigma, \Sigma'}^{\#}/2^n$. Here $\#$ denotes the \mathbf{Z} -linear map from $\mathbf{Z}[G]$ to itself given by $g \mapsto g^{-1}$ for all $g \in G$. The Galois representation ρ is defined from the Galois representations associated to Hilbert modular cusp forms, with special care taken in the case that $\tilde{\mathbf{T}}$ is not reduced, a new feature which arises here from the insertion of an auxiliary prime l into the set Σ .

In §4 we construct the Ritter–Weiss modules $\nabla_{\Sigma'}^{\Sigma'}(H)$ and establish their properties, as outlined earlier in §2. We provide a less computational approach than given in our previous work [11], which allows for the more general version required here and establishes a connection with Galois cohomology even at the prime $p = 2$. This last feature requires a new duality theorem on class formations that is proven in §5.

We would like to thank Frank Calegari, David Loeffler, and Sug Woo Shin for helpful discussions regarding the constructions of §3, particularly the results in §3.3. We would also like to thank Cristian Popescu, Masato Kurihara, and Andreas Nickel for helpful discussions.

The first named author is supported by NSF grant DMS–2200787. The second named author is supported by DST-SERB grant SB/SJF/2020-21/11, SERB MATRICS grant MTR/2020/000215, SERB SUPRA grant SPR/2019/000422, and DST FIST program - 2021 [TPN - 700661]. The fourth named author is supported by NSF grant DMS 2201346.

2 Ritter–Weiss Modules and Brumer–Stark

In this section we first state precisely the properties satisfied by the generalized Ritter–Weiss modules $\nabla_{\Sigma'}^{\Sigma}(H)$ that we construct in §4. We then explain how Theorem 1.2 implies the Brumer–Stark conjecture. Next, we will establish some preliminaries and outline the proof of Theorem 1.2.

2.1 Properties of Ritter–Weiss Modules

Consider an abelian extension of global fields H/F , with $G = \text{Gal}(H/F)$. Fix finite disjoint sets of places S, T of F , such that $S \cup T$ contains all infinite places of F . We do *not* require that all infinite places lie in S , unlike previous constructions of Ritter–Weiss [24] and of the first two named authors [11].

We will construct a $\mathbf{Z}[G]$ -module $\nabla_S^T(H)$, the *generalized Ritter–Weiss module*, as the cokernel of a map $V^{\theta} \rightarrow B^{\theta}$ of finitely generated $\mathbf{Z}[G]$ -modules. In this section we state the main properties of the modules $\nabla_S^T(H)$. These properties will be proved in §4.

Consider the following conditions:

- (A) $S \cup T$ contains all finite places of F that ramify in H .
- (B) For all finite places $v \in T$, H_w/F_v is tamely ramified.
- (B_p) For all finite places $v \in T$, H_w/F_v is p -tamely ramified (the wild inertia subgroup of H_w/F_v has order coprime to p).
- (C) The reduction map

$$\mu(H) \longrightarrow \mathbf{F}_T^* = \prod_{\text{finite } w \in T_H} (\mathcal{O}_H/w)^*$$

is injective, i.e. $\mathcal{O}_{H,S,T}^*$ is torsion-free. Here $\mathcal{O}_{H,S,T}^*$ is the set of all S -units congruent to 1 modulo all primes above primes in T .

Although we typically assume (A), (B_p), and (C) in the body of this paper, in §2.2 we briefly consider a Ritter–Weiss module where condition (A) does not hold. The following theorems also distinguish between results for general extensions H/F where S contains all the infinite places, versus results for CM extensions H/F where $S \cup T$ contains all the infinite places. The former results have largely been proven in [24] and [11], and we reprove them for the sake of completeness.

Theorem 2.1. *We have:*

1. The module $\nabla_S^T(H)$ contains $\text{Cl}_S^T(H)$ as a $\mathbf{Z}[G]$ -submodule, giving an exact sequence

$$0 \longrightarrow \text{Cl}_S^T(H) \longrightarrow \nabla_S^T(H) \longrightarrow X \longrightarrow 0. \quad (5)$$

Under assumption (A), $X = X_{H,S}$.

2. If H/F is a CM extension and S contains a place v such that the complex conjugation $c \in G$ belongs to the decomposition group G_v , then

$$0 \longrightarrow \mathrm{Cl}_S^T(H)_- \longrightarrow \nabla_S^T(H)_- \longrightarrow X_- \longrightarrow 0 \quad (6)$$

is an exact sequence of $\mathbf{Z}[G]_-$ modules.

Theorem 2.2. Assume (B) and (C).

1. When S contains all the infinite places, the modules V^θ and B^θ are projective $\mathbf{Z}[G]_-$ modules. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_{H,S,T}^* \longrightarrow V^\theta \longrightarrow B^\theta \longrightarrow \nabla_S^T(H) \longrightarrow 0.$$

If we additionally assume (A), this is a quadratic presentation.

2. Suppose that H/F is a CM extension. When $S \neq \emptyset$, there exists a map of finitely-generated projective $\mathbf{Z}[G]_-$ -modules $V' \longrightarrow (B^\theta)_-$, giving the following presentation of $\nabla_S^T(H)_-$:

$$0 \longrightarrow (\mathcal{O}_{H,S,T}^*)^- \longrightarrow V' \longrightarrow (B^\theta)_- \longrightarrow \nabla_S^T(H)_- \longrightarrow 0. \quad (7)$$

If we additionally assume (A), this is a quadratic presentation.

If we assume (B_p) instead of (B), analogous results hold after tensoring (over \mathbf{Z}) all the modules above with $\mathbf{Z}_{(p)}$.

We will prove that $\nabla_S^T(H)$ has the following relationship to Galois cohomology.

Theorem 2.3. Assume (A) and either (B) or (B_p) . Assume $S \neq \emptyset$, and fix $v_0 \in S$. Let N be a finite G -module, and if we are in case (B_p) assume that N is a p -group. Suppose we are given a 1-cocycle $\kappa: G_F \longrightarrow N$ and $x_v \in N$ for $v \in S \setminus \{v_0\}$ such that

1. $\kappa(G_{F_{v_0}}) = 0$
2. $\kappa|_{G_{F_v}}$ is unramified for $v \notin (S \cup T)$
3. $\kappa|_{G_{F_v}}$ is tamely ramified for finite $v \in T$
4. for $v \in S$, $\kappa(\sigma_v) = (\sigma_v - 1)x_v$ for all $\sigma_v \in G_{F_v}$.

Then there exists a G -equivariant map $\nabla_S^T(H) \longrightarrow N$ whose image is the $\mathbf{Z}[G]_-$ -submodule generated by $\kappa(G_F)$ and the x_v .

2.2 Brumer–Stark

As mentioned in the introduction, we will prove in a forthcoming paper that Theorem 1.2 implies the Equivariant Tamagawa Number Conjecture for the minus part of the Tate motive associated to H/F . This is a powerful statement with many corollaries, including the Brumer–Stark conjecture. Because of its singular importance, however, we would like to prove here that the Brumer–Stark conjecture follows directly from Theorem 1.2.

Let S and T be as in the statement of Theorem 1.1.

Theorem 2.4. *We have*

$$\Theta_{S,T}^{\#}/2^{n-1} \in \text{Ann}_{\mathbf{Z}[G]_-}((\text{Cl}^T(H)_-)^{\vee}), \quad (8)$$

or equivalently

$$\Theta_{S,T}/2^{n-1} \in \text{Ann}_{\mathbf{Z}[G]_-}(\text{Cl}^T(H)_-). \quad (9)$$

The equivalence of the statements follows since the annihilators of a module and its Pontryagin dual are related by $\#$.

Proof. The statement (8) over $\mathbf{Z}[1/2]$ follows from the result

$$\Theta_{S,T}^{\#} \in \text{Fitt}_{\mathbf{Z}[1/2][G]_-}(\text{Cl}^T(H)^{\vee} \otimes_{\mathbf{Z}[G]} \mathbf{Z}[1/2][G]_-)$$

known as the Strong Brumer–Stark conjecture that is proved in [11]. So it remains to consider the statement (8) after tensoring with \mathbf{Z}_2 . We fix a real place w of F and define

$$\Sigma = \{w\} \cup \{v \in S_{\text{ram}} : v \nmid 2\}, \quad \Sigma' = T \cup \{v \in S_{\text{ram}}, v \nmid 2\} \cup S_{\infty} \setminus \{w\}.$$

Then Theorem 1.2 applies and we have

$$\text{Fitt}_{\mathbf{Z}_2[G]_-}(\nabla_{\Sigma}^{\Sigma'}(H)_{2,-}) = (\Theta_{\Sigma,\Sigma'}/2^{n-1}).$$

Let $\text{Sel}_{\Sigma}^{\Sigma'}(H)_-$ denote the transpose of $\nabla_{\Sigma}^{\Sigma'}(H)_-$ over $\mathbf{Z}[G]_-$ associated to the presentation (7) defining this latter $\mathbf{Z}[G]_-$ -module, as described in §4.

Let $s = \#\{v \in S_{\text{ram}}, v \nmid 2\}$. The argument of [11, Theorem 3.7] applies directly to yield

$$\begin{aligned} \text{Fitt}_{\mathbf{Z}_2[G]_-}(\text{Sel}_{\Sigma}^T(H)_{2,-}) &= \text{Fitt}_{\mathbf{Z}_2[G]_-}^s(\nabla_{\Sigma}^T(H)_{2,-})^{\#} \\ &= (\Theta_{\Sigma,T}/2^{n-1})^{\#} \prod_{v \in S_{\text{ram}}, v \nmid 2} (NI_v, 1 - \sigma_v e_v). \end{aligned}$$

Let $s' = \#S_{\text{ram}}$. The arguments of [11, Appendix B] yield

$$\begin{aligned} \text{Fitt}_{\mathbf{Z}_2[G]_-}(\text{Sel}_{\{w\}}^T(H)_{2,-}) &= \text{Fitt}_{\mathbf{Z}_2[G]_-}^{s'}(\nabla_{\{w\}}^T(H)_{2,-})^{\#} \\ &= (\Theta_{\emptyset,T}/2^{n-1})^{\#} \prod_{v \in S_{\text{ram}}} (NI_v, 1 - \sigma_v e_v) \\ &\supset (\Theta_{S,T}/2^{n-1})^{\#}. \end{aligned}$$

It follows that $\Theta_{S,T}^\# / 2^{n-1}$ annihilates $\text{Sel}_{\{w\}}^T(H)_{2,-}$. By Lemma 4.21, we have a short exact sequence

$$0 \longrightarrow (\nabla_{\{w\}}^T(H)_-)^{\vee}_{\text{tors}} \longrightarrow \text{Sel}_{\{w\}}^T(H)_- \longrightarrow (\text{Hom}_{\mathbf{Z}}(\mathcal{O}_{H,\{w\},T}^*, \mathbf{Z}))^{-\vee} \longrightarrow 0.$$

By Proposition 4.7, $(\nabla_{\{w\}}^T(H)_-)^{\vee}_{\text{tors}}$ has $(\text{Cl}^T(H)_-)^{\vee}$ as a quotient. It follows that $\Theta_{S,T}^\# / 2^{n-1}$ annihilates $(\text{Cl}^T(H)_-)^{\vee}$ as desired. \square

The Brumer–Stark conjecture follows easily from Theorem 2.4; in fact we get a stronger statement with $\Theta_{S,T}$ replaced by $\Theta_{S,T} / 2^{n-2}$.

Proof of Theorem 1.1. Let \mathfrak{P} be as in the statement of Theorem 1.1. Theorem 2.4 implies that $\Theta_{S,T} / 2^{n-1}$ annihilates the class of \mathfrak{P} in $\text{Cl}^T(H)_-$. We may therefore write

$$\mathfrak{P}^{\Theta_{S,T} / 2^{n-1}} = (z)\mathfrak{a}^{1+c} \tag{10}$$

where $z \equiv 1 \pmod{T}$ and $\mathfrak{a} \in I_T(H)$ is a fractional ideal of H coprime to T . Applying $1 - c$ to (10), writing $u = z^{1-c}$, and noting $(1 - c)\Theta_{S,T} = 2\Theta_{S,T}$, we obtain

$$\mathfrak{P}^{\Theta_{S,T} / 2^{n-2}} = (u) \tag{11}$$

where $u \equiv 1 \pmod{T}$ and u has absolute value 1 under every complex embedding. Clearly u is a \mathfrak{p} -unit and satisfies (3) with $\Theta_{S,T}$ replaced by $\Theta_{S,T} / 2^{n-2}$. Our result is proven as long as $n \geq 2$. But $n = 1$ happens when $F = \mathbf{Q}$, and the Brumer–Stark conjecture here is Stickelberger’s classical theorem (see also the remark below). \square

Remark 2.5. In [12] we show how our results here can be used to prove that the Brumer–Stark conjecture holds with $\Theta_{S,T}$ replaced by $\Theta_{S,T} / 2^{n-1}$ when H/F is ramified at any finite prime. In particular, we recover the (known) result for $F = \mathbf{Q}$.

2.3 Minimal Sets

It will be convenient in our constructions with modular forms to consider certain minimal sets Σ . To relate the modules ∇ and Stickelberger elements Θ as Σ, Σ' vary, we note the following result proven in §4.7.

Lemma 2.6. *Assume (A1)–(A5). We have the following:*

- *Let $w \in \Sigma'$ be a real place. Then*

$$\text{Fitt}_{\mathbf{Z}_p[G]}(\nabla_{\Sigma \cup \{w\}}^{\Sigma' \setminus w}(H)) = 2 \text{Fitt}_{\mathbf{Z}_p[G]}(\nabla_{\Sigma}^{\Sigma'}(H)).$$

- Let $\mathfrak{l} \notin \Sigma \cup \Sigma'$, so by the assumptions, \mathfrak{l} is a finite prime unramified in H . Put $\sigma_{\mathfrak{l}}$ for the associated Frobenius element. Then

$$\text{Fitt}_{\mathbf{Z}_p[G]}(\nabla_{\Sigma \cup \{\mathfrak{l}\}}^{\Sigma'}(H)) = (1 - \sigma_{\mathfrak{l}}^{-1}) \text{Fitt}_{\mathbf{Z}_p[G]}(\nabla_{\Sigma}^{\Sigma'}(H))$$

and

$$\Theta_{\Sigma \cup \{\mathfrak{l}\}, \Sigma'} = (1 - \sigma_{\mathfrak{l}}^{-1}) \Theta_{\Sigma, \Sigma'}.$$

Similarly we have

$$\text{Fitt}_{\mathbf{Z}_p[G]}(\nabla_{\Sigma}^{\Sigma' \cup \{\mathfrak{l}\}}(H)) = (1 - \sigma_{\mathfrak{l}}^{-1} N\mathfrak{l}) \text{Fitt}_{\mathbf{Z}_p[G]}(\nabla_{\Sigma}^{\Sigma'}(H))$$

and

$$\Theta_{\Sigma, \Sigma' \cup \{\mathfrak{l}\}} = (1 - \sigma_{\mathfrak{l}}^{-1} N\mathfrak{l}) \Theta_{\Sigma, \Sigma'}.$$

Lemma 2.6 shows that if $\Sigma \subset \Sigma_1$ and $\Sigma' \subset \Sigma'_1$, then Theorem 1.2 for (Σ, Σ') implies the result for (Σ_1, Σ'_1) . Furthermore the distribution of real places between Σ, Σ' does not affect the validity of the conjecture (as long as we always maintain the assumption that Σ is nonempty). In what follows, we will therefore fix certain pairs (Σ, Σ') with Σ minimal and prove the result in this setting. As in [11], it is convenient to establish two cases that will appear throughout the paper.

Case 1. There are no primes above p ramified in H/F .

In this case, we fix a prime \mathfrak{l} of F that is unramified in H such that the associated Frobenius $\sigma_{\mathfrak{l}} \in G$ is the complex conjugation c . We then define

$$\begin{aligned} \Sigma &= \{\mathfrak{l}\}, \\ \Sigma' &= T \cup S_{\infty} \cup S_{\text{ram}}. \end{aligned}$$

Any other pair (Σ, Σ') with minimal Σ in Case 1 will either have the form

$$\begin{aligned} \Sigma_1 &= \{\mathfrak{l}'\}, \\ \Sigma'_1 &= T \cup S_{\infty} \cup S_{\text{ram}} \end{aligned}$$

for another finite prime \mathfrak{l}' or have the form

$$\begin{aligned} \Sigma_2 &= \{v\}, \\ \Sigma'_2 &= T \cup (S_{\infty} \setminus \{v\}) \cup S_{\text{ram}} \end{aligned}$$

for a real place v . Lemma 2.6 yields

$$\begin{aligned} \Theta_{\Sigma_1, \Sigma'_1} &= \frac{1}{2} \Theta_{\{\mathfrak{l}', \mathfrak{l}\}, \Sigma'_1} = \frac{1 - \sigma_{\mathfrak{l}'}^{-1}}{2} \Theta_{\Sigma, \Sigma'} \\ \text{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_{\Sigma_1}^{\Sigma'_1}(H)_{p,-}) &= \frac{1}{2} \text{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_{\{\mathfrak{l}, \mathfrak{l}'\}}^{\Sigma'_1}(H)_{p,-}) = \frac{1 - \sigma_{\mathfrak{l}'}^{-1}}{2} \text{Fitt}_{\mathbf{Z}_p[G]_-}((\nabla_{\Sigma}^{\Sigma'})_{p,-}). \end{aligned}$$

Therefore the statements of Theorem 1.2 for the pairs (Σ, Σ') and (Σ_1, Σ'_1) are equivalent. A similar calculation holds for (Σ_2, Σ'_2) ; by Lemma 2.6, moving an infinite place from Σ' to Σ multiplies the Fitting ideal of ∇ by 2. Therefore, in Case 1, proving Theorem 1.2 for the particular pair (Σ, Σ') above implies the result for all pairs satisfying our conditions.

Case 2. There exists a prime above p ramified in H/F .

In this case, there is a unique minimal Σ , and it suffices to prove Theorem 1.2 in this setting:

$$\begin{aligned}\Sigma &= \{v \in S_{\text{ram}} : v \mid p\}, \\ \Sigma' &= T \cup S_{\infty} \cup \{v \in S_{\text{ram}} : v \nmid p\}.\end{aligned}$$

2.4 Character Group Rings

Let \mathcal{O} denote the ring of integers of a finite extension of \mathbf{Q}_p containing the values of all characters $\chi \in \hat{G}$. There is a \mathbf{Z}_p -algebra injection

$$\mathbf{Z}_p[G] \longrightarrow \prod_{\chi \in \hat{G}} \mathcal{O}, \quad x \mapsto (\chi(x)).$$

In our constructions with modular forms, we would like to work with local rings (as opposed to the semilocal ring $\mathbf{Z}_p[G]_{-}$). Furthermore, it will be convenient if the Stickelberger element $\Theta_{\Sigma, \Sigma'}$ is a nonzerodivisor in our ring. To this end, for every $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -stable set of characters $\Psi \subset \hat{G}$ we define the associated *character group ring*

$$R_{\Psi} = \text{Image}(\mathbf{Z}_p[G] \longrightarrow \prod_{\chi \in \Psi} \mathcal{O}), \quad x \mapsto (\chi(x)).$$

If we define

$$\Psi = \{\chi \in \hat{G} : \chi \text{ is odd and } \chi(G_v) \neq 1 \text{ for all } v \in \Sigma\}$$

then the image of the Stickelberger element $\Theta_{\Sigma, \Sigma'}$ in R_{Ψ} is a nonzerodivisor. If M is any $\mathbf{Z}_p[G]$ -module, we write $M_{\Psi} = M \otimes_{\mathbf{Z}_p[G]} R_{\Psi}$.

Write $G = G_p \times G'$ where G_p is a p -group and G' has order relatively prime to p . For each $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -conjugacy class of characters Φ of G' , we define

$$\Phi_0 = \{\chi \in \hat{G} : \chi \text{ is odd, } \chi|_{G'} \in \Phi, \text{ and } \chi(G_v) \neq 1 \text{ for all } v \in \Sigma\}. \quad (12)$$

Then each R_{Φ_0} is a local ring in which $\Theta_{\Sigma, \Sigma'}$ is a nonzerodivisor, and we have a decomposition

$$R_{\Psi} = \prod_{\Phi} R_{\Phi_0}, \quad (13)$$

where Φ runs through all $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -conjugacy classes of characters of G' .

The following result was proven in [11, Lemmas 2.4 and 2.5].

Lemma 2.7. *Let M be a quadratically presented module over a character group ring R_Ψ such that $\text{Fitt}_{R_\Psi}(M) = (x)$. Then M is finite if and only if x is a nonzerodivisor, and in this case we have*

$$\#M = \#(R_\Psi/(x)) = \#(\mathbf{Z}_p / \prod_{\psi \in \Psi} \psi(x)).$$

Here $\#$ denotes size.

2.5 Inclusion Implies Equality

We recall the setting. We have a totally real field F and a CM abelian extension H/F . We write $G = \text{Gal}(H/F)$. Fix a prime p and define

$$\begin{aligned} \Sigma &= S_\infty \cup \{v \in S_{\text{ram}} : v \mid p\}, \\ \Sigma' &= T \cup \{v \in S_{\text{ram}} : v \nmid p\}. \end{aligned}$$

Let Φ_0 be as in (12). Our goal in this section is to prove:

Theorem 2.8. *Suppose that for every setting as above, we have*

$$\text{Fitt}_{R_{\Phi_0}}(\nabla_{\Sigma}^{\Sigma'}(H)_{\Phi_0}) \subset (\Theta_{\Sigma, \Sigma'}). \quad (14)$$

Then every such inclusion is actually an equality, and

$$\text{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_{\Sigma}^{\Sigma'}(H)_{p,-}) = (\Theta_{\Sigma, \Sigma'}). \quad (15)$$

Remark 2.9. In the previous section we defined certain minimal sets (Σ, Σ') . As mentioned, moving t infinite primes from Σ' to Σ multiplies the Fitting ideal of ∇ by 2^t . Furthermore removing a prime \mathfrak{l} from Σ whose Frobenius equals c (when $\Sigma \setminus \{\mathfrak{l}\}$ is nonempty) has the effect of dividing both Θ and the Fitting ideal of ∇ by 2. So Theorem 2.8 implies that if we always have

$$\text{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_{\Sigma}^{\Sigma'}(H)_{p,-}) \subset (\Theta_{\Sigma, \Sigma'} / 2^t)$$

for our minimal (Σ, Σ') , then these inclusions are all equalities, and (15) holds.

Theorem 2.8 was proven for p odd in [11], so we will assume for the remainder of this section that $p = 2$. Let

$$\Psi = \{\chi \in \hat{G} : \chi \text{ is odd and } \chi(G_v) \neq 1 \text{ for all } v \in \Sigma\}$$

and let R_Ψ be the associated character group ring. In view of the product decomposition (13), the following lemma implies that the first assertion of Theorem 2.8 implies the second.

Lemma 2.10. *With notation as above, if*

$$\text{Fitt}_{R_\Psi}(\nabla_{\Sigma}^{\Sigma'}(H)_\Psi) = (\Theta_{\Sigma, \Sigma'}),$$

then

$$\text{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_{\Sigma}^{\Sigma'}(H)_{p,-}) = (\Theta_{\Sigma, \Sigma'}).$$

Lemma 2.10 was proven in [11, Lemma 7.1]. The key point is that if $\overline{\Psi}$ denotes the set of odd characters of G not lying in Ψ , then

$$\text{Fitt}_{R_{\overline{\Psi}}}(\nabla_{\Sigma}^{\Sigma'}(H)_{\overline{\Psi}}) = 0 = \Theta_{\Sigma, \Sigma'} R_{\overline{\Psi}}.$$

We now prove the first assertion of Theorem 2.8. By Proposition 4.13 the module $\nabla_{\Sigma}^{\Sigma'}(H)_{\Psi}$ is quadratically presented and therefore its Fitting ideal is principal. Write

$$\text{Fitt}_{R_{\Psi}}(\nabla_{\Sigma}^{\Sigma'}(H)_{\Psi}) = (x).$$

Since $R_{\Psi} = \prod_{\Phi} R_{\Phi_0}$, (14) implies that $(x) \subset (\Theta_{\Sigma, \Sigma'})$ in R_{Ψ} and hence it suffices to prove that

$$\#R_{\Psi}/(x) = \#R_{\Psi}/(\Theta_{\Sigma, \Sigma'}).$$

Now it follows from Lemma 2.7 that

$$\#\nabla_{\Sigma}^{\Sigma'}(H)_{\Psi} = \#R_{\Psi}/(x) = \#\mathbf{Z}_2 \Big/ \prod_{\alpha \in \Psi} \alpha(x)$$

and

$$\#R_{\Psi}/(\Theta_{\Sigma, \Sigma'}) = \#\mathbf{Z}_2 \Big/ \prod_{\alpha \in \Psi} L_{\Sigma, \Sigma'}(\alpha^{-1}, 0) = 2\text{-part of } \prod_{\alpha \in \Psi} L_{\Sigma, \Sigma'}(\alpha^{-1}, 0).$$

We must therefore show

$$\#\nabla_{\Sigma}^{\Sigma'}(H)_{\Psi} \doteq \prod_{\alpha \in \Psi} L_{\Sigma, \Sigma'}(\alpha^{-1}, 0), \quad (16)$$

where \doteq indicates equality up to a 2-adic unit. This will be achieved through the analytic class number formula. We prove (16) by partitioning Ψ into $\text{Gal}(\overline{\mathbf{Q}}_2/\mathbf{Q}_2)$ conjugacy classes. For such a conjugacy class Φ , we have

$$\#\nabla_{\Sigma}^{\Sigma'}(H)_{\Phi} \doteq \prod_{\alpha \in \Phi} \alpha(x),$$

and we will show:

Theorem 2.11. *With notation as above, under the assumption of Theorem 2.8, we have*

$$\#\nabla_{\Sigma}^{\Sigma'}(H)_{\Phi} \doteq \prod_{\alpha \in \Phi} L_{\Sigma, \Sigma'}(\alpha^{-1}, 0). \quad (17)$$

Taking the product of (17) over all possible $\text{Gal}(\overline{\mathbf{Q}}_2/\mathbf{Q}_2)$ conjugacy classes $\Phi \subset \Psi$ yields (16) and completes the proof of Theorem 2.8. In the remainder of the section, we prove Theorem 2.11. Let H_{Φ} denote the fixed field in H of the kernel of any character in Φ .

Lemma 2.12. *We have*

$$\nabla_{\Sigma}^{\Sigma'}(H)_{\Phi} \cong \nabla_{\Sigma}^{\Sigma'}(H_{\Phi})_{\Phi}. \quad (18)$$

Proof. Since $\nabla_{\Sigma}^{\Sigma'}(H)_{\Phi} \cong (\nabla_{\Sigma}^{\Sigma'}(H)_{\text{Gal}(H/H_{\Phi})})_{\Phi}$, this follows from Lemma 4.6. \square

Write $G_{\Phi} = \text{Gal}(H_{\Phi}/F)$. The conjugacy class Φ can be written $\Phi = \phi_0\phi_1$ where ϕ_0 is a conjugacy class of 2-power order characters and ϕ_1 is a conjugacy class of odd order characters. This corresponds to the decomposition $G_{\Phi} = G_2 \times G'$ of G_{Φ} as a product of a 2-group and an odd order group. Since G' has odd order and we are working over \mathbf{Z}_2 , we have a decomposition

$$\mathbf{Z}_2[G_{\Phi}]_{-} = \prod_{\psi} R_{\psi}, \quad (19)$$

where the product ranges over the Galois conjugacy classes of characters of G that can be written $\phi_0\phi'$, where ϕ' is a conjugacy class of characters of G' . The essential point here is that every odd character of G_{Φ} has restriction to G_2 lying in ϕ_0 . (This is the key point where $p = 2$ is being used in this proof, and explains why the argument of [11, §5] for p odd was different.)

In view of (19), we have

$$\text{Cl}^{\Sigma'}(H_{\Phi})_{-} = \prod_{\psi} \text{Cl}^{\Sigma'}(H_{\Phi})_{\psi}.$$

By [11, Lemma 5.2], we have

$$\text{Cl}^{\Sigma'}(H_{\Phi})_{\psi} \cong \text{Cl}^{\Sigma'}(H_{\psi})_{\psi}. \quad (20)$$

To avoid trivial zeroes, we need to consider

$$\Sigma_c = \{v \in \Sigma : c \in G_v\} \supset S_{\infty}.$$

Let

$$\Sigma_{\psi} = \Sigma_c \cup \{v \in S_{\text{ram}}(H_{\psi}/F), v \mid 2\}.$$

We have the short exact sequence

$$0 \longrightarrow \text{Cl}_{\Sigma_{\psi}}^{\Sigma'}(H_{\psi})_{\psi} \longrightarrow \nabla_{\Sigma_{\psi}}^{\Sigma'}(H_{\psi})_{\psi} \longrightarrow (X_{\Sigma_{\psi}, H_{\psi}})_{\psi} \longrightarrow 0.$$

The inclusion (14) for the field H_{ψ} implies that

$$\prod_{\alpha \in \psi} L_{\Sigma_{\psi}, \Sigma'}(\alpha^{-1}, 0) \mid \# \nabla_{\Sigma_{\psi}}^{\Sigma'}(H_{\psi})_{\psi} = \# \text{Cl}_{\Sigma_{\psi}}^{\Sigma'}(H_{\psi})_{\psi} \cdot \#(X_{\Sigma_{\psi}, H_{\psi}})_{\psi}, \quad (21)$$

where the divisibility occurs in \mathbf{Z}_2 .

Lemma 2.13. *We have*

- $L_{\Sigma_{\psi}, \Sigma'}(\alpha^{-1}, 0) = L_{\Sigma_c, \Sigma'}(\alpha^{-1}, 0)$.
- $\text{Cl}_{\Sigma_{\psi}}^{\Sigma'}(H_{\psi})_{\psi} \cong \text{Cl}_{\Sigma_c}^{\Sigma'}(H_{\psi})_{\psi}$.

- $(X_{\Sigma_\psi, H_\psi})_\psi \cong (X_{\Sigma_c, H_\psi})_\psi \cong (X_{\Sigma_c, H_\Phi})_\psi$

Proof. The first bullet point is clear, since by definition, if $v \in \Sigma_\psi \setminus \Sigma_c$ then any $\alpha \in \psi$ is ramified at v . For the second bullet point, the kernel of

$$\text{Cl}_{\Sigma_c}^{\Sigma'}(H_\psi) \longrightarrow \text{Cl}_{\Sigma_\psi}^{\Sigma'}(H_\psi)$$

is the subgroup generated by the primes lying above $v \in \Sigma_\psi \setminus \Sigma_c$. Such a prime is fixed by the inertia subgroup I_v , but since any $\alpha \in \psi$ is ramified at v there exists a $\sigma \in I_v$ such that $\alpha(\sigma) \neq 1$ for all $\alpha \in \psi$. Since $c \notin G_v$, I_v has odd order and $\alpha(\sigma)$ is an odd order root of unity. Thus $1 - \alpha(\sigma)$ is a 2-adic unit, and the image of the class represented by v in $\text{Cl}_{\Sigma_c}^{\Sigma'}(H_\psi)_\psi$ is trivial.

The proof of the first isomorphism in the third bullet point is similar, and the second isomorphism is elementary. \square

Combining (20), (21), and Lemma 2.13, and taking the product over all ψ , we obtain

$$\prod_{\alpha \in \hat{G}_\Phi, \alpha \text{ odd}} L_{\Sigma_c, \Sigma'}(\alpha^{-1}, 0) \mid \# \text{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)_- \cdot \#(X_{\Sigma_c, H_\Phi})_-. \quad (22)$$

Lemma 2.14. *The divisibility in (22) is an equality up to sign.*

Proof. By the Artin formalism for L -functions, the left side of (22) may be written

$$\prod_{\alpha \in \hat{G}_\Phi, \alpha \text{ odd}} L_{\Sigma_c, \Sigma'}(\alpha^{-1}, 0) = L_{\Sigma_c, \Sigma'}(H_\Phi/H_\Phi^+, \epsilon, 0)$$

where H_Φ^+ denotes the maximal totally real subfield of the CM field H_Φ , and ϵ is the non-trivial character of $\text{Gal}(H_\Phi/H_\Phi^+)$. Dividing the analytic class number formulas for $\zeta_{H_\Phi, \Sigma_c, \Sigma'}(s)$ and $\zeta_{H_\Phi^+, \Sigma_c, \Sigma'}(s)$ at $s = 0$ (see [11, Equation (25)]) yields

$$L_{\Sigma_c, \Sigma'}(H_\Phi/H_\Phi^+, \epsilon, 0) = \pm \frac{\# \text{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)}{\# \text{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi^+)} \cdot \frac{R_{\Sigma_c}^{\Sigma'}(H_\Phi)}{R_{\Sigma_c}^{\Sigma'}(H_\Phi^+)}. \quad (23)$$

The groups associated to the indicated regulators are the same, namely

$$\mathcal{O}_{H_\Phi, \Sigma_c, \Sigma'}^* = \mathcal{O}_{H_\Phi^+, \Sigma_c, \Sigma'}^*,$$

but the normalization of the absolute values is off by a factor of 2 in each entry of the regulator matrix. As a result, the ratio of the regulators on the right side of (23) is $2^{\#(\Sigma_c)_{H_\Phi} - 1}$. It remains to prove that

$$\frac{\# \text{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)}{\# \text{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi^+)} = \# \text{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)_- \quad (24)$$

and

$$\#(X_{\Sigma_c, H_\Phi})_- = 2^{\#(\Sigma_c)_{H_\Phi} - 1}. \quad (25)$$

The second of these is straightforward and follows from the short exact sequence

$$0 \longrightarrow (X_{\Sigma_c, H_\Phi})_- \longrightarrow (Y_{\Sigma_c, H_\Phi})_- \longrightarrow \mathbf{Z}_- = \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

(The left exactness of this sequence can be established directly, and also follows from Lemma 4.8, which implies $\mathrm{Tor}_1^{\mathbf{Z}[G]}(\mathbf{Z}, \mathbf{Z}[G]_-) = \mathbf{Z}_+[2] = 0$.) Indeed, for $v \in \Sigma_c$ we have $c \in G_v$ so the module $(Y_{v, H_\Phi})_- \cong \mathbf{Z}[G_\Phi/G_v]/2$ has size $2^{\#\{w|v\}}$, where the exponent is the number of places w of H_Φ dividing v . This proves (25).

To prove (24), we note there is a short exact sequence

$$1 \longrightarrow \mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)^- \longrightarrow \mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi) \xrightarrow{N} \mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi^+) \longrightarrow 1, \quad (26)$$

where the labelled arrow is given by the norm. Here the *superscript* on $\mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)^-$ denotes the largest *subgroup* (rather than quotient, which is indicated by a subscript) on which c acts as -1 . It is clear that $\mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)^-$ contains the kernel of the norm but a small observation is needed to deduce equality: suppose $\mathbf{a}\mathbf{a}^c = (x)\mathbf{b}$ where $x \in (H_\Phi)_{\Sigma'}^*$ and \mathbf{b} is in the subgroup of fractional ideals generated by the primes above Σ_c . Dividing this equation by its conjugate, we see that $(x/x^c) = (1)$. Therefore $x/x^c \in \mathcal{O}_{H_\Phi, S_\infty, \Sigma'}^* = \mathcal{O}_{H_\Phi^+, \emptyset, \Sigma'}^*$. But x/x^c has absolute value 1 in every complex embedding, and $\mathcal{O}_{H_\Phi^+, S_\infty, \Sigma'}^*$ contains no nontrivial roots of unity, so $x = x^c$, i.e. $x \in (H_\Phi^+)_{\Sigma'}^*$. Thus \mathbf{a} lies in the kernel of the norm.

We should also comment on the surjectivity of the norm map in (26). By class field theory, the cokernel of this map is identified with the Galois group of the largest subextension of H_Φ/H_Φ^+ in which the primes in Σ_c split completely. Since this includes the infinite primes, the largest such subextension is trivial (the point here is that in the definition of $\mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi^+)$ one takes the quotient by principal ideals generated by all elements of $(H_\Phi^+)_{\Sigma'}^*$, not just the totally positive elements).

To conclude the proof, we just need to note the equality $\#\mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)^- = \#\mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)_-$, which follows from the tautological short exact sequence

$$1 \longrightarrow \mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)^- \longrightarrow \mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi) \xrightarrow{1+c} \mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi) \longrightarrow \mathrm{Cl}_{\Sigma_c}^{\Sigma'}(H_\Phi)_- \longrightarrow 1.$$

□

In view of Lemma 2.14, it follows that all the divisibilities (21) used to deduce (22) are equalities up to 2-adic units. For $\psi = \Phi$ this says

$$\#\nabla_{\Sigma_\Phi}^{\Sigma'}(H_\Phi)_\Phi \doteq \prod_{\alpha \in \Phi} L_{\Sigma_\Phi, \Sigma'}(\alpha^{-1}, 0). \quad (27)$$

Now, the primes in $\Sigma \setminus \Sigma_\Phi$ are unramified for the characters in H_Φ . Adding unramified primes to the smoothing set simply multiplies the Fitting ideal of ∇ by the Euler factor $(1 - \sigma_v^{-1})$ at those primes, so we immediately obtain from (27)

$$\#\nabla_{\Sigma}^{\Sigma'}(H_\Phi)_\Phi \doteq \prod_{\alpha \in \Phi} L_{\Sigma, \Sigma'}(\alpha^{-1}, 0).$$

In view of Lemma 2.12, this concludes the proof of Theorem 2.11 and hence of Theorem 2.8.

2.6 Summary

The following result summarizes the results obtained in §2 so far.

Theorem 2.15. *Fix a prime p . In Case 1 (no primes above p are ramified in H) let*

$$\Sigma = \{\mathfrak{l}\}$$

for some finite unramified prime $\mathfrak{l} \nmid p$ whose Frobenius in G equals the complex conjugation c . In Case 2, let

$$\Sigma = \{v \in S_{ram} : v \mid p\}.$$

In both cases let

$$\Sigma' = T \cup \{v \in S_{ram} : v \nmid p\} \cup S_\infty.$$

For each $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -conjugacy class of characters Φ of G' , let

$$\Phi_0 = \{\chi \in \hat{G} : \chi \text{ is odd, } \chi(G_v) \neq 1 \text{ for all } v \in \Sigma, \text{ and } \chi|_{G'} \in \Phi\}$$

and let $R = R_{\Phi_0}$ be the associated character group ring. Suppose that for each Φ we have an R -module N and a cocycle $\kappa \in Z^1(G_F, N)$ satisfying the conditions of Theorem 2.3 such that

$$\text{Fitt}_R(N) \subset (\Theta_{\Sigma, \Sigma'} / 2^n)R. \quad (28)$$

Then we have

$$\text{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_{\Sigma}^{\Sigma'}(H)_{p,-}) = (\Theta_{\Sigma, \Sigma'} / 2^n) \quad (29)$$

and in particular, the p -part of the Brumer–Stark conjecture holds.

Proof. Theorem 2.3 yields a surjective map $\nabla_{\Sigma}^{\Sigma'}(H)_R \rightarrow N$, which in conjunction with the inclusion (28) yields

$$\text{Fitt}_R(\nabla_{\Sigma}^{\Sigma'}(H)_R) \subset (\Theta_{\Sigma, \Sigma'} / 2^n)R.$$

By Theorem 2.8, the equality (29) holds. The p -part of the Brumer–Stark conjecture holds from the discussion of §2.2. \square

2.7 Ribet’s Lemma

The following version of Ribet’s Lemma applicable to our setting is proved in [13]. Of central importance is the fact that the condition of residual distinguishability $\chi \not\equiv \psi \pmod{\mathfrak{m}}$ is not assumed.

Theorem 2.16. *Let $\mathbf{T} \subset \tilde{\mathbf{T}}$ be an inclusion of commutative Noetherian rings, with \mathbf{T} local. Suppose that \mathbf{T} and $\tilde{\mathbf{T}}$ are complete with respect to the maximal ideal of \mathbf{T} . Let $\tilde{I} \subset \tilde{\mathbf{T}}$ be a nontrivial ideal and let $I = \tilde{I} \cap \mathbf{T}$. Let $K = \text{Frac}(\tilde{\mathbf{T}})$ be the total ring of fractions of $\tilde{\mathbf{T}}$ and assume that the maximal ideals of K are principal. Suppose we are given a continuous representation*

$$\rho : G_F \longrightarrow \text{GL}_2(K)$$

satisfying the following conditions.

- For $\sigma \in G_F$, the characteristic polynomial $P_{\rho(\sigma)}(x)$ lies in $\mathbf{T}[x]$. Furthermore we have

$$P_{\rho(\sigma)}(x) \equiv (x - \chi(\sigma))(x - \psi(\sigma)) \pmod{I}$$

for two characters $\chi, \psi: G_F \rightarrow \mathbf{T}^*$.

- Let $K_0 = \text{red}(K)$ denote the maximal reduced quotient of K . Write $K_0 = \prod_{i=1}^m k_i$ as a product of fields. For every projection $K \rightarrow K_0 \rightarrow k_i$, the projection of ρ to $\text{GL}_2(k_i)$ is an irreducible representation of G_F over k_i .
- There is a set of primes S such that for all $v \in S$, there exists a basis in which the restriction of ρ to a decomposition group $G_v \subset G_F$ has the form

$$\rho|_{G_v} \cong \begin{pmatrix} \eta_v & 0 \\ * & \xi_v \end{pmatrix} \quad (30)$$

for two characters $\xi_v, \eta_v: G_v \rightarrow \tilde{\mathbf{T}}^*$.

- There is a subset $\Sigma \subset S$ such that for each $v \in \Sigma$, we have $\xi_v \equiv \psi|_{G_v} \pmod{\tilde{I}}$.
- Let $\mathcal{P} = S \setminus \Sigma$. For all $v \in \mathcal{P}$, we have $(\xi_v)|_{I_v} \equiv \chi|_{I_v} \pmod{\tilde{I}}$. Here $I_v \subset G_v$ denotes the inertia group at v .

If Σ is nonempty, fix $v_0 \in \Sigma$. Choose an element $\sigma_v \in G_v$ for each $v \in \mathcal{P}$. Then there exists a finitely generated \mathbf{T} -module N and a continuous cocycle

$$\kappa \in Z^1(G_F, N(\chi\psi^{-1}))$$

satisfying the following conditions.

- The module N is generated over \mathbf{T} by $\kappa(G_F)$ and the y_v , $v \in \Sigma \setminus \{v_0\}$.
- The cohomology class represented by κ is unramified at any prime for which ρ is unramified.
- If Σ is nonempty, we have $\kappa(G_{v_0}) = 0$ and for each $v \in \Sigma \setminus \{v_0\}$, there exists $y_v \in N$ such that

$$\kappa(\sigma) = (\chi\psi^{-1}(\sigma) - 1)y_v$$

for all $\sigma \in G_v$.

- For each $v \in \mathcal{P}$, we have $\kappa(\sigma) = 0$ for all $\sigma \in I_v$.
- We have

$$\prod_{v \in \mathcal{P}} (\xi_v(\sigma_v) - \chi(\sigma_v)) \text{Fitt}_{\mathbf{T}}(N) \subset \tilde{I}. \quad (31)$$

We conclude this section by combining Theorems 2.15 and 2.16 to show that in order to prove our main result (Theorem 1.2), it suffices to construct a Galois representation with the desired properties. As explained in [11, §4.1], we may assume that T (and hence also Σ') contains no primes above p , as removing such primes alters neither of the ideals $(\Theta_{\Sigma, \Sigma'}/2^n)$ nor $\text{Fitt}_R(\nabla_{\Sigma'}^{\Sigma'}(H)_R)$.

Theorem 2.17. *Let the notation be as in the statement of Theorem 2.15. Suppose we are given an inclusion of commutative Noetherian rings $\mathbf{T} \subset \tilde{\mathbf{T}}$, with \mathbf{T} local. Suppose that \mathbf{T} and $\tilde{\mathbf{T}}$ are complete with respect to the maximal ideal of \mathbf{T} . Suppose further we have a continuous Galois representation*

$$\rho: G_F \longrightarrow \text{GL}_2(\text{Frac}(\tilde{\mathbf{T}}))$$

satisfying the properties of Theorem 2.16 where:

- The set Σ is as in Theorem 2.15 and \mathcal{P} is equal to the set of primes above p that are not in Σ .
- The representation ρ is unramified outside $\Sigma \cup \Sigma' \cup \mathcal{P}$.
- We have $\psi = \boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is the canonical character $\boldsymbol{\psi}: G_F \longrightarrow G \longrightarrow \mathbf{T}^*$. We have $\chi \equiv 1 \pmod{p^m}$ for some positive integer m large enough that $(\Theta_{\Sigma, \Sigma'}^{\#}/2^n)$ divides p^m in R .
- \mathbf{T}/I is cyclic as an R -module, i.e. the structure map $R \longrightarrow \mathbf{T}/I$ is surjective.
- If $y \in R$ and $(\prod_{v \in \mathcal{P}} (\xi_v(\sigma_v) - \chi(\sigma_v)))y \in \tilde{I}$, then $y \in (\Theta_{\Sigma, \Sigma'}^{\#}/2^n)R$.

Then Theorem 1.2 follows.

Proof. Theorem 2.16 provides a \mathbf{T} -module N and a cocycle $\kappa \in Z^1(G_F, N(\chi\boldsymbol{\psi}^{-1}))$. We consider $\bar{N} = N/(IN, p^m N)$. The projection of κ to \bar{N} can be viewed as a cocycle

$$\bar{\kappa} \in Z^1(G_F, \bar{N}(\boldsymbol{\psi}^{-1}))$$

since $\chi \equiv 1 \pmod{p^m}$.

Since \mathbf{T}/I is a cyclic R -module, the \mathbf{T} -module generators of N reduce to R -module generators of \bar{N} . All of the properties required by Theorem 2.3 are directly seen to be satisfied by the corresponding properties in Theorem 2.16, except for possibly the discussion of ramification. The Galois representation ρ is unramified outside $\Sigma \cup \Sigma' \cup \mathcal{P}$, so $\bar{\kappa}$ is unramified outside this set. Furthermore, we are given in Theorem 2.16 that $\bar{\kappa}$ is unramified at \mathcal{P} , and that the local conditions at Σ hold. Finally, $\bar{\kappa}$ is tamely ramified at the finite primes in Σ' since our module is pro- p and Σ' contains no primes above p .

To conclude, we must explain why $\text{Fitt}_R(\overline{N}(\psi^{-1})) \subset (\Theta_{\Sigma, \Sigma'}/2^n)$, or equivalently,

$$\text{Fitt}_R(\overline{N}) \subset (\Theta_{\Sigma, \Sigma'}^\# / 2^n). \quad (32)$$

We have

$$\text{Fitt}_R(\overline{N}) \subset \text{Fitt}_R(N/IN) + p^m R \subset \text{Fitt}_{\mathbf{T}}(N/IN) + p^m R \subset \text{Fitt}_{\mathbf{T}}(N) + I + p^m R.$$

Therefore, if $y \in \text{Fitt}_R(\overline{N})$, there exists $r \in R$ and $i \in I$ such that $y + i + p^m r \in \text{Fitt}_{\mathbf{T}}(N)$, so by (31) we have

$$\prod_{v \in \mathcal{P}} (\xi_v(\sigma_v) - \chi(\sigma_v))(y + p^m r) \in \tilde{I}.$$

The last assumption of the theorem then yields $y + p^m r \in (\Theta_{\Sigma, \Sigma'}^\# / 2^n)$ and since $\Theta_{\Sigma, \Sigma'}^\# / 2^n$ divides p^m , we obtain $y \in (\Theta_{\Sigma, \Sigma'}^\# / 2^n)$ as desired. \square

In §3, we will use group-ring valued Hilbert modular forms to construct the Galois representation required by Theorem 2.17.

3 Modular Forms

Let Σ, Σ' , and $R = R_{\Phi_0}$ be as in the statement of Theorem 2.15. Recall that Σ' contains no primes above p (see comment preceding Theorem 2.17).

The goal of this section is to describe certain Hecke algebras $\mathbf{T}_{\mathfrak{m}} \subset \tilde{\mathbf{T}}_{\mathfrak{m}}$, which will be Noetherian R -algebras that are complete with respect to the maximal ideal $\mathfrak{m} \subset \mathbf{T}$. We will associate to this algebra:

1. An R -algebra homomorphism $\varphi: \tilde{\mathbf{T}}_{\mathfrak{m}} \rightarrow W$ where W is another R -algebra, satisfying the following:
 - (a) The homomorphism φ is Eisenstein in the sense that $\varphi(T_{\mathfrak{q}}) = 1 + \psi(\mathfrak{q})$ for primes $\mathfrak{q} \notin \Sigma \cup \Sigma', \mathfrak{q} \nmid p$.
 - (b) The restriction of φ to $\mathbf{T}_{\mathfrak{m}}$ induces an isomorphism $\mathbf{T}_{\mathfrak{m}} \cong R/(x\Theta_{\Sigma, \Sigma'}^\# / 2^n)$ where x is a certain nonzerodivisor in R .
 - (c) For a specific element $U \in \tilde{\mathbf{T}}_{\mathfrak{m}}$ to be defined later, if $y \in R$ and $\varphi(Uy) = 0$, then $y \in (\Theta_{\Sigma, \Sigma'}^\# / 2^n)R$.
2. A continuous Galois representation

$$\rho: G_F \rightarrow \text{GL}_2(K), \quad K = \text{Frac}(\tilde{\mathbf{T}}_{\mathfrak{m}})$$

such that $\text{tr}(\rho(\text{Frob}_{\mathfrak{q}})) = T_{\mathfrak{q}}$ for $\mathfrak{q} \notin \Sigma \cup \Sigma', \mathfrak{q} \nmid p$.

The homomorphism φ and representation ρ will satisfy certain other local properties as required in Theorem 2.17. The construction of φ follows the argument in [11] with one exception—the introduction of the prime $\mathfrak{l} \in \Sigma$ in Case 1 requires an altered definition of the necessary cusp form (Corollary 3.6). Similarly in the construction of the Galois representation ρ , subtleties arising from forms old at \mathfrak{l} arise in Case 1. In this section we will focus only on these new aspects, while stating without proof the results carried over directly from [11]. For simplicity, we will simply write $\Theta^\#$ for $\Theta_{\Sigma, \Sigma'}^\#$ in this section.

3.1 Hecke Algebra

We use the definitions and notations concerning *Hilbert modular forms* and *group-ring valued Hilbert modular forms* as given in [11, §7.2–7.3]. We will follow closely the construction of *loc. cit.* §8. Define

$$\begin{aligned} \mathfrak{n} &= \text{cond}(H/F) \prod_{\mathfrak{q}|T} \mathfrak{q}, \\ \mathfrak{P} &= \text{gcd}(p^\infty, \mathfrak{n}), \\ \mathfrak{P}' &= \prod_{\mathfrak{p}|p, \mathfrak{p} \nmid \mathfrak{P}} \mathfrak{p}. \end{aligned}$$

Recall also the cases that were defined in §2.3 above—Case 1 is when $\mathfrak{P} = 1$ and Case 2 is when $\mathfrak{P} \neq 1$. In Case 1 we introduced an auxiliary prime \mathfrak{l} to ensure that Σ is non-empty. We define

$$\tilde{\mathfrak{n}} = \begin{cases} \mathfrak{n}\mathfrak{l} & \text{in Case 1} \\ \mathfrak{n} & \text{in Case 2.} \end{cases}$$

We will be considering group-ring valued Hilbert modular forms of level $\tilde{\mathfrak{n}}\mathfrak{P}'$ and weight k , where k is a large positive integer congruent to 1 modulo $(p-1)p^N$ for N sufficiently large. We denote by

$$\psi: G_F \longrightarrow G \longrightarrow R^*$$

the canonical character and consider the space of p -ordinary cusp forms $S_k(\tilde{\mathfrak{n}}\mathfrak{P}', R, \psi)^{p\text{-ord}}$.

Let $\mathbf{T} \subset \text{End}_R(S_k(\tilde{\mathfrak{n}}\mathfrak{P}', R, \psi)^{p\text{-ord}})$ denote the Hecke algebra of this space generated over R by the following operators:

- $T_{\mathfrak{q}}$ for $\mathfrak{q} \nmid \tilde{\mathfrak{n}}\mathfrak{P}'$,
- $U_{\mathfrak{p}}$ for $\mathfrak{p} \mid \mathfrak{P}$,
- the diamond operators $S(\mathfrak{a})$ for $(\mathfrak{a}, \tilde{\mathfrak{n}}\mathfrak{P}') = 1$.

Let $\tilde{\mathbf{T}} \supset \mathbf{T}$ denote algebra generated over \mathbf{T} by the operators $U_{\mathfrak{p}}$ for $\mathfrak{p} \mid \mathfrak{P}'$, and in Case 1, the operator $U_{\mathfrak{l}}$.

In the next subsection, we will construct a modular form $F_k(\boldsymbol{\psi}) \in S_k(\tilde{\mathfrak{n}}\mathfrak{P}', R, \boldsymbol{\psi})^{p\text{-ord}}$ that is congruent to an Eisenstein series modulo the maximal ideal of R (in fact we prove a much stronger congruence). The existence of this form implies the following.

Lemma 3.1. *Let k denote the residue field of the local ring R . There is an R -algebra homomorphism $\bar{\varphi} : \mathbf{T} \rightarrow k$ given by*

- $\bar{\varphi}(T_{\mathfrak{q}}) = 1 + \bar{\boldsymbol{\psi}}(\mathfrak{q})$ for $\mathfrak{q} \nmid \tilde{\mathfrak{n}}p$;
- $\bar{\varphi}(U_{\mathfrak{p}}) = 1$ for $\mathfrak{p} \mid \mathfrak{P}$;
- $\bar{\varphi}(S(\mathfrak{a})) = \bar{\boldsymbol{\psi}}(\mathfrak{a})$.

Denote by $\mathfrak{m} \subset \mathbf{T}$ the kernel of $\bar{\varphi}$. Denote by $\mathbf{T}_{\mathfrak{m}}$ and $\tilde{\mathbf{T}}_{\mathfrak{m}} = \tilde{\mathbf{T}} \otimes_{\mathbf{T}} \mathbf{T}_{\mathfrak{m}}$ the \mathfrak{m} -adic completions of \mathbf{T} and $\tilde{\mathbf{T}}$, respectively.

We now describe these Hecke algebras more concretely. Denote by M the set of p -ordinary cuspidal newforms of weight k , level dividing $\tilde{\mathfrak{n}}\mathfrak{P}'$ and nebentypus ψ for all characters $\psi \in \Psi$. For each $f \in M$, denote by f_p the ordinary stabilization of f with respect to all primes $\mathfrak{p} \mid p$. Let $E = E_f$ denote the finite extension of \mathbf{Q}_p generated by the Fourier coefficients of f , let \mathcal{O} be the ring of integers of E , and let π be a uniformizer for E . Let

$$\bar{M} = \{f \in M : c(1, (f_p)|_t) \equiv \bar{\varphi}(t) \pmod{\pi} \text{ for all } t \in \mathbf{T}\}.$$

We can state this another way. For each $f \in M$, let \mathcal{P}_f denote the prime ideal of \mathbf{T} corresponding to the ordinary p -stabilization f_p , i.e. the ideal generated by $t - c(1, (f_p)|_t)$ for all $t \in \mathbf{T}$. Then \bar{M} is the set of $f \in M$ such that $\mathcal{P}_f \subset \mathfrak{m}$.

For $f \in \bar{M}$, write $K = \text{Frac}(\tilde{\mathbf{T}}_{\mathfrak{m}})$ for the total ring of fractions of $\tilde{\mathbf{T}}_{\mathfrak{m}}$. Write K_f for the total ring of fractions of the localization $\tilde{\mathbf{T}}_{\mathcal{P}_f} = \mathbf{T}_{\mathcal{P}_f} \otimes_{\mathbf{T}} \tilde{\mathbf{T}}$. Then we have

$$K = \prod_{f \in \bar{M}} K_f. \tag{33}$$

We can be explicit about K_f . First note that $\text{Frac}(\mathbf{T}_{\mathcal{P}_f}) = E =$ the finite extension of \mathbf{Q}_p generated by the Fourier coefficients of f . Next we note that $U_{\mathfrak{p}} \in \text{Frac}(\mathbf{T})$ for $\mathfrak{p} \mid \mathfrak{P}'$, which results from the discussion of [11, §8.5]. However, the same is not necessarily true in Case 1 for $U_{\mathfrak{l}}$. As we outline below, there are two situations that occur.

- We are in Case 2, or in Case 1 and the newform f has level divisible by \mathfrak{l} . Then $K_f = E$.
- We are in Case 1 and the newform f has level not divisible by \mathfrak{l} . Then

$$K_f = E[U_{\mathfrak{l}}]/g(U_{\mathfrak{l}}) \tag{34}$$

where

$$g(x) = x^2 - c(f, \mathfrak{l})x + \psi(\mathfrak{l})N\mathfrak{l}^{k-1}$$

is the Hecke polynomial at \mathfrak{l} of f . As always there are three possibilities for the quadratic polynomial $g(x)$: it may be irreducible over E , in which case K_f is a quadratic field extension of E ; it may split as a product of distinct linear factors, in which case $K_f \cong E \times E$; it may factor as a square, in which case $K_f \cong E[x]/x^2$ is the ring of dual numbers over E (and is in particular not reduced but has a principal maximal ideal).

3.2 Cusp form and homomorphism

We now define the modular forms whose existence yields the desired homomorphism φ on $\tilde{\mathbf{T}}_{\mathfrak{m}}$. We will work in a weight $k > 1$ such that $k \equiv 1 \pmod{(p-1)p^N}$ for N sufficiently large. For now, it will be convenient to ensure that

$$\epsilon_{\text{cyc}}^{k-1} \equiv 1 \pmod{\Theta^\# / 2^n}, \quad (35)$$

where ϵ_{cyc} is the p -adic cyclotomic character. Therefore we choose a positive integer m large enough such that $\Theta^\# / 2$ divides p^m in R and ensure that $N \geq m$, so that $\epsilon_{\text{cyc}}^{k-1} \equiv 1 \pmod{p^m}$. The integer N will need to be made further sufficiently large in the results below.

We first recall the following theorem of Silliman [25], generalizing an earlier construction of Hida–Wiles.

Theorem 3.2. *Let m be a positive integer. For positive integers $k \equiv 0 \pmod{(p-1)p^N}$ with N sufficiently large, there is a modular form $V_k \in M_k(1, \mathbf{Z}_p, 1)$ such that $V_k \equiv 1 \pmod{p^m}$, and such that the normalized constant term $c_{\mathcal{A}}(0, V_k)$ for each cusp $[\mathcal{A}] \in \text{cusps}(1)$ is congruent to 1 $\pmod{p^m}$.*

We next refer the reader to [11, §8.3, Eqn. (102)], where a certain group-ring family of Eisenstein series $W_k(\boldsymbol{\psi}, 1) \in M_k(\mathfrak{n}, R, \boldsymbol{\psi})$ was defined. In the (easier) Case 2, the construction of [11, Theorem 8.18] works without change for our application. We only remark that the factor of 2^n was ignored in *loc. cit.* since this factor is a unit for p odd, but we must be careful about recording this factor now that we are including $p = 2$.

Theorem 3.3. *Suppose we are in Case 2. For an integer $k > 1$ with $k \equiv 1 \pmod{(p-1)p^N}$ and N sufficiently large, there exists a group ring form $H_k(\boldsymbol{\psi}) \in M_k(\mathfrak{n}, R, \boldsymbol{\psi})$ such that*

$$\tilde{F}_k(\boldsymbol{\psi}) = e_{\mathfrak{q}^\#}^{\text{ord}} \left(W_1(\boldsymbol{\psi}, 1)V_{k-1} - (\Theta^\# / 2^n)H_k(\boldsymbol{\psi}) \right)$$

lies in $S_k(\mathfrak{n}, R, \boldsymbol{\psi})$.

In Case 1, we need to modify the construction of [11] to account for the new prime \mathfrak{l} .

Lemma 3.4. *Let the notation be as in Theorem 3.2, with $N \geq m$. For any prime $\mathfrak{l} \nmid p$, the form*

$$V_k - V_k|_{\mathfrak{l}} \in M_k(\mathfrak{l}, \mathbf{Z}_p, 1)$$

has constant terms divisible by p^m at each cusp.

Proof. We freely borrow notation from [10] here. If we are given a cusp $\mathcal{A} \in C_{\infty}(\mathfrak{l})$ (see [10, §3.5]), then by [10, Lemma 3.13] the constant term of $V_k|_{\mathfrak{l}}$ at \mathcal{A} is equal to the constant term of V_k at the cusp denoted \mathcal{A}' in *loc. cit.* The result is then immediate since the normalized constant term of V_k at every cusp is congruent to 1 (mod p^m).

As we will explain, when $\mathcal{A} \in C_0(\mathfrak{l})$, a similar calculation shows that the constant term of $V_k|_{\mathfrak{l}}$ at \mathcal{A} equals $N\mathfrak{l}^{-k}$ times the constant term of V_k at \mathcal{A}' . This will give the desired result since $k \equiv 0 \pmod{p^N(p-1)}$ and $N \geq m$, so $N\mathfrak{l}^{-k} \equiv 1 \pmod{p^m}$.

Let $\mathcal{A} \in C_0(\mathfrak{l})$. A direct calculation using the definitions [10, equations (4) and (11)] reduces the statement above to proving $\mathfrak{b}_{\mathcal{A}'} = \mathfrak{b}_{\mathcal{A}}\mathfrak{l}^{-1}\alpha_{\mu}$. We have

$$\begin{aligned} \mathfrak{b}_{\mathcal{A}'} &= \alpha\alpha_{\mu}\mathcal{O}_F + \gamma(\mathfrak{t}_{\mu}\mathfrak{d})^{-1} \\ &= \alpha\alpha_{\mu}\mathcal{O}_F + \gamma\alpha_{\mu}(\mathfrak{t}_{\lambda}\mathfrak{d})^{-1}\mathfrak{l}^{-1} \\ &= (\alpha\mathfrak{l} + \gamma(\mathfrak{t}_{\lambda}\mathfrak{d})^{-1})\alpha_{\mu}\mathfrak{l}^{-1}. \end{aligned} \tag{36}$$

By definition of $\mathfrak{b}_{\mathcal{A}}$, we have $\mathcal{O}_F = \alpha\mathfrak{b}_{\mathcal{A}}^{-1} + \gamma(\mathfrak{t}_{\lambda}\mathfrak{d})^{-1}\mathfrak{b}_{\mathcal{A}}^{-1}$. Since $\mathcal{A} \in C_0(\mathfrak{l})$, the ideal $\gamma(\mathfrak{t}_{\lambda}\mathfrak{d})^{-1}\mathfrak{b}_{\mathcal{A}}^{-1}$ is coprime to \mathfrak{l} . Therefore

$$\mathcal{O}_F = \alpha\mathfrak{l}\mathfrak{b}_{\mathcal{A}}^{-1} + \gamma(\mathfrak{t}_{\lambda}\mathfrak{d})^{-1}\mathfrak{b}_{\mathcal{A}}^{-1},$$

or equivalently, $\mathfrak{b}_{\mathcal{A}} = \alpha\mathfrak{l} + \gamma(\mathfrak{t}_{\lambda}\mathfrak{d})^{-1}$. Substituting into (36) gives the desired result $\mathfrak{b}_{\mathcal{A}'} = \mathfrak{b}_{\mathcal{A}}\mathfrak{l}^{-1}\alpha_{\mu}$. □

Theorem 3.5 ([11, Theorem 8.17]). *Suppose we are in Case 1. For $k > 1$ and $k \equiv 1 \pmod{(p-1)p^N}$ with N sufficiently large, there exists a group ring form*

$$\tilde{F}_k(\boldsymbol{\psi}) = xW_1(\boldsymbol{\psi}, 1)V_{k-1} - W_k(\boldsymbol{\psi}, 1) - (x\Theta^{\#}/2^n)H_k(\boldsymbol{\psi})$$

that lies in $S_k(\mathfrak{n}, R, \boldsymbol{\psi})$, where $x = \Theta_{S_{\infty}}(1-k)/\Theta_{S_{\infty}}(0) \in R$ is a nonzero-divisor.

The modification we need in Case 1 to incorporate the prime \mathfrak{l} is the following.

Corollary 3.6. *Suppose we are in Case 1. For $k > 1$ and $k \equiv 1 \pmod{(p-1)p^N}$ with N sufficiently large, there exists a group ring form $H_k(\boldsymbol{\psi}_{\mathfrak{l}}) \in M_k(\mathfrak{n}\mathfrak{l}, R, \boldsymbol{\psi})$ such that*

$$\tilde{F}_k(\boldsymbol{\psi}_{\mathfrak{l}}) = xW_1(\boldsymbol{\psi}_{\mathfrak{l}}, 1)V_{k-1} - W_k(\boldsymbol{\psi}_{\mathfrak{l}}, 1) - (x\Theta^{\#}/2^n)H_k(\boldsymbol{\psi}_{\mathfrak{l}})$$

lies in $S_k(\mathfrak{n}\mathfrak{l}, R, \boldsymbol{\psi})$. Here $x \in R$ is as above.

Proof. Put

$$\tilde{F}_k(\boldsymbol{\psi}_\mathfrak{l}) = \tilde{F}_k(\boldsymbol{\psi}) - \tilde{F}_k(\boldsymbol{\psi})|_\mathfrak{l}, \quad \tilde{H}_k(\boldsymbol{\psi}_\mathfrak{l}) = H_k(\boldsymbol{\psi}) - H_k(\boldsymbol{\psi})|_\mathfrak{l}. \quad (37)$$

Then

$$\begin{aligned} \tilde{F}_k(\boldsymbol{\psi}_\mathfrak{l}) &= xW_1(\boldsymbol{\psi}, 1)V_{k-1} - W_k(\boldsymbol{\psi}, 1) - (x\Theta^\# / 2^n)H_k(\boldsymbol{\psi}) - \\ &\quad (xW_1(\boldsymbol{\psi}, 1)V_{k-1} - W_k(\boldsymbol{\psi}, 1) - x\Theta^\# H_k(\boldsymbol{\psi}))|_\mathfrak{l} \\ &= x(W_1(\boldsymbol{\psi}, 1)V_{k-1} - W_1(\boldsymbol{\psi}, 1)|_\mathfrak{l}V_{k-1}|_\mathfrak{l}) - W_k(\boldsymbol{\psi}_\mathfrak{l}, 1) - (x\Theta^\# / 2^n)\tilde{H}_k(\boldsymbol{\psi}_\mathfrak{l}) \\ &= xW_1(\boldsymbol{\psi}_\mathfrak{l}, 1)V_{k-1} - W_k(\boldsymbol{\psi}_\mathfrak{l}, 1) - (x\Theta^\# / 2^n)\tilde{H}_k(\boldsymbol{\psi}_\mathfrak{l}) + xW_1(\boldsymbol{\psi}, 1)|_\mathfrak{l}(V_{k-1} - V_{k-1}|_\mathfrak{l}) \end{aligned}$$

By Lemma 3.4 this form has constant terms divisible by xp^m . One can then finish the proof as in [11, Theorem 8.17]. \square

Remark 3.7. Here $W_1(\boldsymbol{\psi}_\mathfrak{l}, 1)$ and $W_k(\boldsymbol{\psi}_\mathfrak{l}, 1)$ are Eisenstein series with $\boldsymbol{\psi}$ viewed as having modulus divisible by the prime \mathfrak{l} . On the other hand, $\tilde{F}_k(\boldsymbol{\psi}_\mathfrak{l})$ and $H_k(\boldsymbol{\psi}_\mathfrak{l})$ are just notations, defined by (37).

Recall that our ring R is associated to a $\text{Gal}(\overline{\mathbf{Q}}_p / \mathbf{Q}_p)$ -conjugacy class of characters Φ of $G' \subset G$. Let $\chi \in \Phi$ and define

$$\mathfrak{P}'' = \prod_{\mathfrak{p} | \mathfrak{P}', \chi(\mathfrak{p}) \neq 1} \mathfrak{p}.$$

This is of course independent of the choice of $\chi \in \Phi$. The proof of Corollaries 8.19 and 8.21 of [11] applies directly to deduce the following from Corollary 3.6.

Corollary 3.8. *Suppose we are in Case 1. For $k > 1$ and $k \equiv 1 \pmod{(p-1)p^N}$ with N sufficiently large, there exists a cuspidal group ring family $F_k(\boldsymbol{\psi}_\mathfrak{l}) \in S_k(\mathfrak{n}\mathfrak{P}\mathfrak{l}, R, \boldsymbol{\psi})^{p\text{-ord}}$ such that*

$$F_k(\boldsymbol{\psi}_\mathfrak{l}) \equiv \begin{cases} xW_1(\boldsymbol{\psi}_\mathfrak{l}, 1) - W_k(\boldsymbol{\psi}_\mathfrak{l}, 1_p) \pmod{x\Theta^\# / 2^n} & \text{if } \mathfrak{P}' \neq \mathfrak{P}'', \\ W_1(\boldsymbol{\psi}_{\mathfrak{l}p}, 1) \pmod{\Theta^\# / 2^n} & \text{if } \mathfrak{P}' = \mathfrak{P}''. \end{cases} \quad (38)$$

In Case 2 there exists a cuspidal group ring family $F_k(\boldsymbol{\psi}) \in S_k(\mathfrak{n}\mathfrak{P}', R, \boldsymbol{\psi})^{p\text{-ord}}$ such that

$$F_k(\boldsymbol{\psi}) \equiv W_1(\boldsymbol{\psi}_{\mathfrak{P}\mathfrak{P}'}, 1) \pmod{\Theta^\# / 2^n}.$$

Lemma 3.1 stated above follows directly from Corollary 3.8 as in [11, Lemma 8.22].

Theorem 3.9. *In both Cases 1 and 2, there exists a non-zerodivisor $x \in R$, an $R/(x\Theta^\# / 2^n)$ -algebra W , and a surjective R -algebra homomorphism $\varphi : \tilde{\mathbf{T}}_m \rightarrow W$ satisfying the following properties. We have $x = 1$ unless we are in Case 1 and $\mathfrak{P}' \neq \mathfrak{P}''$.*

- *The structure map $R/(x\Theta^\# / 2^n) \rightarrow W$ is an injection.*
- *The restriction of φ to \mathbf{T}_m takes values in $R/(x\Theta^\# / 2^n) \subset W$. More precisely,*

$$\begin{aligned} \varphi(S(\mathfrak{a})) &= \boldsymbol{\psi}(\mathfrak{a}) \text{ for } \mathfrak{a} \in G_n^+, \\ \varphi(T_{\mathfrak{q}}) &= \epsilon_{\text{cyc}}^{k-1}(\mathfrak{q}) + \boldsymbol{\psi}(\mathfrak{q}) \text{ for } \mathfrak{q} \nmid \mathfrak{np}, \end{aligned} \quad (39)$$

$$\varphi(U_{\mathfrak{p}}) = 1 \text{ for } \mathfrak{p} \mid \mathfrak{P}. \quad (40)$$

- In Case 1 we have

$$\varphi(U_{\mathfrak{l}}) = \begin{cases} \epsilon_{\text{cyc}}^{k-1}(\mathfrak{l}) & \text{if } \mathfrak{P}' \neq \mathfrak{P}'' \\ 1 & \text{if } \mathfrak{P}' = \mathfrak{P}'' \end{cases} \quad (41)$$

- Let

$$U = \prod_{\mathfrak{p}|\mathfrak{P}'} (U_{\mathfrak{p}} - \varphi(\mathfrak{p})) \in \tilde{\mathbf{T}}_{\mathfrak{m}}.$$

If $y \in R$ and $\varphi(U)y = 0$ in W , then $y \in (\Theta^{\#}/2^n)$.

Proof. This result is proved using the forms $F_k(\boldsymbol{\psi}_{\mathfrak{l}})$ and $F_k(\boldsymbol{\psi})$ exactly as in the proof of [11, Theorem 8.23]. The only extra feature is that in Case 1, we must verify that

$$F_k(\boldsymbol{\psi}_{\mathfrak{l}})|_{U_{\mathfrak{l}}} \equiv \begin{cases} \text{Nt}^{k-1} F_k(\boldsymbol{\psi}_{\mathfrak{l}}) \pmod{x\Theta^{\#}/2^n} & \text{if } \mathfrak{P}' \neq \mathfrak{P}'' \\ F_k(\boldsymbol{\psi}_{\mathfrak{l}}) \pmod{\Theta^{\#}/2^n} & \text{if } \mathfrak{P}' = \mathfrak{P}'' \end{cases}.$$

This follows directly from (38). (Note that the first case uses (35).) \square

3.3 Galois Representation

It remains now to construct the Galois representation $\rho: G_F \rightarrow \text{GL}_2(K)$. We do this on each factor of $K = \prod_{f \in \overline{M}} K_f$ in the decomposition (33). To each f (say with nebentypus ψ), Hida and Wiles [26, Theorems 1 and 2] attach a continuous irreducible Galois representation

$$\rho_f: G_F \rightarrow \text{GL}_2(E) \quad (42)$$

satisfying the following properties:

- ρ_f is unramified outside $\tilde{\mathfrak{n}}p$.
- For all primes $\mathfrak{q} \nmid \tilde{\mathfrak{n}}p$, the characteristic polynomial of $\rho(\text{Frob}_{\mathfrak{q}})$, where $\text{Frob}_{\mathfrak{q}}$ denotes arithmetic Frobenius, is given by

$$\text{char}(\rho_f(\text{Frob}_{\mathfrak{q}}))(x) = x^2 - c(\mathfrak{q}, f)x + \psi(\mathfrak{q})\epsilon_{\text{cyc}}^{k-1}(\mathfrak{q}), \quad (43)$$

- For all $\mathfrak{p} \mid p$, we have

$$\rho_f|_{G_{\mathfrak{p}}} \sim \begin{pmatrix} \eta_{\mathfrak{p}} & 0 \\ * & \xi_{\mathfrak{p}} \end{pmatrix} \quad (44)$$

where $\eta_{\mathfrak{p}}, \xi_{\mathfrak{p}}: G_{\mathfrak{p}} \rightarrow E^*$ are characters with $\eta_{\mathfrak{p}}$ unramified and satisfying $\eta_{\mathfrak{p}}(\text{Frob}_{\mathfrak{p}}) = c(\mathfrak{p}, f_p)$. Here $\text{Frob}_{\mathfrak{p}}$ denotes a lifting to $G_{\mathfrak{p}}^{\text{ab}}$ of the Frobenius element on the maximal unramified extension of $F_{\mathfrak{p}}$.

We now define

$$\tilde{\rho}_f = \rho_f \otimes_E K_f: G_F \rightarrow \text{GL}_2(K_f).$$

We need to show that $\tilde{\rho}_f$ satisfies a statement analogous to (44) for $(\rho_f)|_{G_{\mathfrak{l}}}$ in Case 1. Before stating this, we need the following lemma which will allow us to refine our choice of \mathfrak{l} .

Lemma 3.10. *Let f be a Hilbert modular newform of weight greater than 1. Then there is a density one set of primes \mathfrak{q} of F such that $\rho_f|_{G_{\mathfrak{q}}}$ is not a scalar.*

Proof. As the weight of f is greater than 1, the projectivization of ρ_f has infinite image. The Čebotarev density theorem gives a density one set of primes of F with non-trivial image under the projectivization of ρ_f . \square

Until now, in Case 1 we have put no assumption on \mathfrak{l} except that it is coprime to $n\mathfrak{p}$ and that its Frobenius is the complex conjugation in $\text{Gal}(H/F)$. Using Lemma 3.10 we now choose \mathfrak{l} such that it additionally satisfies the following condition: for each $f \in \overline{M}$ with level not divisible by \mathfrak{l} , we have that $\rho_f|_{G_{\mathfrak{l}}}$ is not scalar. Note that this is not circular, since the set of $f \in \overline{M}$ with level not divisible by \mathfrak{l} is a fixed finite set that is independent of \mathfrak{l} .

Lemma 3.11. *In Case 1 we have*

$$\tilde{\rho}_f|_{G_{\mathfrak{l}}} \sim \begin{pmatrix} \eta_{\mathfrak{l}} & 0 \\ * & \xi_{\mathfrak{l}} \end{pmatrix} \quad (45)$$

where $\eta_{\mathfrak{l}}, \xi_{\mathfrak{l}} : G_{\mathfrak{l}} \rightarrow E^*$ are characters with $\eta_{\mathfrak{l}}$ unramified and satisfying $\eta_{\mathfrak{l}}(\text{Frob}_{\mathfrak{l}}) = U_{\mathfrak{l}}$.

Proof. If \mathfrak{l} divides the level of f , then since \mathfrak{l}^2 does *not* divide the level of f and the nebentypus ψ is unramified at \mathfrak{l} , the desired form (45) follows from a result of Carayol [6]. Indeed, the local automorphic representation associated to f is an unramified twist of Steinberg (see [7] or [18, Proposition 2.8(2)]), and the Galois representation corresponding to this under local Langlands is described in [6, §6.6].

Now suppose that \mathfrak{l} does not divide the level of f . Then the representation ρ_f is unramified at \mathfrak{l} , and equation (43) is satisfied for $\mathfrak{q} = \mathfrak{l}$. We must therefore simply show that $\rho(\text{Frob}_{\mathfrak{l}})$ is conjugate to a matrix of the form $\begin{pmatrix} U_{\mathfrak{l}} & 0 \\ * & * \end{pmatrix}$ over $\text{GL}_2(K_f)$, where K_f is as in (34). This follows from the following elementary result in linear algebra. \square

Lemma 3.12. *Let E be a field and $A \in M_2(E)$. Suppose that A has minimal polynomial $g(x) = x^2 - cx + d$ and let $K = E[U]/g(U)$. Then A is conjugate via $\text{GL}_2(K)$ to a matrix of the form $\begin{pmatrix} U & 0 \\ * & * \end{pmatrix}$.*

Proof. Let $U' = c - U$ denote the other root in the factorization $g(x) = (x - U)(x - U')$ over K . We must simply show that A has an eigenvector $v \in K^2$ with eigenvalue U' such that v can be completed to a basis of K^2 . The rational canonical form shows that A is conjugate via $\text{GL}_2(E)$ to $\begin{pmatrix} 0 & 1 \\ -d & c \end{pmatrix}$. Let $\{v_1, v_2\}$ be the associated basis. Then $v = v_1 + v_2U'$ is the desired vector. \square

Remark 3.13. Lemmas 3.11 and 3.12 make it clear why we must insist that $\rho_f(\text{Frob}_{\mathfrak{l}})$ is not scalar in Case 1 when \mathfrak{l} does not divide the level of f —a scalar matrix is only conjugate to

itself. Interestingly, the situation where $\rho(\text{Frob}_l)$ has one eigenvalue with multiplicity 2 but is not scalar is conjectured to never occur. It is curious and somewhat fortuitous that we can still handle this conjecturally vacuous case via Lemma 3.12, and that we *cannot* handle via similar means the case where $\rho(\text{Frob}_l)$ is scalar (which *does* occur), but that we can avoid the scalar case by the Čebotarev argument of Lemma 3.10.

We define ρ by putting the $\tilde{\rho}_f$ together:

$$\rho = \prod_{f \in \bar{M}} \tilde{\rho}_f : G_F \longrightarrow \text{GL}_2(K). \quad (46)$$

By construction, the representation ρ satisfies the following properties:

- ρ is unramified outside $\tilde{\mathfrak{n}}p$.
- For all primes $\mathfrak{q} \nmid \tilde{\mathfrak{n}}p$, the characteristic polynomial of $\rho(\text{Frob}_{\mathfrak{q}})$ is given by

$$\text{char}(\rho(\text{Frob}_{\mathfrak{q}}))(x) = x^2 - T_{\mathfrak{q}}x + \boldsymbol{\psi}(\mathfrak{q})\epsilon_{\text{cyc}}^{k-1}(\mathfrak{q}). \quad (47)$$

- For all primes $\mathfrak{p} \mid p$, and for $\mathfrak{p} = \mathfrak{l}$ in Case 1, we have

$$\rho|_{G_{\mathfrak{p}}} \sim \begin{pmatrix} \eta_{\mathfrak{p}} & 0 \\ * & \xi_{\mathfrak{p}} \end{pmatrix},$$

where $\eta_{\mathfrak{p}}, \xi_{\mathfrak{p}} : G_{\mathfrak{p}} \longrightarrow \tilde{\mathbf{T}}^*$ are characters with $\eta_{\mathfrak{p}}$ unramified and $\eta_{\mathfrak{p}}(\text{Frob}_{\mathfrak{p}}) = U_{\mathfrak{p}}$.

Next we show that ρ satisfies all the properties required by Theorem 2.16, as specified further in Theorem 2.17. Let $\mathcal{P} = \{\mathfrak{p} \mid p, \mathfrak{p} \notin \Sigma\}$ and $S = \Sigma \cup \mathcal{P}$.

Theorem 3.14. *Let $\chi = \epsilon_{\text{cyc}}^{k-1}$. The representation ρ satisfies the following.*

1. For $\sigma \in G_F$, the characteristic polynomial $P_{\rho(\sigma)}(x)$ lies in $\mathbf{T}[x]$. Furthermore we have

$$P_{\rho(\sigma)}(x) \equiv (x - \chi(\sigma))(x - \boldsymbol{\psi}(\sigma)) \pmod{I}.$$

2. Let $K_0 = \text{red}(K)$ denote the maximal reduced quotient of K . Write $K_0 = \prod_{i=1}^m k_i$ as a product of fields. For every projection $K \rightarrow K_0 \rightarrow k_i$, the projection of ρ to $\text{GL}_2(k_i)$ is an irreducible representation of G_F over k_i .
3. For each $v \in \Sigma$, we have $\xi_v \equiv \boldsymbol{\psi}|_{G_v} \pmod{\tilde{I}}$.
4. For all $v \in \mathcal{P}$, we have $(\xi_v)|_{I_v} = \chi|_{I_v}$.

Proof. We verify the statements in turn.

(1) Since $\varphi(T_{\mathfrak{q}}) = \chi(\mathfrak{q}) + \boldsymbol{\psi}(\mathfrak{q})$ by (39), we have

$$\text{char}(\rho(\text{Frob}_{\mathfrak{q}}))(x) \equiv (x - \epsilon_{\text{cyc}}^{k-1}(\text{Frob}_{\mathfrak{q}}))(x - \boldsymbol{\psi}(\text{Frob}_{\mathfrak{q}})) \pmod{I}$$

for all $\mathfrak{q} \nmid \tilde{\mathfrak{n}}p$. Since the $\text{Frob}_{\mathfrak{q}}$ are dense in G_F by Čebotarev, we have $\text{char}(\rho(\sigma))(x) \in \mathbf{T}[x]$ for all $\sigma \in G_F$ and

$$\text{char}(\rho(\sigma))(x) \equiv (x - \epsilon_{\text{cyc}}^{k-1}(\sigma))(x - \boldsymbol{\psi}(\sigma)) \pmod{I}.$$

(2) The projections of ρ to the field factors of the reduced quotient of K are just the representations ρ_f , which are irreducible.

(3) For $v \in \Sigma$, we have $\xi_v = \chi\boldsymbol{\psi}\eta_v^{-1}$. We must show that $\chi|_{I_v} \equiv \eta_v \pmod{\tilde{I}}$. In Case 1, we have $\Sigma = \{\mathfrak{l}\}$. Then $\eta_{\mathfrak{l}}$ and χ are both unramified at \mathfrak{l} . Furthermore $\eta_{\mathfrak{l}}(\text{Frob}_{\mathfrak{l}}) = U_{\mathfrak{l}}$. If $\mathfrak{P}' \neq \mathfrak{P}''$, then $U_{\mathfrak{l}} \equiv \chi(\mathfrak{l}) \pmod{\tilde{I}}$ by (41), so we are done. If $\mathfrak{P}' = \mathfrak{P}''$ then (41) states $U_{\mathfrak{l}} \equiv 1 \pmod{\tilde{I}}$. But in this case we have $x = 1$ so $\Theta^{\#}/2^n \in \tilde{I}$, whence the congruence $\chi \equiv 1 \pmod{\Theta^{\#}/2^n}$ specified in (35) yields the desired result. In Case 2, we have $\Sigma = \{\mathfrak{p} \mid \mathfrak{P}\}$ and a similar argument holds. We again have $x = 1$ so $\chi \equiv 1 \pmod{\tilde{I}}$, and (40) shows that $\eta_{\mathfrak{p}} \equiv 1 \pmod{\tilde{I}}$ as well.

(4) For $v \in \mathcal{P}$, we again have $\xi_v = \chi\boldsymbol{\psi}\eta_v^{-1}$, which equals χ on I_v since η_v and $\boldsymbol{\psi}$ are unramified at v . □

We conclude this section by noting that in the application of Theorems 2.16 and 2.17 we choose for $\mathfrak{p} \in \mathcal{P}$ the elements $\sigma_{\mathfrak{p}} \in G_{\mathfrak{p}}$ to be lifts of $\text{Frob}_{\mathfrak{p}} \in G_{\mathfrak{p}}^{\text{ab}}$. Then

$$\xi_{\mathfrak{p}}(\sigma_{\mathfrak{p}}) - \chi(\sigma_{\mathfrak{p}}) = \epsilon_{\text{cyc}}^{k-1}(\sigma_{\mathfrak{p}})(\boldsymbol{\psi}(\mathfrak{p})U_{\mathfrak{p}}^{-1} - 1)$$

is a unit multiple of $(U_{\mathfrak{p}} - \boldsymbol{\psi}(\mathfrak{p}))$. Hence the implication

$$y \in R \text{ and } \left(\prod_{v \in \mathcal{P}} (\xi_v(\sigma_v) - \chi(\sigma_v)) \right) y \in \tilde{I} \implies y \in (\Theta^{\#}/2^n)R$$

required by Theorem 2.17 follows from the last bullet point of Theorem 3.9. This concludes the proof that the Hecke algebras $\mathbf{T}_{\mathfrak{m}} \subset \tilde{\mathbf{T}}_{\mathfrak{m}}$, the homomorphism φ , and the Galois representation ρ satisfy the properties used in §2.

4 Construction of Generalized Ritter–Weiss Modules

In the remainder of the paper we construct our generalized Ritter–Weiss modules $\nabla_S^T(H)$ and establish the properties stated in §2.1. We will use the language of class formations; for the benefit of the reader, we recall this terminology in §4.1.

4.1 Class modules and class formations

We recall the basic properties of class modules following [21]. Let G be a finite group. A discrete G -module C is a *class module* if

1. $H^1(G, C) = 0$,
2. $H^2(G, C)$ is cyclic of order $\#G$.

For class module C , a choice of generator $\gamma_G \in H^2(G, C)$ is known as a *fundamental class*. For any G -module C , an element $\gamma \in H^2(G, C)$ defines a 2-extension of the form

$$0 \longrightarrow C \longrightarrow C(\gamma) \longrightarrow \mathbf{Z}[G] \longrightarrow \mathbf{Z} \longrightarrow 0. \quad (48)$$

Theorem 4.1 (Tate). *Let C be a discrete G -module and $\gamma \in H^2(G, C)$. The following are equivalent:*

1. C is a class module with fundamental class γ .
2. $C(\gamma)$ is G -cohomologically trivial.
3. For all $i \in \mathbf{Z}$ and all subgroups $H \subset G$, the cup product on Tate cohomology

$$\cup \text{res}_{G,H}(\gamma) : \widehat{H}^i(H, \mathbf{Z}) \longrightarrow \widehat{H}^{i+2}(H, C)$$

is an isomorphism.

In particular, if C is a class module for G , cup-product with γ defines the *Nakayama map*:

$$G^{ab} = \widehat{H}^{-2}(G, \mathbf{Z}) \cong \widehat{H}^0(G, C) = C^G / N_G(C).$$

The *reciprocity map* is then defined to be

$$\text{rec} : C^G \longrightarrow C^G / N_G(C) \cong G^{ab}.$$

Now, consider a profinite group G and a discrete G -module C . The pair (G, C) is called a *class formation* if, for all open subgroups $H' \subset H$ of G with H' normal in H , the H/H' -module $C^{H'}$ is a class module for H/H' equipped with a choice of fundamental class $\gamma_{H/H'}$. The fundamental classes are required to satisfy certain compatibilities under inflation and restriction ([21], Definition 3.1.8). When working with a fixed class formation, we will abbreviate $C(H/H') := C(\gamma_{H/H'})$.

For a profinite class formation (G, C) and an open subgroup H of G , there is a reciprocity map $\text{rec} : C^H \longrightarrow H^{ab}$, defined as the inverse limit of the reciprocity maps $C^H \longrightarrow (H/H')^{ab}$ over open normal subgroups $H' \subset H$. The reciprocity map induces an isomorphism

$$\text{rec} : \widehat{(C^H)}_{\text{norm}} \cong H^{ab},$$

where $(\widehat{C^H})_{\text{norm}} := \varprojlim_{H'} C^H/N_{H/H'}(C^{H'})$ is the completion in the norm topology on C^H .

We will also consider, for any open normal subgroup $H \subset G$, the 2-extension

$$0 \longrightarrow H^{ab} \longrightarrow E(G/H) \longrightarrow \mathbf{Z}[G/H] \longrightarrow \mathbf{Z} \longrightarrow 0 \quad (49)$$

corresponding to pushing out the 2-extension

$$0 \longrightarrow C^H \longrightarrow C(G/H) \longrightarrow \mathbf{Z}[G/H] \longrightarrow \mathbf{Z} \longrightarrow 0$$

along the map $C^H \longrightarrow H^{ab}$. The module $E(G/H)$ is topologized using the profinite topology on H^{ab} and the discrete topology on $\mathbf{Z}[G/H]$.

An *inclusion of class formations* $(G', C') \longrightarrow (G, C)$ consists of inclusions $G' \subset G$, $C' \subset C$, such that for all open subgroups $H' \subset H$ of G with H' normal in H , the image of $\gamma_{H/H'}$ along

$$H^2(H/H', C^H) \longrightarrow H^2((G' \cap H)/(G' \cap H'), C^H)$$

equals the image of $\gamma_{G'}$ along

$$H^2((G' \cap H)/(G' \cap H'), (C')^{(G' \cap H')}) \longrightarrow H^2((G' \cap H)/(G' \cap H'), C^H).$$

For an open normal subgroup $H \subset G$, let $H' = G' \cap H$. The methods of [24, §2 and §4] define a map $C'(G'/H') \longrightarrow C(G/H)$ such that the following diagram commutes:

$$\begin{array}{ccccc} (C')^{H'} & \longrightarrow & C'(G'/H') & \longrightarrow & \mathbf{Z}[G'/H'] \\ \downarrow & & \downarrow & & \downarrow \\ C^H & \longrightarrow & C(G/H) & \longrightarrow & \mathbf{Z}[G/H]. \end{array}$$

We obtain a commutative diagram

$$\begin{array}{ccccc} (H')^{ab} & \longrightarrow & E(G'/H') & \longrightarrow & \mathbf{Z}[G'/H'] \\ \downarrow & & \downarrow & & \downarrow \\ H^{ab} & \longrightarrow & E(G/H) & \longrightarrow & \mathbf{Z}[G/H]. \end{array}$$

4.2 Class field theory

The main results of class field theory can be summarized as:

Theorem 4.2. *We have*

1. For F_v a local field, $(G_{F_v}, \bigcup_{H_v/F_v} H_v^*)$ is a class formation.
2. For F a global field, $(G_F, \bigcup_{H/F} \mathbf{A}_H^*/H^*)$ is a class formation.

3. *There is an inclusion of class formations*

$$(G_{F_v}, \bigcup_{H_v/F_v} H_v^*) \longrightarrow (G_F, \bigcup_{H/F} \mathbf{A}_H^*/H^*).$$

Proof. For (1), (2), see [2]. For (3), see [8, pp. 195–196]. \square

We will also need the following well-known facts about the local and global reciprocity maps.

Theorem 4.3. 1. *For H_w an archimedean local field, we have an exact sequence*

$$0 \longrightarrow (H_w^*)^\circ \longrightarrow H_w^* \xrightarrow{\text{rec}_w} G_{H_w}^{ab} \longrightarrow 0,$$

where $(H_w^*)^\circ$ denotes the connected component of the identity.

2. *For H_w a nonarchimedean local field, we have an exact sequence*

$$0 \longrightarrow H_w^* \xrightarrow{\text{rec}_w} G_{H_w}^{ab} \longrightarrow \widehat{\mathbf{Z}}/\mathbf{Z} \longrightarrow 0.$$

3. *For H a global field, the following diagram commutes:*

$$\begin{array}{ccc} \prod'_w H_w^* & \longrightarrow & \mathbf{A}_H^*/H^* \\ \downarrow \text{rec}_w & & \downarrow \\ \prod'_w G_{H_w}^{ab} & \longrightarrow & G_H^{ab}. \end{array}$$

Let S be a finite set of places of H containing the archimedean places. Restricting the local terms from H_w^* to \mathcal{O}_w^* for $w \notin S$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \prod_{w \in S} H_w^* \prod_{w \notin S} \mathcal{O}_w^* & \longrightarrow & \mathbf{A}_H^*/H^* & \longrightarrow & \text{Cl}_S(H) & \longrightarrow & 0 \\ \downarrow \text{rec}_w & & \downarrow \text{rec} & & \downarrow = & & \\ \prod_{w \in S} \text{rec}(H_w^*) \prod_{w \notin S} \text{rec}(\mathcal{O}_w^*) & \longrightarrow & G_H^{ab} & \longrightarrow & \text{Cl}_S(H) & \longrightarrow & 0. \end{array}$$

Now we fix a finite Galois extension of number fields H/F and let $G = \text{Gal}(H/F)$. The fundamental class $\gamma_G \in H^2(G, \mathbf{A}_H^*/H^*)$ defines the *global Tate sequence*

$$0 \longrightarrow \mathbf{A}_H^*/H^* \longrightarrow V \longrightarrow B \longrightarrow \mathbf{Z} \longrightarrow 0, \quad (50)$$

an exact sequence of $\mathbf{Z}[G]$ -modules, where $B = \mathbf{Z}[G]$. For w a place of H with decomposition group $G_w \subset G$, the fundamental class $\gamma_{G_w} \in H^2(G_w, H_w^*)$ defines the *local Tate sequence*

$$0 \longrightarrow H_w^* \longrightarrow V_w \longrightarrow B_w \longrightarrow \mathbf{Z} \longrightarrow 0, \quad (51)$$

an exact sequence of G_w -modules, where $B_w = \mathbf{Z}[G_w]$.

There is a map from the local Tate sequence to the global Tate sequence:

$$\begin{array}{ccc} V_w & \longrightarrow & B_w \\ \downarrow & & \downarrow \\ V & \longrightarrow & B \end{array}$$

(see §4.1). This is a diagram of $\mathbf{Z}[G_w]$ -modules, giving rise to a diagram of $\mathbf{Z}[G]$ -modules

$$\begin{array}{ccc} \mathrm{Ind}_{G_w}^G(V_w) & \longrightarrow & \mathrm{Ind}_{G_w}^G(B_w) \\ \downarrow & & \downarrow \\ V & \longrightarrow & B, \end{array} \tag{52}$$

where the induced complex only depends on the place v of F under w .

As in Theorem 4.3, there are compatible *reciprocity maps*:

$$\begin{array}{ccc} H_w^* & \xrightarrow{\mathrm{rec}_w} & G_{H_w}^{ab} \\ \downarrow & & \downarrow \\ \mathbf{A}_H^*/H^* & \xrightarrow{\mathrm{rec}} & G_H^{ab}. \end{array}$$

If we push out the local and global Tate sequences along these reciprocity maps, we obtain prodiscrete modules $V_{w,2}$ and V_2 sitting in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_{H_w}^{ab} & \longrightarrow & V_{w,2} & \longrightarrow & B_w & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G_H^{ab} & \longrightarrow & V_2 & \longrightarrow & B & \longrightarrow & \mathbf{Z} & \longrightarrow & 0, \end{array}$$

4.3 The Ritter–Weiss complex

For each place v of F fix a place $w \mid v$ of H . We will define, for each place v of F , a complex of $\mathbf{Z}[G_w]$ -modules. Unless mentioned otherwise, all the two term complexes below are in degree 0 and 1.

4.3.1 Global complex

The global Tate sequence corresponds to the two-term complex

$$V \longrightarrow B, \tag{53}$$

where $B = \mathbf{Z}[G]$. By Theorems 4.1 and 4.2, V is a cohomologically trivial G -module. Of course, the same is true for B as well.

4.3.2 Local complexes

Fix finite disjoint sets of places S, T of F , such that $S \cup T$ contains all infinite places of F .

Case $v \in S$. The local Tate sequence corresponds to the two-term complex

$$V_w \longrightarrow B_w, \quad (54)$$

where $B_w = \mathbf{Z}[G_w]$. As in the global case, both V_w and B_w are cohomologically trivial G_w -modules.

Case $v \in T$ (finite). Define $U_w = \mathcal{O}_{w,1}^*$ (the 1-units) for $w \in T_H$ finite. Consider the two-term complex

$$U_w \longrightarrow 0. \quad (55)$$

Note that U_w is G_w -cohomologically trivial in case (B), and $U_w \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ is G_w -cohomologically trivial in case (B _{p}) (see [11, Lemma A.4]).

Case $v \in T$ (infinite). Define $U_w = H_w$ for $w \in T_H$ infinite. Consider the two-term complex

$$U_w \longrightarrow 0. \quad (56)$$

This maps to $V_w \longrightarrow B_w$ (and hence to $V \longrightarrow B$) via $H_w \xrightarrow{\text{exp}} H_w^* \longrightarrow V_w$. It is easy to see that U_w ($= \mathbf{R}$ or \mathbf{C}) is cohomologically trivial for G_w ($= \{1\}$ or $\mathbf{Z}/2\mathbf{Z}$).

Case $v \notin S \cup T$ (ramified). Consider the module $W_w = V_w/\mathcal{O}_w^*$. Let ΔG_w denote the augmentation ideal of $\mathbf{Z}[G_w]$, and let $I_w \subset G_w$ denote the inertia subgroup at w . The norm $N(I_w) = \sum_{\tau \in I_w} \tau$ defines a map

$$N(I_w): \mathbf{Z}[G_w/I_w] \longrightarrow \mathbf{Z}[G_w].$$

Let $\sigma_w \in G_w/I_w$ denote the arithmetic Frobenius. The module W_w is shown in [24, §3] to have the following description:

$$W_w \cong \{(x, y) \in \Delta G_w \times \mathbf{Z}[G_w/I_w] \mid \bar{x} = (1 - \sigma_w^{-1})y\}. \quad (57)$$

There are two maps $\pi_1, \pi_2: W_w \longrightarrow \mathbf{Z}[G_w]$, given by $\pi_1((x, y)) = x$ and $\pi_2((x, y)) = N(I_w)y$. Thus we obtain two maps $\pi_1, \pi_2: V_w \longrightarrow \mathbf{Z}[G_w]$. Consider the two-term complex

$$V_w \xrightarrow{(\pi_1, \pi_2)} \mathbf{Z}[G_w]^2. \quad (58)$$

The terms of this complex are G_w -cohomologically trivial, and it has $H^0 = \mathcal{O}_w^*$ and $H^1 = W_w^* = \text{Hom}_{\mathbf{Z}}(W_w, \mathbf{Z})$ (see [11] (138)–(139)).

Case $v \notin S \cup T$ (unramified). When H_w/F_v is unramified, the complex

$$\mathcal{O}_w^* \longrightarrow 0 \quad (59)$$

is quasi-isomorphic to the complex

$$V_w \longrightarrow W_w, \quad (60)$$

and the terms of either complex are G_w -cohomologically trivial (see [11] Lemma A.4).

4.4 Construction of the Ritter–Weiss complex

Let S' be a finite set of places of F unramified in H such that S' is disjoint from $S \cup T$. Let

$$S'' = S \cup S' \cup \{v : v \notin (S \cup T), v \text{ ramified in } H\}.$$

We define

$$\begin{aligned} V_{\text{loc}} &:= \prod_{v \in S''} \text{Ind}_{G_w}^G(V_w) \times \prod_{w \in T} \text{Ind}_{G_w}^G(U_w) \times \prod_{v \notin (S'' \cup T)} \text{Ind}_{G_w}^G(\mathcal{O}_w^*), \\ B_{\text{loc}} &:= \prod_{v \in S} \text{Ind}_{G_w}^G(B_w) \times \prod_{v \in S'} \text{Ind}_{G_w}^G(W_w) \times \prod_{w \in (S'' \setminus (S \cup S'))} \text{Ind}_{G_w}^G(\mathbf{Z}[G_w]^2), \\ C_{\text{loc}}^* &: V_{\text{loc}} \longrightarrow B_{\text{loc}}, \\ C_{\text{global}}^* &: V \longrightarrow B. \end{aligned}$$

Note that C_{loc}^* is simply the product, over all places w of H , of the inductions of the local complexes defined above, each of which comes with a map to C_{global}^* (see (52)). The set S' determines which of the two quasi-isomorphic complexes to use for unramified places $v \notin S \cup T$. Thus we have a map

$$\theta : C_{\text{loc}}^* \longrightarrow C_{\text{global}}^*. \quad (61)$$

We define the *Ritter–Weiss complex*:

$$C_{\text{RW}, S'}^* = \text{cone}(\theta)[-1]. \quad (62)$$

Lemma 4.4. *Up to canonical quasi-isomorphism, $C_{\text{RW}, S'}^*$ is independent of the auxiliary set S' .*

Proof. This follows immediately from the fact that the two complexes (59) and (60), which the set S' chooses between, are quasi-isomorphic. \square

The *generalized Ritter–Weiss module* is defined to be

$$\nabla_S^T(H) := H^1(C_{\text{RW}}^*). \quad (63)$$

Define the *logarithmic (S, T) -units*:

$$U_S^T(H) := \left\{ (x_w, y) \in \left(\prod_{w \in (T_H)_\infty} H_w \right) \times \mathcal{O}_{H, S, T}^* : \exp(x_w) = \sigma_w(y) \right\}. \quad (64)$$

Lemma 4.5. *The cohomology of C_{RW}^* is*

$$\begin{aligned} H^0(C_{\text{RW}}^*) &= U_S^T(H), \\ H^1(C_{\text{RW}}^*) &= \nabla_S^T(H), \\ H^2(C_{\text{RW}}^*) &= \text{coker} \left(\prod_{w \in S_H} \mathbf{Z} \times \prod_{\substack{w \notin (S \cup T) \\ w \text{ ramified}}} W_w^* \longrightarrow \mathbf{Z} \right). \end{aligned}$$

If we assume (A), we have $H^2(C_{\text{RW}}^*) = \begin{cases} \mathbf{Z} & \text{if } S = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$

There is a short exact sequence

$$0 \longrightarrow \text{Cl}_S^T(H) \longrightarrow \nabla_S^T(H) \longrightarrow X \longrightarrow 0 \quad (65)$$

where

$$X = \ker \left(\prod_{w \in S_H} \mathbf{Z} \times \prod_{\substack{w \notin (S \cup T) \\ w \text{ ramified}}} W_w^* \longrightarrow \mathbf{Z} \right). \quad (66)$$

If we assume (A), we have $X = X_{S,H}$.

Proof. We may assume $S' = \emptyset$. There is an exact triangle

$$C_{\text{loc}}^* \longrightarrow C_{\text{global}}^* \longrightarrow C_{\text{RW}}^*.$$

Computing cohomologies of the first two complexes, the associated long exact sequence in cohomology becomes

$$\begin{array}{ccccccc} & & & & & & 0 \longrightarrow H^0(C_{\text{RW}}^*) \\ & & & & & \searrow & \\ & & & & & \longrightarrow & \\ & & & & & \longrightarrow & \\ \prod_{w \in S_H} H_w^* \times \prod_{w \in T_H} U_w \times \prod_{w \notin (S_H \cup T_H)} \mathcal{O}_w^* & \longrightarrow & \mathbf{A}_H^*/H^* & \longrightarrow & H^1(C_{\text{RW}}^*) & \\ & & & & \searrow & \\ & & & & \longrightarrow & \\ \prod_{w \in S_H} \mathbf{Z} \times \prod_{\substack{w \notin (S \cup T) \\ w \text{ ramified}}} W_w^* & \longrightarrow & \mathbf{Z} & \longrightarrow & H^2(C_{\text{RW}}^*) & \longrightarrow 0. \end{array}$$

The calculation of $H^*(C_{\text{RW}}^*)$ follows, as does the short exact sequence (65). The stated calculations under the assumption (A) are immediate. \square

Lemma 4.6. *Assume condition (A) holds and that $S \neq \emptyset$. Let K be a subfield of H containing F . Then $\nabla_S^T(H)_{\text{Gal}(H/K)} = \nabla_S^T(K)$, where the left side denotes $\text{Gal}(H/K)$ -coinvariants.*

Proof. First, note that the Ritter–Weiss complex for K is isomorphic to the coinvariants of the complex for H : $C_{\text{RW}}^*(H)_{\text{Gal}(H/K)} = C_{\text{RW}}^*(K)$. This follows from Lemma 5.2 applied to the local and global Tate sequences.

Fix $v_0 \in S$. Since $\text{Ind}_{G_{v_0}}^G(B_{w_0}) \longrightarrow B$ is an isomorphism, we obtain the following presentation of $\nabla_S^T(H)$:

$$\nabla_S^T(H) \cong \frac{V \oplus \prod_{v \in S \setminus v_0} \text{Ind}_{G_w}^G(B_w)}{V_{\text{loc}}}. \quad (67)$$

By the right-exactness of coinvariants,

$$\nabla_S^T(H)_{\text{Gal}(H/K)} \cong \frac{(V \oplus \prod_{v \in S \setminus v_0} \text{Ind}_{G_w}^G(B_w))_{\text{Gal}(H/K)}}{(V_{\text{loc}})_{\text{Gal}(H/K)}} \cong \nabla_S^T(K).$$

□

Proposition 4.7. *If H/F is a CM extension and S contains a place v such that $c \in G_v$, then $\text{Tor}_1^{\mathbf{Z}[G]}(X, \mathbf{Z}[G]_-) = 0$, whence*

$$0 \longrightarrow \text{Cl}_S^T(H)_- \longrightarrow \nabla_S^T(H)_- \longrightarrow X_- \longrightarrow 0 \quad (68)$$

is an exact sequence of $\mathbf{Z}[G]_-$ modules.

Before we prove this proposition, we need a lemma on $\text{Tor}_*^{\mathbf{Z}[G]}(\cdot, N)$ where $N = \mathbf{Z}[G]_-$ or $\mathbf{Z}[G]_+$. We will suppress the ring $\mathbf{Z}[G]$ in our notation.

Lemma 4.8. 1. *For $i \geq 1$ and any $\mathbf{Z}[G]$ -module M , we have*

$$\text{Tor}_i(M, \mathbf{Z}[G]_+) = \text{Tor}_{i+1}(M, \mathbf{Z}[G]_-), \quad \text{Tor}_i(M, \mathbf{Z}[G]_-) = \text{Tor}_{i+1}(M, \mathbf{Z}[G]_+). \quad (69)$$

2. *For every $\mathbf{Z}[G]$ -module M , we have*

$$\text{Tor}_1(M, \mathbf{Z}[G]_+) = M_-[2], \quad \text{Tor}_1(M, \mathbf{Z}[G]_-) = M_+[2].$$

Proof. First, notice that there are two short exact sequences of $\mathbf{Z}[G]$ -modules

$$0 \longrightarrow \mathbf{Z}[G]_+ \xrightarrow{c+1} \mathbf{Z}[G] \longrightarrow \mathbf{Z}[G]_- \longrightarrow 0, \quad (70)$$

and

$$0 \longrightarrow \mathbf{Z}[G]_- \xrightarrow{c-1} \mathbf{Z}[G] \longrightarrow \mathbf{Z}[G]_+ \longrightarrow 0, \quad (71)$$

where the inclusion maps are multiplication by $c+1$ and $c-1$ respectively. To verify that the first of these maps is injective, let $x = \sum \alpha_g g \in \mathbf{Z}[G]$ and note

$$x(c+1) = 0 \iff \alpha_g + \alpha_{gc} = 0 \text{ for all } g \in G \iff x = (c-1) \sum_{g \in G/\langle c \rangle} \alpha_g g.$$

Similarly for the second map.

If we apply $\otimes_{\mathbf{Z}[G]} M$ to the sequence (70), we obtain a long exact sequence

$$\cdots \longrightarrow \text{Tor}_1(M, \mathbf{Z}[G]_-) \longrightarrow M \otimes \mathbf{Z}[G]_+ \longrightarrow M \otimes \mathbf{Z}[G] \longrightarrow M \otimes \mathbf{Z}[G]_- \longrightarrow 0. \quad (72)$$

Since $\mathbf{Z}[G]$ is projective and hence flat, we obtain the first equality of (69):

$$\text{Tor}_i(M, \mathbf{Z}[G]_+) = \text{Tor}_{i+1}(M, \mathbf{Z}[G]_-).$$

Similarly, we also obtain the second equality of (69) using (71). This finishes the proof of the first statement of the lemma.

The long exact sequence (72) together with the vanishing of $\mathrm{Tor}_1(M, \mathbf{Z}[G])$, yields

$$\mathrm{Tor}_1(M, \mathbf{Z}[G]_-) = (M_+)^-.$$

Notice that c acts trivially on M_+ , so $(M_+)^- = M_+[2]$. We calculate $\mathrm{Tor}_1(M, \mathbf{Z}[G]_+)$ similarly from (71). \square

Proof of Proposition 4.7. Recall from (66) the short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow \mathbf{Z} \longrightarrow 0$$

where

$$Y = \prod_{w \in S_H} \mathbf{Z} \times \prod_{\substack{w \notin (S \cup T) \\ w \text{ ramified}}} W_w^*. \quad (73)$$

Applying $\otimes \mathbf{Z}[G]_-$ to the sequence, we obtain a long exact sequence

$$\longrightarrow \mathrm{Tor}_2(Y, \mathbf{Z}[G]_-) \longrightarrow \mathrm{Tor}_2(\mathbf{Z}, \mathbf{Z}[G]_-) \longrightarrow \mathrm{Tor}_1(X, \mathbf{Z}[G]_-) \longrightarrow \mathrm{Tor}_1(Y, \mathbf{Z}[G]_-) \longrightarrow . \quad (74)$$

Writing Y_v for the local components of Y given in (73), we have

$$\mathrm{Tor}_1(Y, \mathbf{Z}[G]_-) = \bigoplus_{v \in \Sigma} \mathrm{Tor}_1(Y_v, \mathbf{Z}[G]_-) = \bigoplus_{v \in \Sigma} (Y_v)_+[2].$$

If $c \in G_v$, then c acts trivially on Y_v , therefore

$$(Y_v)_+[2] = Y_v[2] = 0.$$

If $c \notin G_v$, then $(Y_v)_+ = Y_v/(c-1)Y_v$ is still torsion-free, so again $(Y_v)_+[2] = 0$. Therefore $\mathrm{Tor}_1(Y, \mathbf{Z}[G]_-) = 0$. In view of (74), it remains to prove that

$$\mathrm{Tor}_2(Y, \mathbf{Z}[G]_-) \longrightarrow \mathrm{Tor}_2(\mathbf{Z}, \mathbf{Z}[G]_-) \quad (75)$$

is surjective.

We first note that by Lemma 4.8, we have

$$\mathrm{Tor}_2(\mathbf{Z}, \mathbf{Z}[G]_-) = \mathrm{Tor}_1(\mathbf{Z}, \mathbf{Z}[G]_+) = \mathbf{Z}/2\mathbf{Z}.$$

Next we compute $\mathrm{Tor}_2(Y, \mathbf{Z}[G]_-) = \bigoplus_{v \in \Sigma} (Y_v)_-[2]$ as above. If $c \in G_v$, then c acts trivially and $(Y_v)_-$ is just $Y_v/2Y_v$. The component of (75) is the surjective map $Y_v/2Y_v \longrightarrow \mathbf{Z}/2\mathbf{Z}$ induced by the surjection $Y_v \longrightarrow \mathbf{Z}$. This concludes the proof. (Incidentally, we note that the condition $c \in G_v$ for some $v \in S$ is necessary, since if $c \notin G_v$, then $(Y_v)_-$ is torsion free, thus $(Y_v)_-[2] = 0$.) \square

4.5 Quadratic presentation

Lemma 4.9. *We have*

1. $V^{\text{loc}} \longrightarrow V$ is surjective when $\cup_{v \in S'} G_v = G$ and $\text{Cl}_{S'}^T(H) = 0$.
2. $B^{\text{loc}} \longrightarrow B$ is surjective when $\cup_{v \in S'} G_v = G$ and $S \neq \emptyset$.

Proof. See [24] Page 162 or [11] §A.1. □

Now suppose that S' is sufficiently large to satisfy both conditions in Lemma 4.9, and that $S \neq \emptyset$. Consider

$$V^\theta \longrightarrow B^\theta, \tag{76}$$

where

$$V^\theta = \ker(V_{\text{loc}} \longrightarrow V), \quad B^\theta = \ker(B_{\text{loc}} \longrightarrow B).$$

This complex is quasi-isomorphic to C_{RW}^* .

Corollary 4.10. *Suppose that $S \neq \emptyset$, S' is sufficiently large to satisfy both conditions in Lemma 4.9, and that condition (C) holds.*

1. *The $\mathbf{Z}[G]$ -modules V^θ and B^θ are finitely generated and projective, and there is an exact sequence*

$$0 \longrightarrow U_S^T(H) \longrightarrow V^\theta \longrightarrow B^\theta \longrightarrow \nabla_S^T(H) \longrightarrow 0.$$

2. *If H/F is a CM extension, the $\mathbf{Z}[G]_-$ -modules $(V^\theta)^-$ and B_-^θ are finitely generated and projective, and there is an exact sequence*

$$0 \longrightarrow U_S^T(H)^- \longrightarrow (V^\theta)^- \longrightarrow B_-^\theta \longrightarrow \nabla_S^T(H)^- \longrightarrow 0.$$

Proof. For (1), we note that the modules V^θ and B^θ are G -cohomologically trivial, as they are kernels of surjective maps of G -cohomologically trivial modules. It is clear that B^θ is torsion-free, and if we assume condition (C), then V^θ is as well. Since the modules V^θ and B^θ are finitely generated, torsion free, and cohomologically trivial, they are projective $\mathbf{Z}[G]$ -modules by a result of Nakayama [20]. Statement (2) follows from the fact that $(\cdot)^-$ is left-exact, $(\cdot)_-$ is right-exact, and $N^- \cong N_-$ for G -cohomologically trivial $\mathbf{Z}[G]$ -modules. □

The following two lemmas are an adaptation of the results of [20] to the setting of $\mathbf{Z}[G]_-$ -modules. Write $G = G_p \times G'$ where G_p is the p -Sylow subgroup of G and G' is the subgroup of prime-to- p order elements of G . In the statement of part (1) below, note that if $p \neq 2$ then the composition $\mathbf{Z}_p[G_p] \longrightarrow \mathbf{Z}_p[G] \longrightarrow \mathbf{Z}_p[G]_-$ is injective.

Lemma 4.11. *We have:*

1. If $p \neq 2$, a $\mathbf{Z}_p[G]_-$ -module B is projective if and only if it is projective with respect to the subring $\mathbf{Z}_p[G_p]$.
2. If $p = 2$, a $\mathbf{Z}_p[G]_-$ -module B is projective if and only if it is projective with respect to the subring $\mathbf{Z}_p[G_p]_-$.

Proof. (1) $p \neq 2$. One direction is clear since $\mathbf{Z}_p[G]_-$ is free as a $\mathbf{Z}_p[G_p]$ -module. For the other direction, note that we have an isomorphism of $\mathbf{Z}_p[G]_-$ modules

$$\mathbf{Z}_p[G]_- \otimes_{\mathbf{Z}_p[G_p]} B \cong \mathbf{Z}_p[G']_- \otimes_{\mathbf{Z}_p} B, \quad (77)$$

where

- G acts on the left on the left side
- G acts via G' on the left factor and G_p on the right factor on the right side.

If B is projective over $\mathbf{Z}_p[G_p]$, then the left side of (77) is projective over $\mathbf{Z}_p[G]_-$. The result now follows as in [20, Prop. 0], as B is a direct summand as a $\mathbf{Z}_p[G]_-$ -module of the right side of (77), since the size of G' is coprime to p .

(2) When $p = 2$ the proof is the same using the isomorphism

$$\mathbf{Z}_p[G]_- \otimes_{\mathbf{Z}_p[G_p]_-} B \cong \mathbf{Z}_p[G/G_p] \otimes_{\mathbf{Z}_p} B.$$

□

Lemma 4.12. *Suppose we have a short exact sequence of $\mathbf{Z}[G]_-$ modules*

$$0 \longrightarrow P \longrightarrow Q \longrightarrow A \longrightarrow 0,$$

with P and Q projective and A finitely-generated and \mathbf{Z} -torsion free. Then A is a projective $\mathbf{Z}[G]_-$ -module.

Proof. It will suffice to show that, for all primes p , $A \otimes \mathbf{Z}_p$ is a projective $\mathbf{Z}_p[G]_-$ module. By Lemma 4.11, we may instead show that $A \otimes \mathbf{Z}_p$ is a projective $\mathbf{Z}_p[G_p]_-$ -module (when $p = 2$) or a projective $\mathbf{Z}_p[G_p]$ -module (when $p \neq 2$).

When $p = 2$, the ring $\mathbf{Z}[G_2]_- \otimes \mathbf{Z}/2\mathbf{Z}$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[G_2/\langle c \rangle]$. Since A is 2-torsion-free, the sequence $0 \longrightarrow P/2P \longrightarrow Q/2Q \longrightarrow A/2A \longrightarrow 0$ is exact. The module P and Q are projective as $\mathbf{Z}_2[G_2]_-$ -modules by Lemma 4.11, and so $P/2P$ and $Q/2Q$ are projective $(\mathbf{Z}/2\mathbf{Z})[G_2/\langle c \rangle]$ -modules. Thus we find that $A/2A$ is a cohomologically trivial $G_2/\langle c \rangle$ -module, and hence is a free $(\mathbf{Z}/2\mathbf{Z})[G_2/\langle c \rangle]$ -module by [19, pg. 42]. By Nakayama's Lemma, a $(\mathbf{Z}/2\mathbf{Z})[G_2/\langle c \rangle]$ -basis of $A/2A$ lifts to $\mathbf{Z}_2[G_2]_-$ -generators of $A \otimes \mathbf{Z}_2$. This gives an exact sequence

$$0 \longrightarrow K \longrightarrow \bigoplus_{i=1}^n \mathbf{Z}_2[G_2]_- \longrightarrow A \otimes \mathbf{Z}_2 \longrightarrow 0.$$

Since $A \otimes \mathbf{Z}_2$ is 2-torsion-free, we may tensor with $\mathbf{Z}/2\mathbf{Z}$ to conclude that $K/2K = 0$, and hence by Nakayama's Lemma $K = 0$. Therefore $A \otimes \mathbf{Z}_2$ is a free $\mathbf{Z}_2[G_2]_-$ -module.

When $p \neq 2$, we can simply apply the results of [20] directly to see that the G_p -cohomologically trivial, finitely-generated, p -torsion free module $A \otimes \mathbf{Z}_p$ is a free $\mathbf{Z}_p[G_p]_-$ -module. \square

Proposition 4.13. *Suppose that $S \neq \emptyset$, and that conditions (A), (B) or (B_p) , and (C) hold. In case (B):*

1. *If S contains all the infinite places, then $\nabla_S^T(H)$ is quadratically presented.*

2. *If H/F is a CM extension, then $\nabla_S^T(H)_-$ is quadratically presented.*

In case (B_p) , (1) and (2) hold for $\nabla_S^T(H) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ and $\nabla_S^T(H)_- \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ respectively, as modules over $\mathbf{Z}_{(p)}[G]$ and $\mathbf{Z}_{(p)}[G]_-$, respectively.

Proof. We give the proof in case (B). The proof in case (B_p) is exactly the same—tensor all complexes with $\mathbf{Z}_{(p)}$, and note that $(U_w) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ is cohomologically trivial for all $w \in T_H$.

(1) If S contains all infinite places, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{H,S,T}^* \longrightarrow V^\theta \longrightarrow B^\theta \longrightarrow \nabla_S^T(H) \longrightarrow 0.$$

Since V^θ and B^θ are finitely generated and projective, to show quadratic presentation it remains to show that, on each connected component of $\text{Spec}(\mathbf{Z}[G])$, V^θ and B^θ have the same rank. Since $\mathbf{Z}[G] \otimes_{\mathbf{Z}} \mathbf{C} = \prod_{\chi} \mathbf{C}$, we must show that $\dim_{\mathbf{C}}(\mathcal{O}_{H,S,T}^* \otimes \mathbf{C})^\chi$ equals $\dim_{\mathbf{C}}(\nabla_S^T(H) \otimes \mathbf{C})^\chi$ for all χ . This follows from the fact that

$$\mathcal{O}_{H,S,T}^* \otimes \mathbf{C} \cong X_{S,H} \otimes \mathbf{C} \cong \nabla_S^T(H) \otimes \mathbf{C}.$$

Note that this final isomorphism follows from (65) under condition (A).

(2) Consider the exact sequence of Corollary 4.10 (2):

$$0 \longrightarrow U_S^T(H)^- \longrightarrow (V^\theta)^- \longrightarrow (B^\theta)_- \longrightarrow (\nabla_S^T)_- \longrightarrow 0. \quad (78)$$

The rank of $(V^\theta)^-$ is larger than that of $(B^\theta)_-$, and hence we will need to replace $(V^\theta)^-$ by a quotient in order to obtain a quadratic presentation.

We have an exact sequence

$$0 \longrightarrow \bigoplus_{\substack{v \in T_H \\ \text{complex}}} 2\pi i \mathbf{Z} \longrightarrow U_S^T(H) \longrightarrow \mathcal{O}_{H,S,T}^* \longrightarrow 0.$$

Taking minus-parts, we get

$$0 \longrightarrow \bigoplus_{\substack{v \in T_H \\ \text{complex}}} 2\pi i \mathbf{Z} \longrightarrow U_S^T(H)^- \longrightarrow (\mathcal{O}_{H,S,T}^*)^- \longrightarrow 0,$$

where right-exactness follows from $H^1(\langle c \rangle, \mathbf{Z}) = 0$.

Define

$$V' := (V^\theta)^- / \bigoplus_{\substack{v \in T_H \\ \text{complex}}} 2\pi i \mathbf{Z}.$$

We obtain an exact sequence

$$0 \longrightarrow (\mathcal{O}_{H,S,T}^*)^- \longrightarrow V' \longrightarrow (B^\theta)_- \longrightarrow \nabla_S^T(H)_- \longrightarrow 0. \quad (79)$$

Note that

$$\bigoplus_{\substack{v \in T_H \\ \text{complex}}} 2\pi i \mathbf{Z} \cong \bigoplus_{v \in T_\infty} \mathbf{Z}[G]_-.$$

Therefore V' is a quotient of projective $\mathbf{Z}[G]_-$ -modules. Furthermore V' is torsion free by (79) since $(\mathcal{O}_{H,S,T}^*)^-$ is torsion free by condition (C). By Lemma 4.12, this implies that V' is itself projective.

Since V' and $(B^\theta)_-$ are projective, to show quadratic presentation for $\nabla_S^T(H)_-$ it remains to show that, on each connected component of $\text{Spec}(\mathbf{Z}[G]_-)$, V' and $(B^\theta)_-$ have the same rank. Since

$$\mathbf{Z}[G]_- \otimes_{\mathbf{Z}} \mathbf{C} \cong \prod_{\chi \text{ odd}} \mathbf{C},$$

we must show that $\dim_{\mathbf{C}}(\mathcal{O}_{H,S,T}^* \otimes \mathbf{C})^\chi$ equals

$$\dim_{\mathbf{C}}(\nabla_S^T(H) \otimes \mathbf{C})^\chi = \dim_{\mathbf{C}}(X_{S,H} \otimes \mathbf{C})^\chi$$

for all odd χ . As

$$\mathcal{O}_{H,S,T}^* \otimes \mathbf{C} \cong X_{S \cup T_\infty, H} \otimes \mathbf{C}$$

by Dirichlet's Unit Theorem, the claim follows from the fact that

$$K := X_{S \cup T_\infty, H} / X_{S,H} \cong \bigoplus_{w \in T_{H,\infty}} \mathbf{Z}$$

satisfies $K \otimes \mathbf{C}[G]_- = 0$. □

Remark 4.14. When H/F is a CM extension and conditions (B) and (C) hold, but not (A), we still may construct the module V' and obtain the exact sequence (79).

4.6 $\nabla_S^T(H)$ and Galois cohomology

In this section, we will assume that condition (A) is satisfied. We consider the following variant of C_{RW}^* , using the modified Tate sequences introduced in §4.2:

$$\begin{aligned} V_{\text{loc},2} &:= \prod_{v \in S} \text{Ind}_{G_w}^G(V_{w,2}) \times \prod_{\substack{v \in T \\ v \text{ finite}}} \text{Ind}_{G_w}^G(U_w) \times \prod_{v \notin (S \cup T)} \text{Ind}_{G_w}^G(\mathcal{O}_w^*), \\ B_{\text{loc},2} &:= \prod_{v \in S} \text{Ind}_{G_w}^G(B_w), \\ C_{\text{loc},2}^* &: V_{\text{loc},2} \longrightarrow B_{\text{loc},2}, \\ C_{\text{global},2}^* &: V_2 \longrightarrow B, \\ C_{\text{RW},2}^* &:= \text{cone}(C_{\text{loc},2}^* \longrightarrow C_{\text{global},2}^*)[-1]. \end{aligned}$$

Combining the local and global reciprocity maps, we have a map

$$\text{rec}: C_{\text{RW}}^* \longrightarrow C_{\text{RW},2}^*. \quad (80)$$

Note that we can omit the infinite places in T from $V_{\text{loc},2}$ as $\text{rec}_w(U_w) = 0$ for $w \in (T_H)_\infty$.

Proposition 4.15. *The reciprocity map defines an isomorphism $\nabla_S^T(H) \cong H^1(C_{\text{RW},2}^*)$.*

Proof. We have an exact sequence

$$\prod_{w \in S_H} G_{H_w}^{ab} \times \prod_{w \in T_H} U_w \times \prod_{w \notin (S_H \cup T_H)} \mathcal{O}_w^* \longrightarrow G_H^{ab} \longrightarrow H^1(C_{\text{RW},2}^*) \longrightarrow \prod_{w \in S_H} \mathbf{Z} \longrightarrow \mathbf{Z}$$

from which we deduce the short exact sequence

$$0 \longrightarrow \text{Cl}_S^T(H) \longrightarrow H^1(C_{\text{RW},2}^*) \longrightarrow X \longrightarrow 0$$

as in (65). Therefore the canonical map $\nabla_S^T(H) = H^1(C_{\text{RW}}^*) \longrightarrow H^1(C_{\text{RW},2}^*)$ is an isomorphism. \square

Proposition 4.15 implies that

$$\nabla_S^T(H) \cong \ker \left(\frac{V_2 \oplus \prod_{v \in S} \text{Ind}_{G_w}^G(B_w)}{V_{\text{loc},2}} \longrightarrow B \right). \quad (81)$$

Now assume $S \neq \emptyset$, and fix $v_0 \in S$. Since $\text{Ind}_{G_{v_0}}^G(B_{w_0}) \longrightarrow B$ is an isomorphism, we obtain the following presentation of $\nabla_S^T(H)$:

$$\nabla_S^T(H) \cong \frac{V_2 \oplus \prod_{v \in S \setminus v_0} \text{Ind}_{G_w}^G(B_w)}{V_{\text{loc},2}}. \quad (82)$$

Theorem 4.16. Fix $v_0 \in S$, and let A be a prodiscrete G -module. Then $\mathrm{Hom}_{\mathrm{cts},G}(\nabla_S^T(H), A)$ may be identified with the set of tuples $(\kappa, (x_v)_{v \in S \setminus \{v_0\}})$, where $\kappa: G_F \rightarrow A$ is a 1-cocycle and $x_v \in A$, such that:

1. for $v \in S \setminus \{v_0\}$, $\kappa(\sigma_v) = (\sigma_v - 1)x_v$ for all $\sigma_v \in G_{F_v}$,
2. $\kappa(G_{F_{v_0}}) = 0$,
3. for finite $w \in T_H$, $\kappa|_{G_{H_w}}(U_w) = 0$, i.e. $\kappa|_{G_{H_w}}$ is tamely ramified,
4. for $w \notin (S_H \cup T_H)$, $\kappa|_{G_{H_w}}(\mathcal{O}_{H_w}^*) = 0$, i.e. $\kappa|_{G_{H_w}}$ is unramified.

The image of the map $\nabla_S^T(H) \rightarrow A$ corresponding to $(\kappa, (x_v))$ is generated over $\mathbf{Z}[G]$ by $\kappa(G_F) \cup \{x_v \mid v \in S \setminus \{v_0\}\}$.

The proof of this theorem relies on a technical result on class formations, Theorem 5.5, whose proof will be the topic of §5.

Proof. The presentation (82) implies that $\mathrm{Hom}_{\mathrm{cts},G}(\nabla_S^T(H), A)$ is the kernel of

$$\mathrm{Hom}_{G,\mathrm{cts}}(V_2 \oplus \prod_{v \in S \setminus v_0} \mathrm{Ind}_{G_v}^G(B_w), A) \rightarrow \mathrm{Hom}_{G,\mathrm{cts}}\left(\prod_{v \in S} \mathrm{Ind}_{G_w}^G(V_{w,2}) \oplus \prod_{v \in T \text{ finite}} \mathrm{Ind}_{G_w}^G(U_w) \oplus \prod_{v \notin (S \cup T)} \mathrm{Ind}_{G_w}^G(\mathcal{O}_w^*), A\right).$$

By Theorem 5.5, this is the same as the kernel of

$$Z^1(G_F, A) \oplus \prod_{v \in S \setminus v_0} C^0(G_{F_v}, A) \rightarrow \quad (83)$$

$$\prod_{v \in S} Z^1(G_{F_v}, A) \oplus \prod_{v \in T \text{ finite}} \mathrm{Hom}_{G_w,\mathrm{cts}}(U_w, A) \oplus \prod_{v \notin (S \cup T)} \mathrm{Hom}_{G_w,\mathrm{cts}}(\mathcal{O}_w^*, A). \quad (84)$$

As the maps here are the ones arising from restriction and boundary maps in group cohomology, we obtain the first part of the theorem.

It remains to show the claim regarding the image of $\nabla_S^T(H) \rightarrow A$. This image is generated over \mathbf{Z} by the images of

$$f: V_2 \rightarrow A$$

and

$$g: \bigoplus_{v \in S \setminus v_0} \mathrm{Ind}_{G_w}^G(B_w) \cong \bigoplus_{v \in S \setminus v_0} \mathbf{Z}[G] \rightarrow A.$$

The map g sends 1 in the v -component to $x_v \in A$. The map f is determined by the factorization

$$\kappa: \mathbf{Z}[G_F] \xrightarrow{\kappa_{\mathrm{univ}}} V_2 \xrightarrow{f} A.$$

Since κ_{univ} is surjective (see Theorem 5.5), we conclude that the image of $\nabla_S^T(H) \rightarrow A$ equals the $\mathbf{Z}[G]$ -submodule of A generated by $\kappa(G_F)$ and the elements x_v . \square

Remark 4.17. If we assume that H/F is tamely ramified at $v \in T$, then condition (3) is equivalent to $\kappa|_{G_{F_v}}$ being tamely ramified. If we assume that H/F is p -tamely ramified at v and that A is a pro- p group, then condition (3) is equivalent to $\kappa|_{G_{F_v}}$ being tamely ramified.

Remark 4.18. Condition (4) is equivalent to $\kappa|_{G_{F_v}}$ being unramified (using the fact that H/F is unramified at $v \notin (S \cup T)$).

4.7 Fitting Ideals and Transposes

In this section we establish some elementary properties of the Fitting ideal of $\nabla_{S,T}(H)_-$ as certain places are moved in or out of S and T .

Lemma 4.19. *With the same hypotheses as Proposition 4.13, let H/F be a CM extension, and assume $S \neq \emptyset$. For any infinite place $v \in T$, we have*

$$\text{Fitt}_{\mathbf{Z}[G]_-}(\nabla_{S \cup \{v\}}^{T \setminus \{v\}}(H)_-) = 2 \text{Fitt}_{\mathbf{Z}[G]_-}(\nabla_S^T(H)_-).$$

Proof. Let

$$C_{S,T}^* : V_{S,T} \longrightarrow B_{S,T}$$

be the quadratic presentation of $\nabla_S^T(H)_-$ produced in Proposition 4.13. This complex is the kernel of the map of complexes

$$\theta : \left(V_{\text{loc}}^- / \bigoplus_{w \in T_{H,\infty}} 2\pi i \mathbf{Z} \longrightarrow B_{\text{loc},-} \right) \longrightarrow (V^- \longrightarrow B_-).$$

The local complex for $w \in T_{H,\infty}$ is $\mathbf{C}^- / 2\pi i \mathbf{Z} \longrightarrow 0$. The local complex for $w \in S_{H,\infty}$ is $V_w^- \longrightarrow B_{w,-}$, which has kernel $(\mathbf{C}^*)^-$ and cokernel \mathbf{Z}_- . The natural map between these local complexes, compatible with the global complex, fits into the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{C}^- / 2\pi i \mathbf{Z} & \longrightarrow & \mathbf{C}^- / 2\pi i \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathbf{C}^*)^- & \longrightarrow & V_w^- & \longrightarrow & (B_w)_- & \longrightarrow & \mathbf{Z}_- & \longrightarrow & 0 \end{array}$$

The exponential map gives an isomorphism $\mathbf{C}^- / 2\pi i \mathbf{Z} \cong (\mathbf{C}^*)^-$, and so the cokernel of the map of local complexes, $V_{w,-} / (\mathbf{C}^*)^- \longrightarrow B_{w,-}$, is simply $2\mathbf{Z}[\text{Gal}(\mathbf{C}/\mathbf{R})]_- \longrightarrow \mathbf{Z}[\text{Gal}(\mathbf{C}/\mathbf{R})]_-$ as $\text{Gal}(\mathbf{C}/\mathbf{R})$ -modules.

If we move an infinite place from T to S , we obtain a map between our quadratic presentations: $C_{S,T}^* \longrightarrow C_{S \cup \{v\}, T \setminus \{v\}}^*$. This map is injective, with cokernel given by the induction $\text{Ind}_{G_w}^G$ applied to $V_{w,-} / (\mathbf{C}^*)^- \longrightarrow B_{w,-}$. Therefore the cokernel of $C_{S,T}^* \longrightarrow C_{S \cup \{v\}, T \setminus \{v\}}^*$ is the complex $\text{Ind}_{G_w}^G(2\mathbf{Z}[G_w]_-) \longrightarrow \text{Ind}_{G_w}^G(\mathbf{Z}[G_w]_-)$, i.e. $2\mathbf{Z}[G]_- \longrightarrow \mathbf{Z}[G]_-$, whose $\mathbf{Z}[G]_-$ -determinant is 2. \square

The following lemma is well-known, but since our construction of ∇ is different than what has previously appeared in the literature, we include it for completeness.

Lemma 4.20. *With the same hypotheses as Proposition 4.13, let H/F be a CM extension, and assume $S \neq \emptyset$. Let v be a place of F that is unramified in H/F , $v \notin S \cup T$.*

1. $\text{Fitt}_{\mathbf{Z}[G]_-}(\nabla_{S \cup \{v\}}^T(H)_-) = (1 - \sigma_v^{-1}) \text{Fitt}_{\mathbf{Z}[G]_-}(\nabla_S^T(H)_-)$
2. $\text{Fitt}_{\mathbf{Z}[G]_-}(\nabla_S^{T \cup \{v\}}(H)_-) = (1 - \sigma_v^{-1} N(v)) \text{Fitt}_{\mathbf{Z}[G]_-}(\nabla_S^T(H)_-)$

Proof. Let

$$C_{S,T}^*: V_{S,T} \longrightarrow B_{S,T}$$

be the quadratic presentation of $\nabla_S^T(H)_-$ produced in Proposition 4.13.

For (1), we assume that $v \notin S'$. We consider the map between the relevant local complexes:

$$(U_w \longrightarrow 0) \longrightarrow (\mathcal{O}_{H_w}^* \longrightarrow 0).$$

We find that $C_{S, T \cup \{v\}}^* \longrightarrow C_{S,T}^*$ has cokernel $\text{Ind}_{G_w}^G(\mathbf{F}_w^*) \longrightarrow 0$, which satisfies

$$\text{Fitt}(\text{Ind}_{G_w}^G(\mathbf{F}_w^*)) = (\sigma_v - N(v)).$$

For (2), we assume that $v \in S'$. We consider the map between the relevant local complexes:

$$(V_w \longrightarrow W'_w) \longrightarrow (V_w \longrightarrow B_w).$$

We find that $C_{S,T}^* \longrightarrow C_{S \cup \{v\}, T}^*$ has cokernel $0 \longrightarrow \text{Ind}_{G_w}^G(\mathbf{Z}[G_w]/(\sigma_v - 1)\mathbf{Z}[G_w])$, which satisfies

$$\text{Fitt}(\text{Ind}_{G_w}^G(\mathbf{Z}[G_w]/(\sigma_v - 1)\mathbf{Z}[G_w])) = (\sigma_v - 1).$$

□

Lemma 4.21. *Suppose $S \neq \emptyset$ and H/F is a CM extension. Assume conditions (B) and (C), so that we may consider the presentation (7) of $\nabla_S^T(H)_-$. Let $\text{Sel}_S^T(H)_-$ denote the transpose of $\nabla_S^T(H)_-$ over $\mathbf{Z}[G]_-$ associated to (7) in the sense of Jannsen, i.e., the module defined by the exact sequence*

$$\text{Hom}_{\mathbf{Z}[G]_-}((B^\theta)_-, \mathbf{Z}[G]_-) \longrightarrow \text{Hom}_{\mathbf{Z}[G]_-}(V', \mathbf{Z}[G]_-) \longrightarrow \text{Sel}_S^T(H)_- \longrightarrow 0.$$

We have a short exact sequence

$$0 \longrightarrow (\nabla_S^T(H)_-)^{\vee}_{\text{tors}} \longrightarrow \text{Sel}_S^T(H)_- \longrightarrow \text{Hom}_{\mathbf{Z}}((\mathcal{O}_{H,S,T}^*)^-, \mathbf{Z}) \longrightarrow 0. \quad (85)$$

Proof. Split up (7) into two short exact sequences

$$\begin{aligned} 0 &\longrightarrow (\mathcal{O}_{H,S,T}^*)^- \longrightarrow V' \longrightarrow V'/(\mathcal{O}_{H,S,T}^*)^- \longrightarrow 0, \\ 0 &\longrightarrow V'/(\mathcal{O}_{H,S,T}^*)^- \longrightarrow (B^\theta)_- \longrightarrow \nabla_S^T(H)_-. \end{aligned}$$

Note that there is an identification of functors on the category of $\mathbf{Z}[G]_-$ -modules

$$\mathrm{Hom}_{\mathbf{Z}[G]_-}(-, \mathbf{Z}[G]_-) = \mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{Z}).$$

Applying $\mathcal{F}(-) = \mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{Z})$ to the two sequences above, we obtain

$$0 \longrightarrow \mathcal{F}(V'/(\mathcal{O}_{H,S,T}^*)^-) \longrightarrow \mathcal{F}(V') \longrightarrow \mathcal{F}((\mathcal{O}_{H,S,T}^*)^-) \longrightarrow 0 \quad (86)$$

since $V'/(\mathcal{O}_{H,S,T}^*)^- \subset (B^\theta)_-$ is a free \mathbf{Z} -module, and

$$\mathcal{F}((B^\theta)_-) \longrightarrow \mathcal{F}(V'/(\mathcal{O}_{H,S,T}^*)^-) \longrightarrow \mathrm{Ext}_{\mathbf{Z}}^1(\nabla_S^T(H)_-, \mathbf{Z}) \longrightarrow 0. \quad (87)$$

Modding out the first two nontrivial terms of (86) by the image of $\mathcal{F}((B^\theta)_-)$ in each, applying (87), and noting that

$$\mathrm{Ext}_{\mathbf{Z}}^1(\nabla_S^T(H)_-, \mathbf{Z}) \cong (\nabla_S^T(H)_-)^{\vee}_{\mathrm{tors}},$$

we obtain the desired exact sequence (85). □

5 A Duality Theorem for Class Formations

Consider a class formation (G, C) and an open normal subgroup H . In this section we will prove that, given a prodiscrete $\mathbf{Z}[G/H]$ -module A , the complex

$$\mathrm{Hom}_{\mathbf{Z}[G/H]}(\mathbf{Z}[G/H], A) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[G/H], \mathrm{cts}}(E(G/H), A),$$

dualizing the modified Tate extension (49), is isomorphic to $\tau_{\leq 1}C^*(G, A)$, the truncation of Galois cohomology. This result was used in the proof of Theorem 4.16 above, which gave an interpretation of the module $\nabla_S^T(H)$ in terms of Galois cohomology. Our results in this section may be of independent interest.

5.1 Duality (G finite)

Consider a class formation (G, C) with G finite. Recall the extension (49) obtained by pushing out the extension (48) along $C^H \longrightarrow C^H/N_H(C) \cong H^{ab}$:

$$0 \longrightarrow C^H/N_H(C) \longrightarrow E(G/H) \longrightarrow \mathbf{Z}[G/H] \longrightarrow \mathbf{Z} \longrightarrow 0, \quad (88)$$

where $E(G/H) := C(G/H)/N_H(C)$.

Let

$$C_*^{\mathrm{bar}} : \mathbf{Z}[G] \longleftarrow \mathbf{Z}[G^2] \longleftarrow \cdots$$

be the bar resolution of \mathbf{Z} by G -modules. For a discrete G/H -module A ,

- $C^*(H, A) := \text{Hom}_{\mathbf{Z}[H]}(C_*^{\text{bar}}, A)$ is a complex of G/H -modules whose cohomology equals $H^*(H, A)$ as a G/H -module.
- $C^*(G, A) := \text{Hom}_{\mathbf{Z}[G]}(C_*^{\text{bar}}, A)$ is a complex whose cohomology equals $H^*(G, A)$.

We will restrict our attention to the truncation $\tau_{\leq 1}C^*(G, A)$, i.e.

$$\tau_{\leq 1}C^*(G, A): C^0(G, A) \longrightarrow Z^1(G, A). \quad (89)$$

Theorem 5.1. *Consider a class formation (G, C) with G finite, a normal subgroup $H \subset G$, and a discrete G/H -module A .*

1. *The truncation $\tau_{\leq 1}C^*(H, A)$ is isomorphic to*

$$\text{Hom}(\mathbf{Z}[G/H], A) \longrightarrow \text{Hom}(E(G/H), A),$$

where the map on first cohomology equals the dual of the reciprocity map.

2. *The truncation $\tau_{\leq 1}C^*(G, A)$ is isomorphic to*

$$\text{Hom}_{\mathbf{Z}[G/H]}(\mathbf{Z}[G/H], A) \longrightarrow \text{Hom}_{\mathbf{Z}[G/H]}(E(G/H), A).$$

3. *The inhomogeneous 1-cocycle $\mathbf{Z}[G] \longrightarrow E(G/H)$ corresponding to the universal element $\kappa_{\text{univ}} \in Z^1(G, E(G/H))$ is surjective.*

We begin with some preparatory lemmas.

Lemma 5.2. *Consider a class formation (G, C) . For $H \subset G$ an normal subgroup, we have $C(G)_H \cong C(G/H)$. In particular, $C(G)_G \cong C^G$. There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C(G) & \longrightarrow & \mathbf{Z}[G] & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^H & \longrightarrow & C(G/H) & \longrightarrow & \mathbf{Z}[G/H] & \longrightarrow & \mathbf{Z} & \longrightarrow & 0, \end{array}$$

where the vertical map on the left term is the norm N_H and the map $\mathbf{Z}[G] \longrightarrow \mathbf{Z}[G/H]$ is the canonical projection.

Proof. Write $I_G \subset \mathbf{Z}[G]$ for the augmentation ideal. Multiplying by the norm $N_H = \sum_{h \in H} h$ yields a commutative diagram

$$\begin{array}{ccccccc} C_H & \longrightarrow & C(G)_H & \longrightarrow & (I_G)_H & \longrightarrow & 0 \\ & & \downarrow N_H & & \downarrow N_H & & \downarrow N_H \\ 0 & \longrightarrow & C^H & \longrightarrow & C(G)^H & \longrightarrow & (I_G)^H. \end{array} \quad (90)$$

Let $K(G, H)$ denote the kernel of $N_H: (I_G)_H \longrightarrow (I_G)^H$. The snake lemma applied to (90) yields a map

$$K(G, H) \longrightarrow C^H/N_H(C). \quad (91)$$

Since $C(G)$ is H -cohomologically trivial, the middle vertical arrow in (90) is an isomorphism, whence (91) is an isomorphism as well.

Note that $(I_H)_H \cong H^{ab}$ is contained in $K(G, H)$. In fact, these are equal: since both $\mathbf{Z}[G]$ and $\mathbf{Z}[H]$ are H -cohomologically trivial, we have

$$\widehat{H}^{-1}(H, I_G) \cong \widehat{H}^{-2}(H, \mathbf{Z}) \cong \widehat{H}^{-1}(H, I_H).$$

Therefore the image of the norm map $(I_G)_H \longrightarrow (I_G)^H$ is isomorphic to

$$I_G/(\mathbf{Z}[G]I_H) \cong I_{G/H}.$$

This proves that $C(G)_H$ is an extension

$$0 \longrightarrow C^H \longrightarrow C(G)_H \longrightarrow I_{G/H} \longrightarrow 0.$$

It remains to verify that $C(G/H) \cong C(G)_H$. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C(G) & \longrightarrow & \mathbf{Z}[G] & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow N_H & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^H & \longrightarrow & C(G)_H & \longrightarrow & \mathbf{Z}[G]_H & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & A & \longrightarrow & \mathbf{Z}[G]_H & \longrightarrow & \mathbf{Z} & \longrightarrow & 0. \end{array}$$

The map from the first row to the second row on the middle two terms is the canonical projection. The map from the second row to third row is the pushout along the inclusion $C^H \subset C$. The bottom two rows imply that $C(G)_H \longrightarrow \mathbf{Z}[G]_H$, viewed as an element of $\text{Ext}_{\mathbf{Z}[G/H]}^2(\mathbf{Z}, C^H)$, inflates to the extension $A \longrightarrow \mathbf{Z}[G]_H$ under the map

$$\text{Ext}_{\mathbf{Z}[G/H]}^2(\mathbf{Z}, C^H) \longrightarrow \text{Ext}_{\mathbf{Z}[G]}^2(\mathbf{Z}, C).$$

On the other hand, this diagram implies that $A \longrightarrow \mathbf{Z}[G]_H$ is *equivalent*, as a 2-extension, to the pushout of the first row along the map

$$C \xrightarrow{N_H} C^H \subset C.$$

The map

$$N_H: \text{Ext}_{\mathbf{Z}[G]}^2(\mathbf{Z}, C) \longrightarrow \text{Ext}_{\mathbf{Z}[G]}^2(\mathbf{Z}, C)$$

equals multiplication by $|H|$, as can be verified using a dimension-shifting argument, since N_H gives this multiplication on degree 0.

This implies that the 2-extension $C(G)_H \rightarrow \mathbf{Z}[G]_H$, when inflated to an element of $H^2(G, C)$, equals $|H|\gamma_G$. This is the same as the inflation of $\gamma_{G/H}$. Since inflation

$$H^2(G/H, C^H) \rightarrow H^2(G, C)$$

is an injection, this implies that the 2-extension $C(G)_H \rightarrow \mathbf{Z}[G]_H$ is equivalent, as a 2-extension of (G/H) -modules, to $C(G/H) \rightarrow \mathbf{Z}[G/H]$. Moreover, this implies that the 1-extension

$$0 \rightarrow C^H \rightarrow C(G)_H \rightarrow I_{G/H} \rightarrow 0$$

is isomorphic to

$$0 \rightarrow C^H \rightarrow C(G/H) \rightarrow I_{G/H} \rightarrow 0,$$

and hence that $C(G)_H \cong C(G/H)$. □

Lemma 5.3. *The isomorphisms $C(G)_G \cong C^G$ and $(I_G)_G \cong G^{ab}$ identify the map*

$$C(G)_G \rightarrow (I_G)_G$$

with the reciprocity map $C^G \rightarrow G^{ab}$.

Proof. The exact sequence in Tate cohomology associated to the short exact sequence

$$0 \rightarrow C \rightarrow C(G) \rightarrow I_G \rightarrow 0$$

yields a boundary map

$$\delta: \widehat{H}^{-1}(G, I_G) \cong \widehat{H}^0(G, C).$$

Note that $\widehat{H}^{-1}(G, I_G) = (I_G)_G$ and $\widehat{H}^0(G, C) = C^G/N_G(C)$. After identifying $(I_G)_G \cong G^{ab}$, the map δ is by definition the Nakayama map.

By the construction of boundary maps in Tate cohomology, δ has the following direct description: take an element of $(I_G)_G$, lift it to $C(G)$, apply the norm map to get an element of $C(G)^G$, observe that it is contained in C^G , and project to $C^G/N_G(C)$.

The reciprocity map is therefore given by

$$C^G \cong C(G)^G \xrightarrow[N_G^{-1}]{\sim} C(G)_G \twoheadrightarrow (I_G)_G.$$

The composition of the first two maps is the identification $C^G \cong C(G)_G$ from Lemma 5.2. □

We can now prove the duality theorem.

Proof of Theorem 5.1. (1) Since \mathbf{Z} and $I_{G/H}$ are \mathbf{Z} -free, the sequence (88) remains exact after applying $\text{Hom}(-, A)$. It follows that

$$\text{Hom}(\mathbf{Z}[G/H], A) \rightarrow \text{Hom}(E(G/H), A) \tag{92}$$

has cohomology

$$\begin{aligned} H^0 &= \text{Hom}(\mathbf{Z}, A) = H^0(H, A), \\ H^1 &= \text{Hom}(C^H/N_H(C), A) = \text{Hom}(H^{ab}, A) = H^1(H, A), \end{aligned}$$

where the last equalities on each line hold since A has trivial H -action. Therefore the complex (92) has the same cohomology as $\tau_{\leq 1}C^*(H, A)$, and our goal is to show that these two complexes are in fact quasi-isomorphic as complexes of $\mathbf{Z}[G/H]$ -modules.

Using the lifting property for projective $\mathbf{Z}[G]$ -modules, we may choose a commutative diagram

$$\begin{array}{ccccc} \mathbf{Z}[G^3] & \longrightarrow & \mathbf{Z}[G^2] & \longrightarrow & \mathbf{Z}[G] \\ \downarrow & & \downarrow & & \downarrow = \\ C & \longrightarrow & C(G) & \longrightarrow & \mathbf{Z}[G] \end{array}$$

Applying $\text{Hom}_{\mathbf{Z}[H]}(\cdot, A)$ to the first complex gives the first few terms of the complex $C^*(H, A)$. We obtain a map of chain complexes of $\mathbf{Z}[G/H]$ -modules:

$$\tau_{\leq 1} \text{Hom}_{\mathbf{Z}[H]}(C \longrightarrow C(G) \longrightarrow \mathbf{Z}[G], A) \longrightarrow \tau_{\leq 1}C^*(H, A).$$

By Lemma 5.2, this becomes:

$$\tau_{\leq 1} \text{Hom}(C_H \longrightarrow C(G/H) \longrightarrow \mathbf{Z}[G/H], A) \longrightarrow \tau_{\leq 1}C^*(H, A).$$

The map $C_H \longrightarrow C(G/H)$ factors as $C_H \xrightarrow{N_H} C^H \subset C(G/H)$. Therefore we obtain a map of complexes:

$$\text{Hom}(E(G/H) \longrightarrow \mathbf{Z}[G/H], A) \longrightarrow \tau_{\leq 1}C^*(H, A).$$

It remains to check that the induced map on H^1 , $\text{Hom}(C^H/N_H(C), A) \longrightarrow H^1(H, A)$, is dual to the reciprocity map, and hence is an isomorphism. We can reduce to the case $G = H$, using the map

$$(C \longrightarrow C(H) \longrightarrow \mathbf{Z}[H]) \longrightarrow (C \longrightarrow C(G) \longrightarrow \mathbf{Z}[G])$$

considered as complexes of $\mathbf{Z}[H]$ -modules.

The partial resolution $0 \longrightarrow I_H \longrightarrow \mathbf{Z}[H] \longrightarrow \mathbf{Z} \longrightarrow 0$ of \mathbf{Z} induces a map

$$H^1(H, A) \longrightarrow \text{Hom}_{\mathbf{Z}[H]}(I_H, A) = \text{Hom}_{\mathbf{Z}[H]}(I_H, A) = \text{Hom}(H^{ab}, A)$$

which is the usual isomorphism. This partial resolution can be extended to $C(H) \longrightarrow \mathbf{Z}[H] \longrightarrow \mathbf{Z} \longrightarrow 0$, inducing a map

$$H^1(H, A) \cong \text{Hom}_{\mathbf{Z}[H]}(I_H, A) \longrightarrow \text{Hom}_{\mathbf{Z}[H]}(C(H), A).$$

By Lemma 5.3, the second map is dual to the reciprocity map.

(2) The argument of (1) implies that $\tau_{\leq 1} C^*(G, A)$ is isomorphic to

$$\tau_{\leq 1} \operatorname{Hom}_{\mathbf{Z}[G]}(C \longrightarrow C(G) \longrightarrow \mathbf{Z}[G], A).$$

Using Lemma 5.2, we find that the following sequence is exact:

$$0 \longrightarrow \operatorname{Hom}(E(G/H), A) \longrightarrow \operatorname{Hom}_{\mathbf{Z}[H]}(C(G), A) \longrightarrow \operatorname{Hom}_{\mathbf{Z}[H]}(C, A).$$

Hence, by left-exactness of $H^0(G/H, \cdot)$, the following sequence is exact as well:

$$0 \longrightarrow \operatorname{Hom}_{\mathbf{Z}[G/H]}(E(G/H), A) \longrightarrow \operatorname{Hom}_{\mathbf{Z}[G]}(C(G), A) \longrightarrow \operatorname{Hom}_{\mathbf{Z}[G]}(C, A).$$

Therefore $\tau_{\leq 1} \operatorname{Hom}_{\mathbf{Z}[G]}(C \longrightarrow C(G) \longrightarrow \mathbf{Z}[G], A)$ is isomorphic to

$$\operatorname{Hom}_{\mathbf{Z}[G]}(E(G/H) \longrightarrow \mathbf{Z}[G], A).$$

(3) The inhomogeneous 1-cocycle $\kappa_{\text{univ}}: G \longrightarrow E(G/H)$ in $Z^1(G, E(G/H))$ corresponds to the homogeneous 1-cocycle $\mathbf{Z}[G^2] \longrightarrow E(G/H)$ occurring in:

$$\begin{array}{ccccc} \mathbf{Z}[G^3] & \longrightarrow & \mathbf{Z}[G^2] & \longrightarrow & \mathbf{Z}[G] \\ \downarrow & & \downarrow & & \downarrow \\ H^{ab} & \longrightarrow & E(G/H) & \longrightarrow & \mathbf{Z}[G/H] \end{array}$$

Since we have an exact sequence

$$0 \longrightarrow H^{ab} \longrightarrow E(G/H) \longrightarrow I_{G/H} \longrightarrow 0,$$

we must show that $f: \mathbf{Z}[G^2] \longrightarrow I_{G/H}$ is surjective, and that $\ker(f)$ surjects onto H^{ab} .

The map

$$\mathbf{Z}[G^2] \longrightarrow I_G \longrightarrow \mathbf{Z}[G/H]$$

has image equal to $I_{G/H}$. The H -submodule $\mathbf{Z}[H^2] \subset \mathbf{Z}[G^2]$ is contained in $\ker(f)$. The restriction of $\mathbf{Z}[G^2] \longrightarrow E(G/H)$ to $\mathbf{Z}[H^2]$ is a homogeneous 1-cocycle $\mathbf{Z}[H^2] \longrightarrow E(G/H)$ whose image is H^{ab} , corresponding to

$$\text{id} \in \operatorname{Hom}(H^{ab}, H^{ab}) = H^1(H, H^{ab}).$$

The result follows. □

5.2 Functoriality

Lemma 5.4 (Functoriality). *Suppose we have a map of class formations $(G', C') \rightarrow (G, C)$. Let H be a subgroup of G , and let $H' = G' \cap H$. Let A be a $\mathbf{Z}[G/H]$ -module. The restriction map*

$$Z^1(G, A) \rightarrow Z^1(G', A)$$

is dual to $E(G'/H') \rightarrow E(G/H)$.

Proof. The argument of Theorem 5.1 proves that the complex

$$0 \rightarrow H^{ab} \rightarrow \mathbf{Z}[G^2]_H / \mathbf{Z}[G^3]_H \rightarrow \mathbf{Z}[G/H] \rightarrow \mathbf{Z} \rightarrow 0$$

is isomorphic to

$$0 \rightarrow C^H / N_H(C) \rightarrow E(G/H) \rightarrow \mathbf{Z}[G/H] \rightarrow \mathbf{Z} \rightarrow 0.$$

Since we have a commutative diagram

$$\begin{array}{ccc} (H')^{ab} & \longrightarrow & (C')^{H'} / N_{H'}(C') \\ \downarrow & & \downarrow \\ H^{ab} & \longrightarrow & C^H / N_H(C), \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{Z}[(G')^2]_{H'} / \mathbf{Z}[(G')^3]_{H'} & \longrightarrow & E(G'/H') \\ \downarrow & & \downarrow \\ \mathbf{Z}[G^2]_H / \mathbf{Z}[G^3]_H & \longrightarrow & E(G/H). \end{array}$$

Applying $\text{Hom}_{\mathbf{Z}[G'/H']}(\cdot, A)$ to the first row and $\text{Hom}_{\mathbf{Z}[G/H]}(\cdot, A)$ to the second, we obtain the required compatibility. \square

5.3 Duality theorem (G profinite)

Recall that a prodiscrete group is an inverse limit of discrete groups with surjective transition maps. Note that, for a class formation (G, C) , the $\mathbf{Z}[G/H]$ -module $E(G/H)$ is prodiscrete.

Theorem 5.5. *Let (G, C) be a class formation and let H be an open normal subgroup of G . For any prodiscrete $\mathbf{Z}[G/H]$ -module A :*

1. *The complex $\tau_{\leq 1} C^*(H, A)$ is isomorphic to*

$$\text{Hom}(\mathbf{Z}[G/H], A) \rightarrow \text{Hom}_{\text{cts}}(E(G/H), A).$$

2. The complex $\tau_{\leq 1} C^*(G, A)$ is isomorphic to

$$\mathrm{Hom}_{\mathbf{Z}[G/H]}(\mathbf{Z}[G/H], A) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[G/H], \mathrm{cts}}(E(G/H), A).$$

3. The inhomogeneous 1-cocycle $\mathbf{Z}[G] \longrightarrow E(G/H)$ corresponding to the universal element $\kappa_{\mathrm{univ}} \in Z^1(G, E(G/H))$ is surjective.

Proof. (1) We may assume that A is discrete, since

$$\varprojlim_n \mathrm{Hom}_{\mathrm{cts}}(E(G/H), A/A_n) \cong \mathrm{Hom}_{\mathrm{cts}}(E(G/H), \varprojlim_n A/A_n) = \mathrm{Hom}_{\mathrm{cts}}(E(G/H), A)$$

and

$$\varprojlim_n Z^1(H, A/A_n) \cong Z^1(H, \varprojlim_n A/A_n) \cong Z^1(H, A).$$

Since A is discrete, $Z^1(H, A) = \varinjlim_{H'} Z^1(H/H', A)$ and

$$\mathrm{Hom}_{\mathrm{cts}}(E(G/H), A) = \varinjlim_{H'} \mathrm{Hom}(E((G/H')/(H/H')), A).$$

As $E((G/H')/(H/H'))$ comes from the *finite* class formation $(G/H', C^{H'})$, we may apply Theorem 5.1 to deduce that $\mathrm{Hom}(E((G/H')/(H/H')), A) \cong Z^1(H/H', A)$.

(2) and (3) follow from Theorem 5.1 via a similar argument. \square

Suppose we have a map of class formations $(G', C') \longrightarrow (G, C)$. Let H be an open subgroup of G , and let $H' = G' \cap H$.

Lemma 5.6 (Functoriality). *For any prodiscrete $\mathbf{Z}[G/H]$ -module A , the map*

$$Z^1(H, A) \longrightarrow Z^1(H', A)$$

is the continuous dual $\mathrm{Hom}_{\mathrm{cts}}(\cdot, A)$ of the map $E(G'/H') \longrightarrow E(G/H)$.

Proof. This follows from the analogous result in the finite case, Lemma 5.4. \square

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