**Gross–Stark units, Stark–Heegner points, and class fields of real
quadratic fields**

by

Samit Dasgupta

A.B. (Harvard University) 1999

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

GRADUATE DIVISION
of the
UNIVERSITY OF CALIFORNIA, BERKELEY

Committee in charge:
Professor Kenneth Ribet, Chair
Professor Bjorn Poonen
Professor David Wagner

Spring 2004

The dissertation of Samit Dasgupta is approved:

Chair   Date

                                                                         Date

---
                                                                         Date

University of California, Berkeley

Spring 2004

Gross–Stark units, Stark–Heegner points, and class fields of real quadratic fields

Copyright 2004

by

Samit Dasgupta

Abstract

Gross–Stark units, Stark–Heegner points, and class fields of real quadratic fields

by

Samit Dasgupta

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Kenneth Ribet, Chair

We present two generalizations of Darmon's construction of Stark–Heegner points on elliptic curves defined over $\mathbf{Q}$. First, we provide a lifting of Stark–Heegner points from elliptic curves to certain modular Jacobians which parameterize them. This construction involves a generalization of a theorem of Greenberg and Stevens that proves the Mazur–Tate–Teitelbaum conjecture.

Next, we replace the modular symbols attached to an elliptic curve with those attached to a modular unit α of level $N > 1$. For a real quadratic field K in which the rational prime p is inert, this allows the definition of certain numbers $u(\alpha, \tau) \in K_p^\times$ attached to α and $\tau \in K - \mathbf{Q}$. The elements $u(\alpha, \tau)$ are analogous to classical elliptic units arising from α. In this vein, we conjecture that the elements $u(\alpha, \tau)$ belong to specific abelian extensions of K. Although this conjecture is still open, we prove a formula relating the p-adic valuation and p-adic logarithm of $u(\alpha, \tau)$ to the leading terms at $s = 0$ of certain partial zeta functions (classical and p-adic, respectively). The existence of units satisfying these properties is equivalent to the p-adic Gross–Stark conjecture; thus our construction gives an analytic construction of Gross's units, minus a proof of their algebraicity.

Professor Kenneth Ribet
Dissertation Committee Chair

To Baba

Contents

List of Tables

Acknowledgments

It is an honor to thank Prof. Henri Darmon of McGill University for the guidance he provided me over the course of my graduate career. I first met Darmon at a conference during the summer of 2001 and became fascinated with his work; our ensuing e-mail correspondence grew into fruitful collaboration. Darmon went beyond my greatest expectations in guiding my mathematical development. I am continually amazed by the energy and passion that he brings to all his endeavors, and I have benefited greatly from my interaction with him. He is an unbelievable mentor, colleague, and friend. I am also truly indebted to my adviser Prof. Kenneth Ribet of University of California-Berkeley, who provided an extraordinary amount of support for me during my graduate school experience. Ken's infallible ability to lead me in the right direction during moments of confusion—mathematical and otherwise— is genuinely remarkable.

Learning during graduate school is necessarily and fortunately a collaborative project, undertaken by those with a common esoteric passion. I am grateful for the companionship and friendship of John Voight and Ronald van Luijk during our mutual struggles to forge a clearing through the forested wilderness of number theory.

The last four years have also witnessed dramatic social changes in my life and my community, and I am blessed to have had the company and support of an amazing circle of friends and family. My mother has always served as a model of strength, love, and tactful jocundity. My brother and Priya have demonstrated by example the virtues of resilience, passion, and humor. My father's values, open-heartedness, and optimism continue to serve as an inspiration to all of us. In Berkeley and beyond, I have been sustained during the last four years by the warmth and friendship of Daniel Biss, Kevin Amer, Nirit Sandman, Danielle Gordon, Saurabh Bhargava, and Grace Lyo.

I would like to acknowledge William Stein for granting me access to the MECCAH cluster at Harvard University, on which the MAGMA programs of Chapter 11 were executed. Finally, I am indebted to Profs. Bjorn Poonen, Kenneth Ribet, and Henri Darmon for many detailed suggestions which greatly improved the quality of this thesis.

Part I

Stark–Heegner Points on Modular Jacobians

Chapter 1

Introduction

The theory of complex multiplication allows the construction of a collection of points on arithmetic curves over $\mathbf{Q}$, defined over abelian extensions of quadratic imaginary fields. Foremost among these are Heegner points on modular curves, as described for example in [15]. By embedding a modular curve in its Jacobian (typically by sending a rational cusp to the origin), one may transfer Heegner points on the curve to each factor of its Jacobian. A study of the arithmetic properties of the points constructed in this fashion has yielded many striking results, most notably, the theorems of Gross–Zagier [18], Kolyvagin [25], and Kolyvagin–Logachëv [26].

The goal of [5] was to define certain points on elliptic curves analogous to Heegner points, except that they would be defined over abelian extensions of real quadratic fields instead of imaginary quadratic fields. In the setting considered, the existence of such points is predicted by the conjecture of Birch and Swinnerton-Dyer. Darmon constructs these "Stark–Heegner points" analytically by replacing complex integration with a certain p-adic integral. The conjecture that Stark–Heegner points are defined over global number fields remains open.

The goal of part I of this thesis is to lift the construction of Stark–Heegner points from elliptic curves to certain modular Jacobians. Let N be a positive integer and let p be a prime not dividing N. Our essential idea is to replace the modular symbol attached to an elliptic curve E of conductor Np (a key tool in [5]) with the universal modular symbol for $\Gamma_0(Np)$. We then construct a certain torus T over $\mathbf{Q}_p$ and sub-lattice L of T, and prove that the quotient T/L is isogenous to the maximal toric quotient $J_0(Np)^{p\text{-new}}$ of the Jacobian of $X_0(Np)$. This theorem generalizes a conjecture of Mazur, Tate, and Teitelbaum [35] on

the p-adic periods of elliptic curves, which was proven by Greenberg and Stevens [12], [13]. Indeed, our proof borrows greatly from theirs.

Our isogeny theorem allows us to define Stark–Heegner points on the abelian variety $J_0(Np)^{p\text{-new}}$. The points we define map to the Stark–Heegner points on E under the projection $J_0(Np)^{p\text{-new}} \to E$. We conjecture that they satisfy the same algebraicity properties. One interesting difference from the case of classical Heegner points is that our points, while lying on modular Jacobians, do not appear to arise from points on the modular curves themselves.

Although the construction of Stark–Heegner points is the most significant arithmetic application of our isogeny theorem, the result is interesting in its own right because it allows the practical computation of the p-adic periods of $J_0(Np)^{p\text{-new}}$.

In Chapter 2 we summarize known uniformization results, beginning with the complex analytic construction of $J_0(N)$ and classical Heegner points. We then discuss p-adic uniformization of Mumford curves via Schottky groups, and present the Manin–Drinfield theorem on the uniformization of the Jacobian of a Mumford curve in the language of p-adic integration. In Chapter 4 we construct our analytic space T/L and state the isogeny theorem. We then use the isogeny theorem to define Stark–Heegner points on $J_0(Np)^{p\text{-new}}$. The remainder of part I is devoted to proving the isogeny theorem. Section 6.1 describes precisely how our result generalizes the Mazur–Tate–Teitelbaum conjecture.

There are some differences to note between our presentation and that of [12]. First, by dealing with the entire Jacobian rather than a component associated to a particular newform, we avoid some technicalities arising in Hida theory. Furthermore, the role of $-2a'_p(2)$ in [12] is played by

$$\mathscr{L}_p := \text{ the "derivative" of } 1 - U_p^2,$$

as defined in Section 6.2; accordingly we treat the cases of split and non-split reduction simultaneously. The proof that the $\mathscr{L}$-invariant of T/L is equal to $\mathscr{L}_p$ is somewhat different from (though certainly bears commonalities with) what appears in [12]. Indeed, the space T/L is constructed from the group

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}_2(\mathbf{Z}[1/p]) \text{ such that } N | c \right\}$$

and a study of its cohomology. The construction of Stark–Heegner points is contingent on the splitting of a certain 2-cocycle for Γ, which is proven by lifting measures on $\mathbf{P}^1(\mathbf{Q}_p)$ to

the $\mathbf{Z}_p^\times$-bundle $\mathbf{X} = (\mathbf{Z}_p \times \mathbf{Z}_p)'$ of primitive vectors over this space. The connection between integrals on $\mathbf{X}$ and p-adic L-functions is described in [2], for which the calculations of part II of this thesis (in particular, Section 10.2) served as the motivation.

Chapter 2

Previous uniformization results

The classical theory of Abel–Jacobi gives a complex analytic uniformization of the Jacobian of a nonsingular proper curve over $\mathbf{C}$. We begin this chapter by recalling this construction for $X_0(N)$ and giving the definition of Heegner points on $J_0(N)$ using this uniformization. Manin and Drinfeld have also given a p-adic uniformization for the Jacobians of Mumford curves. We give a restatement of their result in the language of p-adic integration, which may thus be viewed as a p-adic Abel–Jacobi theory. Unfortunately, the p-adic uniformization of $J_0(p)$ that arises in this fashion does not allow the natural construction of Heegner-type points in an obvious manner. By applying the general integration theory in a more arithmetic setting, we find an alternate p-adic uniformization of $J_0(Np)^{p\text{-new}}$ and use it to construct Stark–Heegner points on this modular Jacobian.

2.1   Archimedean uniformization

The Abel–Jacobi theorem states that the Jacobian of a nonsingular proper curve X over $\mathbf{C}$ is analytically isomorphic to the quotient of the dual of its space of 1-forms by the image of the natural integration map from $H_1(X(\mathbf{C}), \mathbf{Z})$. To execute this uniformization in practice, one often wants to understand the space of 1-forms and the first homology group of X explicitly. A general approach to this problem is given by Schottky uniformization (see [38] for the original work and [22] for a modern summary and generalization). The "retrosection" theorem of [24] states that there exists a Schottky group $\Gamma \subset \mathbf{PGL}_2(\mathbf{C})$ and an open set $\mathcal{H}_\Gamma \subset \mathbf{P}^1(\mathbf{C})$ such that $X(\mathbf{C})$ is analytically isomorphic to $\mathcal{H}_\Gamma/\Gamma$. Among its other properties, the group Γ is free of rank g, the genus of the curve X. Under certain

convergence conditions, one may describe the Jacobian of X as the quotient of a split torus $(\mathbf{C}^{\times})^g$ by the image of an explicit homomorphism from Γ.

While Schottky uniformization is useful as a general theory, it does not necessarily provide a method of constructing rational points on X or its Jacobian in cases of arithmetic interest. Furthermore, if the parameterizing group Γ cannot be found explicitly, one may not even be able to calculate the periods of X in practice.

In our case of study, namely the modular curves $X_0(N)$, it is essential to exploit the "arithmeticity" given by modularity. By its moduli description, the set of complex points of $X_0(N)$ can be identified with the quotient of the extended upper half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$ by the discrete group $\Gamma_0(N)$ acting on the left via linear fractional transformations:

$$X_0(N)(\mathbf{C}) \cong \mathcal{H}^*/\Gamma_0(N). \tag{2.1}$$

Let $X_0(N)$ have genus g, and let $\mathcal{S}_2(N)$ denote the space of cusp forms of level N. For any $\tau_1, \tau_2 \in \mathcal{H}^*$, we can define a homomorphism denoted $\int_{\tau_1}^{\tau_2}$ from $\mathcal{S}_2(N)$ to $\mathbf{C}$ via a complex line integral:

$$\int_{\tau_1}^{\tau_2} : f \mapsto 2\pi i \int_{\tau_1}^{\tau_2} f(z)\,\mathrm{d}z.$$

Since f is a modular form of level N, this value is unchanged if τ_1 and τ_2 are replaced by $\gamma\tau_1$ and $\gamma\tau_2$, respectively, for $\gamma \in \Gamma_0(N)$. Thus if $\mathrm{Div}_0\,\mathcal{H}^*$ denotes the group of degree-zero divisors on the points of the extended upper half plane, we obtain a homomorphism

$$
\begin{aligned}
(\mathrm{Div}_0\,\mathcal{H}^*)_{\Gamma_0(N)} &\to \mathrm{Hom}(\mathcal{S}_2(N), \mathbf{C}) \\
[\tau_1] - [\tau_2] &\mapsto \left(f \mapsto 2\pi i \int_{\tau_1}^{\tau_2} f(z)\,\mathrm{d}z\right).
\end{aligned}
\tag{2.2}
$$

The short exact sequence

$$0 \to \mathrm{Div}_0\,\mathcal{H}^* \to \mathrm{Div}\,\mathcal{H}^* \to \mathbf{Z} \to 0$$

gives rise to a boundary map in homology:

$$\delta : H_1(\Gamma_0(N), \mathbf{Z}) \to (\mathrm{Div}_0\,\mathcal{H}^*)_{\Gamma_0(N)}. \tag{2.3}$$

Denote the composition of the maps in (2.2) and (2.3) by

$$\Phi_1 : H_1(\Gamma_0(N), \mathbf{Z}) \to \mathrm{Hom}(\mathcal{S}_2(N), \mathbf{C}),$$

and let L denote the image of Φ_1. The group L is free abelian of rank $2g$ and is Hecke-stable. For $x \in \mathcal{H}^*$, let $\widetilde{x}$ represent the image of x in $X_0(N)(\mathbf{C}) = \mathcal{H}^*/\Gamma_0(N)$. Under these notations, the Abel–Jacobi theorem may be stated as follows:

Theorem 2.1.1. *The map $[\tilde{x}] - [\tilde{y}] \mapsto \int_y^x$ induces a complex analytic uniformization of the Jacobian of $X_0(N)$:*

$$J_0(N)(\mathbf{C}) \cong \mathrm{Hom}(\mathcal{S}_2(N), \mathbf{C})/L.$$

Let $\tau \in \mathcal{H}^*$ lie in an imaginary quadratic subfield K of $\mathbf{C}$. Then

$$P_\tau := \int_\infty^\tau \in \mathrm{Hom}(\mathcal{S}_2(N), \mathbf{C})/L = J_0(N)(\mathbf{C})$$

is a *Heegner point* on $J_0(N)$. The theory of complex multiplication shows that this analytically defined point is actually defined over an abelian extension of K, and it furthermore prescribes the action of the Galois group of K on this point.

The goal of the remainder of this chapter is to present the theory of p-adic uniformization of Jacobians of degenerating curves via Schottky groups, as studied by Tate, Mumford, Manin, and Drinfeld, in the language of p-adic integration. The standard presentation of this subject (see [11], for example) involves certain theta functions that often have no direct analogue in the complex analytic situation because of convergence issues. Thus our new notation, inspired by [1] (the ideas we have drawn from in [1] appear in the construction of certain p-adic L-functions), allows one to draw a more direct parallel between the complex analytic and p-adic settings.

2.2   Generalities on p-adic measures

To describe a p-adic analogue of the theory described in section 2.1, we first define the p-adic integrals that will play the role of complex line integrals. Let K be a local field (a locally compact field, complete with respect to a discrete valuation). Let Γ be a subgroup of $\mathbf{PGL}_2(K)$, which acts on $\mathbf{P} = \mathbf{P}_K^1$ by linear fractional transformations. Denote by C the completion of an algebraic closure of K.

Definition 2.2.1. The functor of limit points $\mathcal{L}_\Gamma$ associates to each complete field extension F of K contained in C the set of $p \in \mathbf{P}(F)$ such that there exists $q \in \mathbf{P}(F)$ and distinct $\gamma_n \in \Gamma$ with $\lim \gamma_n q = p$. The group Γ is said to be discontinuous if $\mathcal{L}_\Gamma \neq \mathbf{P}$.

Since K is a local field, Γ is discontinuous if and only if it is discrete as a subgroup of $\mathbf{PGL}_2(K)$ and in this case $\mathcal{L}_\Gamma(C) \subset \mathbf{P}(K)$ ([11, 1.6.4]). We will not in general be assuming that Γ is discontinuous. Let $\mathcal{L}_\Gamma(F)$ have the induced topology from F. The space $\mathcal{L}_\Gamma(F)$ is a closed subset of $\mathbf{P}(F)$, and hence compact if F is a local field.

Definition 2.2.2. Let H be a free abelian group of finite rank (endowed with the trivial Γ-action). The group $\mathrm{Meas}(\mathcal{L}_\Gamma(F), H)$ of additive measures on $\mathcal{L}_\Gamma(F)$ with values in H is the group of maps μ that assign to each compact open subset U of $\mathcal{L}_\Gamma(F)$ an element $\mu(U)$ of H, such that

- $\mu(U) + \mu(V) = \mu(U \cup V)$ for disjoint open compacts U and V, and

- $\mu(\mathcal{L}_\Gamma(F)) = 0$.

The group $\mathrm{Meas}(\mathcal{L}_\Gamma(F), H)$ has a natural Γ action, given by $(\gamma\mu)(U) := \mu(\gamma^{-1}U)$ (here and throughout, we will be viewing Γ as an ordinary group, rather than a topological group, when considering Γ-modules and cohomology). Let $\mathcal{M}$ be a Γ-module, and suppose we have a Γ-invariant homomorphism from $\mathcal{M}$ to $\mathrm{Meas}(\mathcal{L}_\Gamma(K), H)$:

$$\mu \in \mathrm{Hom}(\mathcal{M}, \mathrm{Meas}(\mathcal{L}_\Gamma(K), H))^\Gamma.$$

Denote the image of m under μ by μ_m.

Definition 2.2.3. Let $d \in \mathrm{Div}_0 \, \mathcal{H}_\Gamma(C)$ be a degree-zero divisor, and let $m \in \mathcal{M}$. Choose a rational function f_d on $\mathbf{P}^1(C)$ with divisor d, and define the multiplicative double integral:

$$\fint_d \int_m \omega_\mu \;\; := \;\; \fint_{\mathcal{L}_\Gamma(K)} f_d(t) \, \mathrm{d}\mu_m(t)$$
$$:= \;\; \lim_{||\mathcal{U}|| \to 0} \prod_{U \in \mathcal{U}} f_d(t_U) \otimes \mu_m(U) \in C^\times \otimes_{\mathbf{Z}} H. \qquad (2.4)$$

Here the limit is taken over uniformly finer disjoint covers $\mathcal{U}$ of $\mathcal{L}_\Gamma(K)$ by nonempty open compact subsets U, and t_U is an arbitrarily chosen point of U.

Remark 2.2.4. The products in (2.4) are finite since $\mathcal{L}_\Gamma(K)$ is compact. The limit converges since μ_m is a measure.

Also, since $\mu_m(\mathcal{L}_\Gamma(K)) = 0$, the multiplicative integral on $\mathcal{L}_\Gamma(K)$ of a constant (with respect to μ_m) vanishes, so Definition 2.2.3 is independent of the choice of f_d.

For a complete field extension F of K lying in C, denote by $\mathcal{H}_\Gamma(F)$ the space $\mathbf{P}^1(F) - \mathcal{L}_\Gamma(K)$. It is clear that if $d \in \mathrm{Div}_0 \, \mathcal{H}_\Gamma(F)$, then

$$\fint_d \int_m \omega_\mu \in F^\times \otimes H.$$

Proposition 2.2.5. *The multiplicative double integral is* Γ*-invariant:*

$$\fint_d \int_m \omega_\mu = \fint_{\gamma d} \int_{\gamma m} \omega_\mu \in C^\times \otimes H$$

for $d \in \mathcal{H}_\Gamma(C)$, $m \in \mathcal{M}$, *and* $\gamma \in \Gamma$.

Proof. From the Γ-invariance of μ, we have

$$\fint_{\gamma d} \int_{\gamma m} \omega_\mu = \fint_{\mathcal{L}_\Gamma(K)} f_{\gamma d}(t) \, \mathrm{d}\mu_{\gamma m}(t) = \fint_{\mathcal{L}_\Gamma(K)} f_{\gamma d}(\gamma u) \, \mathrm{d}\mu_m(u), \tag{2.5}$$

where $u = \gamma^{-1} t$. Since we may choose $f_{\gamma d}(\gamma u) = f_d(u)$, the result follows. $\qquad\square$

Thus the multiplicative double integral defines a map

$$\fint\!\!\int \omega_\mu : ((\mathrm{Div}_0 \, \mathcal{H}_\Gamma) \otimes \mathcal{M})_\Gamma \to \mathbf{G}_m \otimes H \tag{2.6}$$

Here we view $\mathcal{H}_\Gamma$ and $\mathbf{G}_m$ as functors on the category of complete field extensions of K contained in C.

Remark 2.2.6. If $\tau_1, \tau_2 \in \mathcal{H}_\Gamma(C)$ and $\tau_i \neq \infty$, we write

$$\fint_{\tau_1}^{\tau_2} \int_m \omega_\mu = \fint_{[\tau_2] - [\tau_1]} \int_m \omega_\mu = \fint_{\mathcal{L}_\Gamma(K)} \left(\frac{t - \tau_2}{t - \tau_1} \right) \mathrm{d}\mu_m(t)$$

as in [5].

As we now describe, for each Γ-module $\mathcal{M}$ there is a universal group H admitting a Γ-invariant homomorphism $\mu : \mathcal{M} \to \mathrm{Meas}(\mathcal{L}_\Gamma(K), H)$, in the sense that for any other group H' admitting a Γ-invariant homomorphism $\mu' : \mathcal{M} \to \mathrm{Meas}(\mathcal{L}_\Gamma(K), H')$, there exists a unique homomorphism $f_{\mu'} : H \to H'$ such that $\mu'_m(U) = f_{\mu'}(\mu_m(U))$.

We first recall the Bruhat–Tits tree $\mathcal{T}$ of $\mathbf{PGL}_2(K)$. Denote by $\mathcal{O}$ the ring of integers of K, by π a uniformizer of $\mathcal{O}$, and by $k = \mathcal{O}/\pi\mathcal{O}$ the residue field. The vertices of $\mathcal{T}$ are equivalence classes of free rank-two $\mathcal{O}$-submodules of $K \oplus K$, where two such modules are considered equivalent if they are homothetic by an element of $K^\times$. Two vertices are connected by an edge if they can be represented by modules M and N with $N \subset M$ and $M/N \cong k$; this is clearly a symmetric relation. The unoriented graph $\mathcal{T}$ that results from these definitions is a regular tree of degree $\#\mathbf{P}^1(k)$. The group $\mathbf{PGL}_2(K)$ acts naturally on the tree.

Let v^* denote the vertex corresponding to $\mathcal{O} \oplus \mathcal{O}$ and let w^* denote the vertex corresponding to $\mathcal{O} \oplus \pi\mathcal{O}$. The stabilizer of v^* in $\mathbf{PGL}_2(K)$ is $\mathbf{PGL}_2(\mathcal{O})$. The matrix

$P = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ sends w^* to v^*, and hence the stabilizer of w^* is $P^{-1}\mathbf{PGL}_2(\mathcal{O})P$. Let e^* denote the oriented edge from w^* to v^*. The stabilizer of e^* in $\mathbf{PGL}_2(K)$ is the intersection of the stabilizers of v^* and w^*, namely, the set of matrices of $\mathbf{PGL}_2(\mathcal{O})$ that are upper triangular modulo π. This group equals the stabilizer of $\mathcal{O}$ in $\mathbf{PGL}_2(K)$ under linear fractional transformations. Thus if we associate to the oriented edge e^* the compact open set $U_{e^*} := \mathcal{O} \subset \mathbf{P}^1(K)$, this extends to an assignment of a compact open subset of $\mathbf{P}^1(K)$ to each oriented edge of the tree via $\mathbf{PGL}_2(K)$-invariance:

$$U_{\gamma e^*} := \gamma \mathcal{O} \text{ for all } \gamma \in \mathbf{PGL}_2(K).$$

We note some essential properties of this assignment:

- For an oriented edge e, the oppositely oriented edge $\bar{e}$ satisfies $U_{\bar{e}} = \mathbf{P}^1(K) - U_e$.

- For each vertex v, the sets U_e as e ranges over the edges emanating from v form a disjoint cover of $\mathbf{P}^1(K)$.

- The sets U_e form a basis of compact open subsets of $\mathbf{P}^1(K)$.

Definition 2.2.7. The Bruhat–Tits tree of Γ is the subtree $\mathcal{T}_\Gamma \subset \mathcal{T}$ spanned by all edges such that both open sets corresponding to the two possible orientations of the edge contain an element of $\mathcal{L}_\Gamma(K)$.

The group Γ acts on the tree $\mathcal{T}_\Gamma$. To each oriented edge e of $\mathcal{T}_\Gamma$ we associate the compact open set $U_e(\Gamma) = U_e \cap \mathcal{L}(K)$. The sets $U_e(\Gamma)$ satisfy the properties above with $\mathbf{P}^1(K)$ replaced by $\mathcal{L}_\Gamma(K)$.

An *end* of a tree is a path without backtracking that is infinite in exactly one direction, modulo the relation that two such paths are equivalent if they are eventually equal[1]. The ends of $\mathcal{T}_\Gamma$ are naturally in bijection with $\mathcal{L}_\Gamma(K)$, by sending an end to the unique point in the intersection of all $U_e(\Gamma)$ for the oriented edges e of the end.

The space $\mathcal{H}_\Gamma$ may be viewed as a thickening of the tree $\mathcal{T}_\Gamma$ by means of the reduction map

$$\text{red} : \mathcal{H}_\Gamma \to \mathcal{T}_\Gamma.$$

---

[1] Rigorously, an end is an infinite sequence $v_0, v_1, \ldots$ of distinct vertices of the tree such that (v_i, v_{i+1}) is an oriented edge, modulo the relation that $\{v_i\} \sim \{w_i\}$ if there exist n, m such that $v_{n+i} = w_{m+i}$ for all $i \geq 0$.

We will define the reduction map only on the points of $\mathcal{H}_\Gamma$ defined over finite unramified extensions F of K. In this case, the tree $\mathcal{T}$ of $\mathbf{PGL}_2(K)$ is naturally a subtree of the Bruhat–Tits tree $\mathcal{T}_F$ of $\mathbf{PGL}_2(F)$, and hence the tree $\mathcal{T}_\Gamma$ may be viewed as a subtree of $\mathcal{T}_F$ as well. A point $u \in \mathcal{H}_\Gamma(F)$ corresponds to an end of $\mathcal{T}_F$; this end may be represented by a unique path originating from a vertex v_u in $\mathcal{T}_\Gamma$ and intersecting $\mathcal{T}_\Gamma$ only at v_u. The vertex v_u is defined to be the reduction of u.

The Bruhat–Tits tree of Γ allows one to understand measures on $\mathcal{L}_\Gamma(K)$ combinatorially. Denote by E_Γ (resp. V_Γ) the set of all oriented edges (resp. vertices) of the tree $\mathcal{T}_\Gamma$. Denote by C_E the group $\operatorname{Div} E_\Gamma/(e+\overline{e})$, the abelian group generated freely by the oriented edges of $\mathcal{T}_\Gamma$ modulo the relation that oppositely oriented edges add to zero. Denote by C_V the group $\operatorname{Div} V_\Gamma$. Define a trace map $\operatorname{Tr} : C_V \to C_E$ by sending each vertex v to the sum of the edges of $\mathcal{T}_\Gamma$ with source vertex v. The trace map is injective, and we define C_Q by the short exact sequence

$$0 \longrightarrow C_V \xrightarrow{\operatorname{Tr}} C_E \longrightarrow C_Q \longrightarrow 0. \tag{2.7}$$

The correspondence $e \mapsto U_e(\Gamma)$ shows that for each H, the group $\operatorname{Meas}(\mathcal{L}_\Gamma(K), H)$ is the kernel of the dual of the trace map:

$$0 \longrightarrow \operatorname{Meas}(\mathcal{L}_\Gamma(K), H) \longrightarrow \operatorname{Hom}(C_E, H) \xrightarrow{\operatorname{Tr}^*} \operatorname{Hom}(C_V, H) \longrightarrow 0. \tag{2.8}$$

Thus $\operatorname{Meas}(\mathcal{L}_\Gamma(K), H) \cong \operatorname{Hom}(C_Q, H)$, which is called the group of H-valued harmonic cocycles on the tree $\mathcal{T}_\Gamma$. For each Γ-module $\mathcal{M}$,

$$\operatorname{Hom}(\mathcal{M}, \operatorname{Meas}(\mathcal{L}_\Gamma(K), H)) \cong \operatorname{Hom}(C_Q \otimes \mathcal{M}, H).$$

In all the cases we will consider, the maximal torsion-free quotient of $(C_Q \otimes \mathcal{M})_\Gamma$ will be free of finite rank. Hence this group will be the universal free abelian group H admitting a Γ-invariant homomorphism from $\mathcal{M}$ to $\operatorname{Meas}(\mathcal{L}_\Gamma(K), H))$.

Definition 2.2.8. Denote by $\tilde{H}$ the maximal torsion-free quotient of $(C_Q \otimes \mathcal{M})_\Gamma$, and by $\tilde{T}$ the torus $\mathbf{G}_m \otimes \tilde{H}$. Denote by $\mu^{\operatorname{univ}}$ the universal Γ-invariant homomorphism from $\mathcal{M}$ to $\operatorname{Meas}(\mathcal{L}_\Gamma(K), \tilde{H}))$, with corresponding multiplicative double integral

$$\rlap{\,\subset}{\int\!\!\!\int} \omega_{\operatorname{univ}} : ((\operatorname{Div}_0 \mathcal{H}_\Gamma) \otimes \mathcal{M})_\Gamma \to \tilde{T}. \tag{2.9}$$

Consider the short exact sequence of Γ-modules defining $\mathrm{Div}_0\,\mathcal{H}_\Gamma$:

$$0 \to \mathrm{Div}_0\,\mathcal{H}_\Gamma \to \mathrm{Div}\,\mathcal{H}_\Gamma \to \mathbf{Z} \to 0.$$

After tensoring with $\mathcal{M}$, the long exact sequence in homology yields a boundary map

$$\delta : H_1(\Gamma, \mathcal{M}) \to ((\mathrm{Div}_0\,\mathcal{H}_\Gamma) \otimes \mathcal{M})_\Gamma. \tag{2.10}$$

Definition 2.2.9. Let Φ_1 denote the composition of the homomorphisms in (2.9) and (2.10):

$$\Phi_1 : H_1(\Gamma, \mathcal{M}) \to K^\times \otimes \tilde{H}. \tag{2.11}$$

Each element in the image of Φ_1 may be expressed in terms of double integrals involving divisors $d \in \mathcal{H}_\Gamma(F)$ for any nontrivial extension F of K. By the independence of the integral from the choice of F, it follows that the image of Φ_1 indeed lies in $K^\times \otimes \tilde{H}$.

2.3   Mumford curves and Schottky groups

A discrete subgroup Γ of $\mathbf{PGL}_2(K)$ is called a Schottky group if it is finitely generated and has no nontrivial elements of finite order. Such a group Γ acts freely on its Bruhat–Tits tree $\mathcal{T}_\Gamma$ and is hence free [11, §I.3]. A curve X over K is called a Mumford curve if the stable reduction of X contains only rational curves over k. Mumford proved [36] that for such a curve there exists a Schottky group $\Gamma \subset \mathbf{PGL}_2(K)$ and a $\mathrm{Gal}(C/K)$-equivariant rigid analytic isomorphism

$$X(C) \cong \mathcal{H}_\Gamma(C)/\Gamma. \tag{2.12}$$

Furthermore, the Schottky group Γ satisfying (2.12) is unique up to conjugation by an element of $\mathbf{PGL}_2(K)$. It is free of rank g, the genus of X.

We will apply the definitions of the previous section with the trivial Γ-module $\mathcal{M} = \mathbf{Z}$. In this case we see that $\tilde{H} = (C_E)_\Gamma / \mathrm{Tr}((C_V)_\Gamma)$. But $(C_E)_\Gamma \cong C_{E'}$ and $(C_V)_\Gamma \cong C_{V'}$ where $C_{E'}$ and $C_{V'}$ are the corresponding groups for the finite quotient graph $\mathcal{T}_\Gamma/\Gamma$. Define an inner product on $C_{E'}$ with values in $\mathbf{Z}$ by declaring that two oriented edges e, e' are orthogonal unless they are equal as unoriented edges, in which case their inner product is 1 (resp. -1) if $e' = e$ (resp. $e' = \overline{e}$, the oppositely oriented edge). One easily checks that under this inner product, the quotient $C_{E'}/\mathrm{Tr}(C_{V'})$ is dual to the kernel of the boundary map

$$\partial : C_{E'} \to C_{V'} \text{ defined by } \partial([e]) = [t(e)] - [s(e)],$$

sending an edge to the difference of its terminal and source vertices. Moreover, basic combinatorics identifies this kernel with the topological first homology group (with $\mathbf{Z}$ coefficients) of the quotient $\mathcal{T}_\Gamma/\Gamma$. Since Γ acts freely on the tree $\mathcal{T}_\Gamma$, this homology group is canonically identified with the abelianization of Γ. (Alternatively, this follows directly from the long exact sequence (3.3) that we describe later.) Thus we have

$$\tilde{H} \cong \operatorname{Hom}(H_1(\Gamma, \mathbf{Z}), \mathbf{Z}) = \operatorname{Hom}(\Gamma, \mathbf{Z}).$$

Thus our multiplicative integral (2.6) becomes a map

$$(\operatorname{Div}_0 \mathcal{H}_\Gamma)_\Gamma \to \operatorname{Hom}(\Gamma, K^\times).$$

Recall from [11, §II.2] the definition of the theta function $\Theta(a, b; z)$ for $a, b, z \in \mathcal{H}_\Gamma(C)$ and $z \notin \Gamma a, \Gamma b$:

$$\Theta(a, b; z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma a}{z - \gamma b} \in C^\times.$$

The function Θ satisfies various automorphy properties, which we do not list here. The following result relates our multiplicative integral to values of theta functions. (The ideas of the proof are present already in [1].)

Proposition 2.3.1. *Let a, b be elements of $\mathcal{H}_\Gamma(F)$. Viewing the universal multiplicative integral as a homomorphism from Γ to $F^\times$, we have*

$$\left(\fint_b^a \omega_{\mathrm{univ}} \right)(\delta) = \frac{\Theta(a, b; \delta z)}{\Theta(a, b; z)} \tag{2.13}$$

for any $z \in \mathcal{H}_\Gamma(C) - (\Gamma a \cup \Gamma b)$ and $\delta \in \Gamma$. (The aforementioned automorphy properties of Θ imply that the right-hand side is independent of z.)

Proof. For a vertex v of the Bruhat–Tits tree of $\mathbf{PGL}_2(K)$, define the level $\ell(v)$ to be the length of the unique path from v to the central vertex v^*. Define $\ell(e)$ for an edge e to be the maximum level of its two boundary vertices. Write $\langle \cdot, \cdot \rangle$ for the inner product on $(C_E)_\Gamma$ defined above. Choose a vertex v_0 of $\mathcal{T}_\Gamma$ and let P be a path from v_0 to δv_0. Let $f(t) = \frac{t-a}{t-b}$. Then writing $\bar{P}$ for the image of P in $(C_E)_\Gamma$, we have

$$\left(\fint_b^a \omega_{\mathrm{univ}} \right)(\delta) = \lim_{n \to \infty} \prod_{\substack{e \in \mathcal{T}_\Gamma \\ \ell(e) = n}} f(t_e)^{\langle e, \bar{P} \rangle}, \tag{2.14}$$

where e is oriented outwards from v^* and t_e is an arbitrarily chosen point of $U_e(\Gamma)$.

To calculate the right side of (2.13), let z_0 be an element of $\mathcal{H}_\Gamma(C)$ reducing to the vertex v_0. We have

$$
\begin{aligned}
\frac{\Theta(a,b;\delta z_0)}{\Theta(a,b;z_0)} &= \prod_{\gamma\in\Gamma} \frac{\delta z_0 - \gamma a}{\delta z_0 - \gamma b} \cdot \frac{z_0 - \gamma b}{z_0 - \gamma a} \\
&= \prod_{\gamma\in\Gamma} \frac{\gamma^{-1}\delta z_0 - a}{\gamma^{-1}\delta z_0 - b} \cdot \frac{\gamma^{-1} z_0 - b}{\gamma^{-1} z_0 - a} \tag{2.15} \\
&= \prod_{\gamma\in\Gamma} \frac{\gamma\delta z_0 - a}{\gamma\delta z_0 - b} \cdot \frac{\gamma z_0 - b}{\gamma z_0 - a}, \tag{2.16}
\end{aligned}
$$

where (2.15) follows from the invariance of cross-ratios under linear fractional transformations. Let the vertices in the path P be $v_0, v_1, \ldots, v_s = \delta v_0$ and let the corresponding edges be written $e_1, \ldots, e_s$. For each vertex v_i, choose an element $z_i \in \mathcal{H}_\Gamma$ reducing to v_i, such that $z_s = \delta z_0$. The product (2.16) may be broken up as

$$
\prod_{\gamma\in\Gamma} \frac{f(\gamma\delta z_0)}{f(\gamma z_0)} = \prod_{\gamma\in\Gamma}\prod_{i=1}^{s} \frac{f(\gamma z_i)}{f(\gamma z_{i-1})}. \tag{2.17}
$$

Each of the products

$$
\prod_{\gamma\in\Gamma} \frac{f(\gamma z_i)}{f(\gamma z_{i-1})}
$$

converges since Γ acts freely on the tree, so the values $\ell(\gamma e_i)$ increase as γ ranges over Γ and the quotients of the values of f on the endpoints of these edges tend to 1. Thus we may rewrite (2.17) as

$$
\prod_{i=1}^{s}\prod_{\gamma\in\Gamma} \frac{f(\gamma z_i)}{f(\gamma z_{i-1})}.
$$

In fact for fixed n, if $\Gamma(i)$ denotes the set of $\gamma \in \Gamma$ such that $\ell(\gamma e_i) \le n$, then we have

$$
\prod_{\gamma\in\Gamma} \frac{f(\gamma z_i)}{f(\gamma z_{i-1})} \equiv \prod_{\gamma\in\Gamma(i)} \frac{f(\gamma z_i)}{f(\gamma z_{i-1})} \quad (\mathrm{mod}^* \ \pi^n), \tag{2.18}
$$

where $a \equiv b \ (\mathrm{mod}^* \ \pi^n)$ if $\frac{a}{b} - 1$ is divisible by π^n. The right side of (2.18) may be further decomposed as

$$
\prod_{\ell(\gamma v_{i-1})<n} f(\gamma z_{i-1})^{-1} \prod_{\ell(\gamma v_i)<n} f(\gamma z_i) \prod_{\substack{\ell(\gamma v_i)=n-1 \\ \ell(\gamma v_{i-1})=n}} f(\gamma z_{i-1})^{-1} \prod_{\substack{\ell(\gamma v_i)=n \\ \ell(\gamma v_{i-1})=n-1}} f(\gamma z_i). \tag{2.19}
$$

In the product over $i = 1, \ldots, s$, the first two terms of (2.19) cancel out. The last two yield

$$
\prod_{i=1}^{s}\prod_{\substack{\gamma\in\Gamma \\ \ell(\gamma e_i)=n}} f(\gamma z_*)^{\langle \gamma e_i, \bar{P}\rangle}, \tag{2.20}
$$

where $z_* \in \{z_{i-1}, z_i\}$ and γz_* reduces to the endpoint of γe_i of level n. Expression (2.20) is equivalent to (2.14) $\mathrm{mod}^* \pi^n$, and the result follows. $\qquad\square$

For $x \in \mathcal{H}_\Gamma(C)$, let $\widetilde{x}$ represent the image of x in $X(C) = \mathcal{H}_\Gamma(C)/\Gamma$.

Theorem 2.3.2 (Manin–Drinfeld). *The map*

$$[\widetilde{x}] - [\widetilde{y}] \mapsto \fint_y^x \omega_{\mathrm{univ}}$$

induces a $\mathrm{Gal}(C/K)$*-equivariant rigid analytic isomorphism between the rigid analytic space associated to the* C*-valued points of the Jacobian of the curve* X *and* $(C^\times \otimes \tilde{H})/\mathrm{Im}(\Phi_1)$.

Proof. The original statement of the Manin–Drinfeld theorem is given in terms of theta functions [11], [30]. Proposition 2.3.1 relates the classical version to our formulation. $\qquad\square$

Theorem 2.3.2 is the p-adic analogue of the Abel–Jacobi theorem. However, it does not allow the obvious construction of Heegner-type points in the case $X = X_0(p)$ over $\mathbf{Q}_p$. In fact, since Mumford's group Γ is not given in an explicit way and is probably not an arithmetic group, the question of computing the periods of $J_0(p)$ (i.e., calculating the lattice $\mathrm{Im}(\Phi_1)$) using this uniformization seems difficult. (See [9] for the calculation of this lattice modulo p using this theory, however.) Accordingly, one needs to find an alternative p-adic uniformization for modular Jacobians that uses arithmetic groups in a crucial way; this is taken up in Chapter 4.

Chapter 3

Properties of the multiplicative double integral

In this chapter we establish some fundamental properties of the multiplicative double integral.

3.1 Behavior of the integral under conjugation

Let $\delta \in \mathbf{PGL}_2(K)$, and let $\Gamma' = \delta\Gamma\delta^{-1}$. The Γ-module $\mathcal{M}$ gives rise to a Γ'-module $\mathcal{M}'$; the module $\mathcal{M}'$ is isomorphic to $\mathcal{M}$ as an abelian group, and Γ' acts on $\mathcal{M}'$ by

$$\delta\gamma\delta^{-1} : m \mapsto \gamma m,$$

where the action on the right side is given by the action of Γ on $\mathcal{M}$. In this setting, we have

$$\mathcal{L}_{\Gamma'}(K) = \delta\mathcal{L}_\Gamma(K) \text{ and } \mathcal{T}_{\Gamma'} = \delta\mathcal{T}_\Gamma.$$

We also have an isomorphism

$$c_\delta : \widetilde{H} = (C_Q \otimes \mathcal{M})_\Gamma \cong \widetilde{H'} = (C_{Q'} \otimes \mathcal{M}')_{\Gamma'}$$

given by $e \otimes m \mapsto \delta e \otimes m$. We omit the proof of the following proposition, which shows how the multiplicative double integral behaves under conjugation.

Proposition 3.1.1. *For $\tau_1, \tau_2 \in \mathcal{H}_\Gamma$ and $m \in \mathcal{M}$, we have*

$$(\mathrm{Id} \otimes c_\delta) \left(\fint_{\tau_1}^{\tau_2} \int_m \omega_{\mathrm{univ}} \right) = \fint_{\delta\tau_1}^{\delta\tau_2} \int_m \omega_{\mathrm{univ}},$$

where the integral on the left is with the universal measure for $(\Gamma, \mathcal{M})$, and the integral on the right is with the universal measure for $(\Gamma', \mathcal{M}')$.

3.2   An exact sequence in homology

Continuing with the notation of Section 2.2, define homomorphisms

$$\partial \ : \ C_E \to C_V, \quad \partial([y]) = [t(y)] - [s(y)], \tag{3.1}$$

$$\epsilon \ : \ C_V \to \mathbf{Z}, \quad \epsilon([x]) = 1.$$

Since $\mathcal{T}_\Gamma$ is a tree, the sequence

$$0 \longrightarrow C_E \xrightarrow{\ \partial\ } C_V \xrightarrow{\ \epsilon\ } \mathbf{Z} \longrightarrow 0, \tag{3.2}$$

is exact. Since $\mathbf{Z}$ is free, we may tensor with $\mathcal{M}$ without losing exactness, and then taking homology gives the the long exact sequence

$$\cdots \to H_i(\Gamma, C_E \otimes \mathcal{M}) \to H_i(\Gamma, C_V \otimes \mathcal{M}) \to H_i(\Gamma, \mathcal{M}) \to H_{i-1}(\Gamma, C_E \otimes \mathcal{M}) \to \cdots. \tag{3.3}$$

We would like to give an explicit description of the map

$$\phi : H_1(\Gamma, \mathcal{M}) \to \ker(\partial : H_0(\Gamma, C_E \otimes \mathcal{M}) \to H_0(\Gamma, C_V \otimes \mathcal{M})) \tag{3.4}$$

given in (3.3). We first recall the presentation of $H_1(\Gamma, \mathcal{M})$ given by "dimension shifting." The short exact sequence defining $\operatorname{Div}_0 \Gamma$, upon tensoring with $\mathcal{M}$, becomes:

$$0 \to (\operatorname{Div}_0 \Gamma) \otimes \mathcal{M} \to (\operatorname{Div} \Gamma) \otimes \mathcal{M} \to \mathcal{M} \to 0, \tag{3.5}$$

giving the long exact sequence in homology:

$$\cdots \to H_1(\Gamma, (\operatorname{Div} \Gamma) \otimes \mathcal{M}) \to H_1(\Gamma, \mathcal{M}) \to ((\operatorname{Div}_0 \Gamma) \otimes \mathcal{M})_\Gamma \to ((\operatorname{Div} \Gamma) \otimes \mathcal{M})_\Gamma \to \cdots.$$
$$\tag{3.6}$$

Now $(\operatorname{Div} \Gamma) \otimes \mathcal{M}$ being an induced module, the first term of (3.6) vanishes. Furthermore, the map

$$B : ((\operatorname{Div} \Gamma) \otimes \mathcal{M})_\Gamma \to \mathcal{M}, \quad \gamma \otimes m \mapsto \gamma^{-1} m$$

is an isomorphism of abelian groups. Writing B again for the induced map

$$((\operatorname{Div}_0 \Gamma) \otimes \mathcal{M})_\Gamma \to \mathcal{M},$$

we hence obtain

$$H_1(\Gamma, \mathcal{M}) \cong \ker(B : ((\operatorname{Div}_0 \Gamma) \otimes \mathcal{M})_\Gamma \to \mathcal{M}). \tag{3.7}$$

Remark 3.2.1. An element $k \in (\mathrm{Div}_0 \, \Gamma) \otimes \mathcal{M}$ may be written in the form

$$k = \sum_{\gamma \in \Gamma} ([\gamma^{-1}] - [\mathrm{Id}]) \otimes m_\gamma. \tag{3.8}$$

In the literature, elements of $H_1(\Gamma, \mathcal{M})$ are often represented by formal linear combinations $\sum_{\gamma \in \Gamma} m_\gamma \cdot [\gamma]$, where all but finitely many of the $m_\gamma \in \mathcal{M}$ are 0 and $\sum (\gamma m_\gamma - m_\gamma) = 0$; such an expression is called a 1-cycle, and corresponds to the element $k \in H_1(\Gamma, \mathcal{M})$ in (3.8).

Remark 3.2.2. If

$$\sum ([\alpha_i] - [\beta_i]) \otimes m_i \in H_1(\Gamma, \mathcal{M}),$$

the expression

$$\sum ([\alpha_i \gamma] - [\beta_i \gamma]) \otimes m_i$$

for $\gamma \in \Gamma$ represents the same element of $H_1(\Gamma, \mathcal{M})$, since their difference equals

$$
\begin{aligned}
& \sum ([\alpha_i] - [\alpha_i \gamma]) \otimes m_i - \sum ([\beta_i] - [\beta_i \gamma]) \otimes m_i \\
= \;& \sum ([\mathrm{Id}] - [\gamma]) \otimes \alpha_i^{-1} m_i - \sum ([\mathrm{Id}] - [\gamma]) \otimes \beta_i^{-1} m_i \\
= \;& ([\mathrm{Id}] - [\gamma]) \otimes \sum (\alpha_i^{-1} m_i - \beta_i^{-1} m_i) \\
= \;& ([\mathrm{Id}] - [\gamma]) \otimes 0 = 0.
\end{aligned}
$$

We now describe the map ϕ of (3.4) explicitly. Fix $v \in V_\Gamma$ and consider the following commutative diagram between the short exact sequences of Γ-modules in (3.2) and (3.5):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Div}_0 \, \Gamma & \longrightarrow & \mathrm{Div} \, \Gamma & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \iota_v} & & \downarrow{\scriptstyle \kappa_v} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & C_E & \overset{\partial}{\longrightarrow} & C_V & \overset{\epsilon}{\longrightarrow} & \mathbf{Z} & \longrightarrow & 0.
\end{array}
\tag{3.9}
$$

Here the map $\kappa_v : \mathrm{Div} \, \Gamma \to C_V$ is defined by $[\gamma] \mapsto [\gamma] v := [\gamma v]$. The map ι_v is defined by the commutativity of the diagram; for each divisor $G \in \mathrm{Div}_0 \, \Gamma$, there exists a unique $G' \in C_E$ such that $\partial(G') = Gv$; we define $\iota_v(G) = G'$. After tensoring all the terms in the diagram (3.9) with $\mathcal{M}$, there is a commutative diagram between the corresponding long exact sequences (3.3) and (3.6). Part of this diagram reads:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H_1(\Gamma, \mathcal{M}) & \longrightarrow & ((\mathrm{Div}_0 \, \Gamma) \otimes \mathcal{M})_\Gamma & \longrightarrow & \mathcal{M} \\
& & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \iota_v} & & \downarrow \\
H_1(\Gamma, C_V \otimes \mathcal{M}) & \longrightarrow & H_1(\Gamma, \mathcal{M}) & \overset{\phi}{\longrightarrow} & H_0(\Gamma, C_E \otimes \mathcal{M}) & \overset{\partial}{\longrightarrow} & H_0(\Gamma, C_V \otimes \mathcal{M}).
\end{array}
\tag{3.10}
$$

Consider an element

$$k = \sum_i G_i \otimes m_i \in ((\mathrm{Div}_0 \, \Gamma) \otimes \mathcal{M})_\Gamma.$$

For each element G_i, there is a unique $C_i \in C_E$ such that $G_i v = \partial(C_i)$. The image of k in under the map ι_v of (3.10) is:

$$\iota_v(k) = \sum_i C_i \otimes m_i \in (C_E \otimes M)_\Gamma. \tag{3.11}$$

One checks that when

$$k \in \ker(B : (\mathrm{Div}_0 \, \Gamma) \otimes \mathcal{M})_\Gamma \to \mathcal{M}) \cong H_1(\Gamma, \mathcal{M}),$$

this value of $\iota_v(k)$ is independent of the choice of vertex v. Equation (3.11) provides an explicit formula for the map ϕ.

3.3 The valuation of the integration map

Let the valuation v_K of K be normalized so that the valuation of the uniformizer π is 1. In this section we will analyze the composite map

$$v_K \circ \Phi_1 : H_1(\Gamma, \mathcal{M}) \xrightarrow{\Phi_1} K^\times \otimes \tilde{H} \xrightarrow{v_K \otimes \mathrm{Id}} \mathbf{Z} \otimes \tilde{H} = \tilde{H}.$$

Since $\tilde{H}$ as defined in Definition 2.2.8 is a quotient of

$$(C_Q \otimes \mathcal{M})_\Gamma = (C_E \otimes \mathcal{M})_\Gamma / \mathrm{Tr}((C_V \otimes \mathcal{M})_\Gamma),$$

we have natural maps

$$H_1(\Gamma, \mathcal{M}) \xrightarrow{\phi} (C_E \otimes \mathcal{M})_\Gamma \longrightarrow \tilde{H}, \tag{3.12}$$

with ϕ as in (3.4). Denote by ι the composite map $H_1(\Gamma, \mathcal{M}) \to \tilde{H}$ in (3.12).

Proposition 3.3.1. *The map $v_K \circ \Phi_1$ is equal to ι.*

We first prove a lemma.

Lemma 3.3.2. *Let $\tau_1, \tau_2 \in \mathcal{H}_\Gamma$ reduce to vertices connected by an oriented edge e of $\mathcal{T}_\Gamma$:*

$$\partial(e) = [\mathrm{red}(\tau_2)] - [\mathrm{red}(\tau_1)].$$

Extend the valuation v_K to the maximal unramified extension of K. Then we have

$$v_K \left(\oint_{\tau_1}^{\tau_2} \int_m \omega_{\mathrm{univ}} \right) = \mu_m(U_e).$$

Proof. The group $\mathbf{PGL}_2(K)$ acts transitively on the edges of its Bruhat–Tits tree, and the reduction map is $\mathbf{PGL}_2(K)$-equivariant. Thus Proposition 3.1.1 implies that it suffices to consider the case where τ_1 reduces to the standard vertex corresponding to $\mathcal{O}_K \oplus \mathcal{O}_K$ and τ_2 reduces to the vertex corresponding to $\mathcal{O}_K \oplus \pi\mathcal{O}_K$. In this case, we have $U_e = \mathcal{L}_\Gamma(K) - \mathcal{O}_K$. Let $\tau_1 \in \mathbf{P}^1(F)$ for an unramified extension F of K. The fact that τ_1 reduces to v_* implies that the image of τ_1 in $\mathbf{P}^1(k_F)$ does not equal the image of any point $t \in \mathcal{L}_\Gamma(K)$ in $\mathbf{P}^1(k_F)$, where k_F is the residue field of F. In particular, the fact that the edge e connecting τ_2 and τ_1 lies in $\mathcal{T}_\Gamma$ implies that $\tau_1 \in \mathcal{O}_F$. Thus for $t \in \mathcal{L}_\Gamma(K)$, we have

$$v_K(\tau_1 - t) = \begin{cases} 0 & \text{if } t \in \mathcal{O}_K \\ v_K(t) & \text{otherwise.} \end{cases}$$

Similarly,

$$v_K(\tau_2 - t) = \begin{cases} -1 & \text{if } t \in \mathcal{O}_K \\ v_K(t) & \text{otherwise.} \end{cases}$$

Without loss of generality, in the definition of the multiplicative integral, we need consider only open coverings $\mathcal{U}$ that refine the open covering $\{U_{\bar{e}}, U_e\}$. For each U in such a covering, the previous calculation shows that $v_K((t_U - \tau_2)/(t_U - \tau_1))$ equals -1 or 0 according to whether $U \subset U_{\bar{e}}$ or not. Thus the valuation of each product inside the limit defining the multiplicative integral equals $-\mu_m(U_{\bar{e}}) = \mu_m(U_e)$. $\square$

By combining Lemma 3.3.2 with the explicit description of ϕ given in Section 3.2, we can now prove Proposition 3.3.1. Let v be a vertex of $\mathcal{T}_\Gamma$, and let $\tau \in \mathcal{H}_\Gamma$ reduce to v. Let

$$k = \sum_i G_i \otimes m_i \in \ker(B : ((\mathrm{Div}_0\,\Gamma) \otimes \mathcal{M})_\Gamma \to \mathcal{M}) \cong H_1(\Gamma, \mathcal{M}).$$

Tracing through definitions, one finds that

$$\Phi_1(k) = \fint_{G_i\tau} \int_{m_i} \omega_{\mathrm{univ}} \in K^\times \otimes \tilde{H}.$$

Let $C_i \in C_E$ be such that $G_i v = \partial(C_i)$. For an oriented edge e of $\mathcal{T}_\Gamma$, $\mu_m(U_e)$ is defined to be the image in $\tilde{H}$ of the element $e \otimes m \in (C_E \otimes \mathcal{M})_\Gamma$. Lemma 3.3.2 thus implies that

$$\sum_i v_K \left(\fint_{G_i\tau} \int_{m_i} \omega_{\mathrm{univ}} \right)$$

is equal to the image of $\sum_i C_i \otimes m_i$ in $\tilde{H}$. Equation (3.11) shows that this is precisely the value $\iota(k)$, thereby proving Proposition 3.3.1.

3.4 Choosing a system of representatives

In (2.9) we defined a universal multiplicative double integral that is canonically associated to a group $\Gamma \subset \mathbf{PGL}_2(K)$ and a Γ-module $\mathcal{M}$. In practice, it will be convenient to make certain choices that allow one to be more explicit with the homology groups that arise. Thus we choose a system of representatives (E, V) for Γ acting on $\mathcal{T}_\Gamma$. This is a subset E of the edges and V of the vertices of $\mathcal{T}_\Gamma$ such that every edge (resp. vertex) of $\mathcal{T}_\Gamma$ is equivalent modulo Γ to exactly one element of E (resp. V). It need not be the case that the endpoints of an edge in E lie in V. Denote by $\mathfrak{o}$ an orientation on each edge of E. Assume that Γ acts without inversion on $\mathcal{T}_\Gamma$, meaning that no oriented edge is in the same Γ-orbit as its opposite. Then extending $\mathfrak{o}$ by Γ-invariance gives an orientation on each edge of the tree $\mathcal{T}_\Gamma$.

Denote by Γ_e the stabilizer of the edge e. The Γ-module C_E is a sum of induced modules:

$$\bigoplus_{e \in E} \operatorname{Div} \Gamma/\Gamma_e \cong C_E, \quad \left(\sum_i a_i \gamma_{e,i} \right)_{e \in E} \mapsto \sum_{i,e} a_i [\gamma_{e,i} e],$$

where on the right side the edge e has the orientation given by $\mathfrak{o}$. Similarly, we have an isomorphism $\bigoplus_{v \in V} \operatorname{Div} \Gamma/\Gamma_v \cong C_V$. Accordingly, for each Γ-module $\mathcal{M}$, Shapiro's Lemma gives isomorphisms

$$H_i(\Gamma, C_E \otimes \mathcal{M}) \cong \bigoplus_{e \in E} H_i(\Gamma_e, \mathcal{M}) \text{ and } H_i(\Gamma, C_V \otimes \mathcal{M}) \cong \bigoplus_{v \in V} H_i(\Gamma_v, \mathcal{M}). \qquad (3.13)$$

Thus the long exact sequence (3.3) becomes

$$\cdots \to \bigoplus_{e \in E} H_i(\Gamma_e, \mathcal{M}) \to \bigoplus_{v \in V} H_i(\Gamma_v, \mathcal{M}) \to H_i(\Gamma, \mathcal{M}) \to \bigoplus_{e \in E} H_{i-1}(\Gamma_e, \mathcal{M}) \to \cdots . \qquad (3.14)$$

In particular, $\tilde{H}$ is the maximal torsion-free quotient of

$$\bigoplus_{e \in E} H_0(\Gamma_e, \mathcal{M}) / \operatorname{Tr} \left(\bigoplus_{v \in V} H_0(\Gamma_v, \mathcal{M}) \right). \qquad (3.15)$$

We will discuss the maps Tr and ∂ in this presentation of our homology groups with an explicit example in the next chapter.

Chapter 4

An arithmetic uniformization and Stark–Heegner points

Let p be a prime number and $N \geq 1$ an integer not divisible by p. Write $M = Np$. In this chapter we will present a p-adic uniformization of the maximal quotient of $J_0(M)$ with toric reduction at p. A key idea, suggested by the definitions of [5], is that the p-adic arithmetic of $J_0(M)$ is intimately linked with the group

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}_2(\mathbf{Z}[1/p]) \text{ such that } N|c \right\} \tag{4.1}$$

and its homology. The group Γ is not discrete as a subgroup of $\mathbf{PGL}_2(K)$ and hence acts with dense orbits on $\mathbf{P}^1$. In this setting, the limit point set equals $\mathcal{L} := \mathcal{L}_\Gamma(\mathbf{Q}_p) = \mathbf{P}^1(\mathbf{Q}_p)$. We also have

$$\mathcal{H}_\Gamma(\mathbf{C}_p) = \mathcal{H}_p := \mathbf{P}^1(\mathbf{C}_p) - \mathbf{P}^1(\mathbf{Q}_p),$$

where $\mathbf{C}_p$ is the completion of an algebraic closure of $\mathbf{Q}_p$.

A measure on $\mathbf{P}^1(\mathbf{Q}_p)$ is given by a harmonic cocycle on the entire Bruhat–Tits tree $\mathcal{T}$ of $\mathbf{PGL}_2(\mathbf{Q}_p)$. From (3.15) with $\mathcal{M} = \mathbf{Z}$, one finds that there are no non-trivial Γ-invariant measures on $\mathbf{P}^1(\mathbf{Q}_p)$ (see (4.3) below with $\mathcal{M} = \mathbf{Z}$). This problem can be remedied by introducing a Γ-invariant measure-valued *modular symbol* as follows.

Let $\mathcal{M} := \mathrm{Div}_0\, \mathbf{P}^1(\mathbf{Q})$ be the group of degree-zero divisors on $\mathbf{P}^1(\mathbf{Q})$, viewed as cusps of the complex upper half plane. The group $\mathcal{M}$ is defined by the exact sequence

$$0 \to \mathcal{M} \to \mathrm{Div}\, \mathbf{P}^1(\mathbf{Q}) \to \mathbf{Z} \to 0. \tag{4.2}$$

The group Γ acts on $\mathcal{M}$ via its action on $\mathbf{P}^1(\mathbf{Q})$ by linear fractional transformations. For a free abelian group H, a $\mathrm{Meas}(\mathbf{P}^1(\mathbf{Q}_p), H)$-valued modular symbol is a homomorphism

$$\mathcal{M} \to \mathrm{Meas}(\mathbf{P}^1(\mathbf{Q}_p), H), \quad m \mapsto \mu_m.$$

The group of modular symbols μ has a Γ-action given by

$$(\gamma^{-1}\mu)_m(U) = \mu_{\gamma m}(\gamma U).$$

In Chapter 2 we constructed a universal Γ-invariant $\mathrm{Meas}(\mathbf{P}^1(\mathbf{Q}_p), \tilde{H})$-valued modular symbol. In the next section we will explore the group $\tilde{H}$.

4.1   The universal modular symbol

The action of Γ on the tree $\mathcal{T}$ is particularly easy to describe ([40, §II]). Each oriented edge of $\mathcal{T}$ is equivalent to either e^* or $\overline{e^*}$; each vertex is equivalent to either v^* or w^*. The stabilizers of v^* and w^* in Γ are $\Gamma_0(N)$ and $P^{-1}\Gamma_0(N)P$, respectively, where P is the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. The stabilizer of e^* is the intersection of these, namely $\Gamma_0(M)$. Hence from (3.15) we have that $\tilde{H}$ is the maximal torsion-free quotient of

$$\mathcal{M}_{\Gamma_0(M)} / \mathrm{Tr}(\mathcal{M}_{\Gamma_0(N)} \oplus \mathcal{M}_{P^{-1}\Gamma_0(N)P}). \tag{4.3}$$

Furthermore, the map $m \mapsto Pm$ defines an isomorphism

$$H_*(P^{-1}\Gamma_0(N)P, \mathcal{M}) \to H_*(\Gamma_0(N), \mathcal{M}).$$

One can interpret the group of co-invariants

$$\mathcal{M}_{\Gamma_0(M)} = H_0(\Gamma_0(M), \mathcal{M})$$

geometrically as follows. Given a divisor $[x] - [y] \in \mathcal{M}$, consider a path from x to y in $\mathcal{H}^*$. If we make the identification $\mathcal{H}^*/\Gamma_0(M) = X_0(M)(\mathbf{C})$, the image of this path gives a well defined element of $H_1(X_0(M), \mathrm{cusps}, \mathbf{Z})$, the singular homology of the Riemann surface $X_0(M)(\mathbf{C})$ relative to the cusps. Manin [29] proves that this map induces an isomorphism between $H_1(X_0(M), \mathrm{cusps}, \mathbf{Z})$ and the maximal torsion-free quotient of $\mathcal{M}_{\Gamma_0(M)}$. This maximal torsion-free quotient will be denoted $H_0(\Gamma_0(M), \mathcal{M})_T$. The torsion of $H_0(\Gamma_0(M), \mathcal{M})$ is finite and supported at 2 and 3. The projection

$$\mathcal{M} \to \mathcal{M}_{\Gamma_0(M)} \to H_1(X_0(M), \mathrm{cusps}, \mathbf{Z})$$

is called the *universal modular symbol for* $\Gamma_0(M)$.

The points of $X_0(M)$ over a scheme S correspond to isomorphism classes of pairs (E, C_M) of generalized elliptic curves E/S equipped with a cyclic subgroup $C_M \subset E$ of order M. To such a pair we can associate two points of $X_0(N)$, namely the points corresponding to the pairs (E, C_N) and $(E/C_p, C_M/C_p)$, where C_p and C_N are the subgroups of C_M of order p and M, respectively. This defines two morphisms of curves

$$f_1 : X_0(M) \to X_0(N) \text{ and } f_2 : X_0(M) \to X_0(N), \tag{4.4}$$

each of which is defined over $\mathbf{Q}$. The map f_2 is the composition of f_1 with the Atkin–Lehner involution W_p on $X_0(M)$. Write $f_* = f_{1*} \oplus f_{2*}$ and $f^* = f_1^* \oplus f_2^*$ (resp. f_*^c and f_c^*) for the induced maps on (relative) singular homology:

$$
\begin{aligned}
f_* : \quad & H_1(X_0(M), \mathbf{Z}) \to H_1(X_0(N), \mathbf{Z})^2, \\
f^* : \quad & H_1(X_0(N), \mathbf{Z})^2 \to H_1(X_0(M), \mathbf{Z}), \\
f_*^c : \quad & H_1(X_0(M), \text{cusps}, \mathbf{Z}) \to H_1(X_0(N), \text{cusps}, \mathbf{Z})^2, \\
f_c^* : \quad & H_1(X_0(N), \text{cusps}, \mathbf{Z})^2 \to H_1(X_0(M), \text{cusps}, \mathbf{Z}).
\end{aligned}
$$

Via the universal modular symbol, the last two maps are identified with maps[1]

$$
\begin{aligned}
f_*^c : \quad & H_0(\Gamma_0(M), \mathcal{M}) \to H_0(\Gamma_0(N), \mathcal{M})^2 \\
f_c^* : \quad & H_0(\Gamma_0(N), \mathcal{M})^2 \to H_0(\Gamma_0(M), \mathcal{M}).
\end{aligned}
$$

The map f_c^* is precisely the map Tr of (4.3). Hence the universal Γ-invariant modular symbol of measures on $\mathbf{P}^1(\mathbf{Q}_p)$ takes values in the cokernel of f_c^*:

$$\tilde{H} \cong \left(H_1(X_0(M), \text{cusps}, \mathbf{Z})/f_c^*(H_1(X_0(N), \text{cusps}, \mathbf{Z})^2) \right)_T.$$

Define also

$$H := \left(H_1(X_0(M), \mathbf{Z})/f^*(H_1(X_0(N), \mathbf{Z})^2) \right)_T. \tag{4.5}$$

The abelian variety $J_0(M)^{p\text{-new}}$ is defined to be the quotient of $J_0(M)$ by the sum of the images of the Picard maps on Jacobians associated to f_1 and f_2. This is the abelian variety with purely toric reduction at p for which we will provide a uniformization (up to isogeny).

---

[1] In purely homological terms, f_{1*} is corestriction, and f_{2*} is the composition $f_{1*} \circ W_p$; similarly for f^* and restriction.

Proposition 4.1.1. *If we write g for the dimension of $J_0(N)^{p\text{-}new}$, the free abelian groups H and $\tilde{H}$ have ranks $2g$ and $2g+1$ respectively, and the natural map $H \to \tilde{H}$ is an injection.*

Proof. It is well known that f^* is injective and that H has rank $2g$. Consider the following commutative diagram of relative homology sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(X_0(N), \mathbf{Z})^2 & \longrightarrow & H_1(X_0(N), \text{cusps}, \mathbf{Z})^2 & \longrightarrow & C(N)^2 & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle f_c^*} & & \Big\downarrow & & \\
0 & \longrightarrow & H_1(X_0(M), \mathbf{Z}) & \longrightarrow & H_1(X_0(M), \text{cusps}, \mathbf{Z}) & \longrightarrow & C(M) & \longrightarrow & 0.
\end{array} \qquad (4.6)$$

Here $C(N)$ and $C(M)$ denote the groups of degree-zero divisors on the set of cusps of $X_0(N)$ and $X_0(M)$, respectively. If c denotes the number of cusps of $X_0(N)$, these are free abelian groups of rank $c-1$ and $2c-1$, respectively. Above each cusp of $X_0(N)$ (under the map f_1) lies two cusps of $X_0(M)$, one of which has ramification index p and the other one of which is unramified. The map W_p on $X_0(M)$ interchanges these two cusps. This implies that the map $C(N)^2 \to C(M)$ of (4.6) is injective and that the torsion subgroup of its cokernel has exponent dividing $p^2 - 1$. Since H and $\tilde{H}$ are the cokernels of f^* and f_c^*, the snake lemma yields the proposition. (Note that we have also shown that f_c^* is injective.) $\qquad \square$

4.2 Statement of the uniformization

As noted in Proposition 4.1.1, the module $\tilde{H}$ contains H with corank 1. To define a modular symbol that takes values in H rather than $\tilde{H}$, we choose a map

$$\psi : \tilde{H} \longrightarrow H.$$

We will require two properties of the map ψ, whose uses will later become evident:

- The groups $\tilde{H}$ and H have natural Hecke actions, described in detail in Chapter 5. We assume that the map ψ is Hecke-equivariant.

- We assume that the composition of ψ with the inclusion $H \subset \tilde{H}$ is an endomorphism of H with finite cokernel.

Associated to ψ we have the multiplicative double integral

$$\fint_{\tau_1}^{\tau_2} \!\!\int_m \omega_\psi = (\text{Id} \otimes \psi)\left(\fint_{\tau_1}^{\tau_2} \!\!\int_m \omega_{\text{univ}}\right) \in T := \mathbf{G}_m^\times \otimes H.$$

The long exact sequence in homology associated to the sequence (4.2) defining $\mathcal{M}$ gives a boundary map

$$\delta : H_2(\Gamma, \mathbf{Z}) \to H_1(\Gamma, \mathcal{M}). \tag{4.7}$$

Denote the composition of $\Phi_1 : H_1(\Gamma, \mathcal{M}) \to T(\mathbf{Q}_p)$ with (4.7) by

$$\Phi_2 : H_2(\Gamma, \mathbf{Z}) \to T(\mathbf{Q}_p).$$

Let L be the image of Φ_2.

We now state the main result of part I of the thesis. The torus T inherits a Hecke action from H.

Theorem 4.2.1. *Let K_p denote the quadratic unramified extension of $\mathbf{Q}_p$. The group L is a discrete, Hecke-stable subgroup of $T(\mathbf{Q}_p)$ of rank $2g$. The quotient T/L admits a Hecke-equivariant isogeny over K_p to the rigid analytic space associated to the product of two copies of $J_0(M)^{p\text{-}new}$.*

During the course of proving Theorem 4.2.1, we will give some control over the set of primes appearing in the degree of this isogeny. Also, we will see that if one lets the nontrivial element of $\mathrm{Gal}(K_p/\mathbf{Q}_p)$ act on T/L by the Hecke operator U_p (defined in Chapter 5), this isogeny is defined over $\mathbf{Q}_p$.

Remark 4.2.2. If we had not used the auxiliary projection $\psi : \tilde{H} \to H$ and continued our construction with integrals valued in $\mathbf{G}_m \otimes \tilde{H}$, the corresponding quotient $\tilde{T}/\tilde{L}$ would be isogenous to two copies of $J_0(M)^{p\text{-new}}$, with one copy of $\mathbf{G}_m$, arising from the Eisenstein quotient $\tilde{H}/H$. However, as the projections to this $\mathbf{G}_m$ of the Stark–Heegner points we will define later bear little arithmetic interest (see Chapter 8), we lose little in employing the projection ψ in exchange for the technical simplicity gained. The Eisenstein quotient $\tilde{H}/H$ has eigenvalue $+1$ for complex conjugation. In part II of the thesis, partial modular symbols are used to construct Eisenstein quotients where complex conjugation acts as -1, and the resulting projections of the Stark–Heegner points to $\mathbf{G}_m$ are analyzed.

Remark 4.2.3. The module H can be expressed up to finite index as a sum $H^+ \oplus H^-$, where the modules H^+ and H^- are the subgroups on which complex conjugation (denoted W_∞) acts as 1 or -1, respectively; these each have rank g over $\mathbf{Z}$. This decomposition of H explains the two components of T/L described in Theorem 4.2.1.

Remark 4.2.4. In Chapter 6.1, we will show how Theorem 4.2.1 is a generalization of the Mazur–Tate–Teitelbaum conjecture [35, Conjecture II.13.1] proven by Greenberg and Stevens [12], [13].

Granting theorem 4.2.1, we next describe the construction of Stark–Heegner points on $J_0(M)^{p\text{-new}}$.

4.3 Stark–Heegner points

Fix $\tau \in \mathcal{H}_p$ and $x \in \mathbf{P}^1(\mathbf{Q})$. Consider the 2-cocycle in $Z^2(\Gamma, T(\mathbf{C}_p))$ given by

$$d_{\tau,x}(\gamma_1, \gamma_2) := \fint_\tau^{\gamma_1\tau} \int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \omega_\psi := \fint_\tau^{\gamma_1\tau} \int_{[\gamma_1 x] - [\gamma_1\gamma_2 x]} \omega_\psi, \tag{4.8}$$

where here as always Γ acts trivially on T. It is an easy verification that the image d of $d_{\tau,x}$ in $H^2(\Gamma, T(\mathbf{C}_p))$ is independent of the choice of τ and x. Since $T(\mathbf{C}_p)$ is divisible and $H_1(\Gamma, \mathbf{Z})$ is finite (see Proposition 4.3.1 below), the universal coefficient theorem identifies d with a homomorphism

$$H_2(\Gamma, \mathbf{Z}) \to T(\mathbf{C}_p);$$

this homomorphism is precisely Φ_2. Thus L, which was defined to be the image of Φ_2, is the minimal subgroup of $T(\mathbf{C}_p)$ such that the image of d in $H^2(\Gamma, T(\mathbf{C}_p)/L)$ is trivial.

Thus there exists a map $\beta_{\tau,x} : \Gamma \to T/L$ such that

$$\beta_{\tau,x}(\gamma_1\gamma_2) - \beta_{\tau,x}(\gamma_1) - \beta_{\tau,x}(\gamma_2) = \fint_\tau^{\gamma_1\tau} \int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \omega_\psi \quad (\text{mod } L). \tag{4.9}$$

The 1-cochain $\beta_{\tau,x}$ is defined uniquely up to an element of $\mathrm{Hom}(\Gamma, T/L)$. The following proposition allows us to deal with this ambiguity.

Proposition 4.3.1. *The abelianization of Γ is finite, and every prime dividing its size divides $6\varphi(N)(p^2 - 1)$.*

Proof. This is a result of Ihara [23]; we provide a quick sketch. In Chapter 3 (see (3.14)) we described an exact sequence that identifies $H_1(\Gamma, \mathbf{Z})$ with the cokernel of the natural map

$$H_1(\Gamma_0(M), \mathbf{Z}) \to H_1(\Gamma_0(N), \mathbf{Z})^2.$$

Since $\Gamma_0(N)$ acts on the complex upper half plane $\mathcal{H}$ with isotropy groups supported at the primes 2 and 3, the group $H_1(\Gamma_0(N), \mathbf{Z})$ may be identified with the corresponding singular

homology of $Y_0(N)(\mathbf{C}) = \mathcal{H}/\Gamma_0(N)$ outside of a finite torsion group supported at 2 and 3. Hence we must show that

$$f_*^Y : H_1(Y_0(M), \mathbf{Z}) \to H_1(Y_0(N), \mathbf{Z})^2 \qquad (4.10)$$

has finite cokernel.

Poincaré duality identifies $H_1(Y_0(N), \mathbf{Z})$ with the $\mathbf{Z}$-dual of the relative homology group $H_1(X_0(N), \mathrm{cusps}, \mathbf{Z})$. We are thus led to reconsider the diagram (4.6) of Proposition 4.1.1. The injectivity of f_c^* implies that the cokernel of (4.10) is finite; furthermore, this cokernel is isomorphic to a subgroup of the cokernel of f_c^*. A result of of Ribet [37] implies that the torsion subgroup of the cokernel of f^* is supported on the set of primes dividing $\varphi(N)$. We saw in the proof of Proposition 4.1.1 that the torsion subgroup of the cokernel of $\mathbf{Z}^{2c-2} \to \mathbf{Z}^{2c-1}$ has exponent $p^2 - 1$. The snake lemma completes the proof. $\qquad\square$

We may now define Stark–Heegner points on $J_0(M)^{p\text{-new}}$. Define the ring

$$R := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}[1/p]) \text{ such that } N \text{ divides } c \right\}. \qquad (4.11)$$

Let K be a real quadratic field such that p is inert in K; choose an embedding σ of K into $\mathbf{R}$, and also an embedding of K into $\mathbf{C}_p$. For each $\tau \in \mathcal{H}_p \cap K$, consider the collection $\mathcal{O}_\tau$ of matrices $g \in R$ satisfying

$$g\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda_g \begin{pmatrix} \tau \\ 1 \end{pmatrix} \text{ for some } \lambda_g \in K. \qquad (4.12)$$

The ring $\mathcal{O}_\tau$ is isomorphic to a $\mathbf{Z}[1/p]$-order in K, via the map $g \mapsto \lambda_g = c\tau + d$. The group of units in $\mathcal{O}_\tau^\times$ of norm 1 is a free abelian group of rank 1. Let γ_τ be the generator such that $\sigma(\lambda_{\gamma_\tau}) > 1$ if $\sigma(\tau) > \sigma(\tau')$, and such that $\sigma(\lambda_{\gamma_\tau}) < 1$ if $\sigma(\tau) < \sigma(\tau')$. Here τ' denotes the conjugate of τ over $\mathbf{Q}$; the definition of γ_τ is independent of choice of σ. Finally, choose an $x \in \mathbf{P}^1(\mathbf{Q})$, and let t denote the exponent of the abelianization of Γ.

Definition 4.3.2. The Stark–Heegner point associated to τ is given by

$$\Phi(\tau) := t \cdot \beta_{\tau,x}(\gamma_\tau) \in T(K_p)/L.$$

Multiplication by t ensures that this definition is independent of choice of $\beta_{\tau,x}$ satisfying (4.9), and one also checks that $\Phi(\tau)$ is independent of x. Furthermore, the point $\Phi(\tau)$ depends only on the Γ-orbit of τ, so we obtain a map

$$\Phi : (\mathcal{H}_p \cap K)/\Gamma \to T(K_p)/L.$$

Let us now denote by $\nu_{\pm}$ the two maps $T/L \to J_0(M)^{p\text{-new}}$ of Theorem 4.2.1, where the $\pm$ sign denotes the corresponding eigenvalue of complex conjugation on H Composing Φ with the maps $\nu_{\pm}$, we obtain

$$\Phi_{\pm} : (\mathcal{H}_p \cap K)_{\Gamma} \to J_0(M)^{p\text{-new}}(K_p).$$

The images of $\Phi_{\pm}$ are the Stark–Heegner points on $J_0(M)^{p\text{-new}}$.

As in [5] and [6], we conjecture that the images of $\Phi_{\pm}$ satisfy explicit algebraicity properties. Fix a $\mathbf{Z}[1/p]$-order $\mathcal{O}$ in K; let us assume that the discriminant of $\mathcal{O}$ is prime to M. Let $K_+^{\times}$ denote the multiplicative group of elements of K of positive norm. Define the *narrow Picard group* $\operatorname{Pic}^+(\mathcal{O})$ to be the group of projective rank one $\mathcal{O}$-submodules of K modulo homothety by $K_+^{\times}$. Class field theory canonically identifies $\operatorname{Pic}^+(\mathcal{O})$ with the Galois group of an extension H^+ of K called the *narrow ring class field* of K attached to $\mathcal{O}$:

$$\operatorname{rec} : \operatorname{Pic}^+(\mathcal{O}) \longrightarrow \operatorname{Gal}(H^+/K).$$

Denote by $\mathcal{H}_p^{\mathcal{O}}$ the set of $\tau \in \mathcal{H}_p \cap K$ such that $\mathcal{O}_{\tau} \cong \mathcal{O}$. The basic fundamental conjecture regarding Stark–Heegner points is:

$$\text{if } \tau \in \mathcal{H}_p^{\mathcal{O}}, \text{ then } \Phi_{\pm}(\tau) \text{ lies in } J_0(M)^{p\text{-new}}(H^+).$$

We now proceed to refine this statement into a "Shimura reciprocity law" for Stark–Heegner points. A $\mathbf{Z}[1/p]$-lattice in K is a $\mathbf{Z}[1/p]$-submodule of K that is free of rank 2. Define

$$\Omega_N(K) = \left\{ (\Lambda_1, \Lambda_2), \text{ with } \begin{array}{l} \Lambda_j \text{ a } \mathbf{Z}[1/p]\text{-lattice in } K, \\ \Lambda_1/\Lambda_2 \simeq \mathbf{Z}/N\mathbf{Z}. \end{array} \right\} / K_+^{\times}.$$

There is a natural bijection $\underline{\tau}$ from $\Omega_N(K)$ to $(\mathcal{H}_p \cap K)/\Gamma$, which to $x = (\Lambda_1, \Lambda_2)$ assigns

$$\underline{\tau}(x) = \omega_1/\omega_2,$$

where $\langle \omega_1, \omega_2 \rangle$ is a $\mathbf{Z}[1/p]$-basis of Λ_1 satisfying

$$\omega_1 \omega_2' - \omega_1' \omega_2 > 0, \quad \operatorname{ord}_p(\omega_1 \omega_2' - \omega_1' \omega_2) \equiv 0 \pmod{2},$$

and $\Lambda_2 = \langle N\omega_1, \omega_2 \rangle$. Here we have written $\omega \mapsto \omega'$ for the action of the nontrivial auto-morphism of $\operatorname{Gal}(K/\mathbf{Q})$. Denote by $\Omega_N(\mathcal{O})$ the set of pairs $(\Lambda_1, \Lambda_2) \in \Omega_N(K)$ such that $\mathcal{O}$

is the largest $\mathbf{Z}[1/p]$-order of K preserving both Λ_1 and Λ_2. Note that $\underline{\tau}(\Omega_N(\mathcal{O})) = \mathcal{H}_p^{\mathcal{O}}/\Gamma$. The group $\mathrm{Pic}^+(\mathcal{O})$ acts naturally on $\Omega_N(\mathcal{O})$ by translation:

$$\mathfrak{a} : (\Lambda_1, \Lambda_2) \mapsto (\mathfrak{a}\Lambda_1, \mathfrak{a}\Lambda_2),$$

and hence it also acts on $\underline{\tau}(\Omega_N(\mathcal{O})) = \mathcal{H}_p^{\mathcal{O}}/\Gamma$. Denote this latter action by

$$(\mathfrak{a}, \tau) \mapsto \mathfrak{a} \star \tau, \qquad \text{for } \mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O}), \quad \tau \in \mathcal{H}_p^{\mathcal{O}}/\Gamma.$$

Our conjectural reciprocity law then states:

Conjecture 4.3.3. *If $\tau \in \mathcal{H}_p^{\mathcal{O}}$, then $\Phi_\pm(\tau) \in J_0(M)^{p\text{-}new}(H^+)$, and*

$$\Phi_\pm(\mathfrak{a} \star \tau) = \mathrm{rec}(\mathfrak{a})^{-1} \Phi_\pm(\tau)$$

for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$.

Remark 4.3.4. Since H^+ is a ring class field, the complex conjugation associated to either real place of K is the same in $\mathrm{Gal}(H^+/K)$. Let $\mathfrak{a}_\infty$ denote an element of $\mathrm{Pic}^+(\mathcal{O})$ corresponding to this complex conjugation. Then for either choice of sign $\epsilon = \pm$, we have

$$\Phi_\epsilon(\mathfrak{a}_\infty \star \tau) = \epsilon \Phi_\epsilon(\tau).$$

The proof of this fact is identical to Proposition 5.13 of [5], since the map ν_ϵ factors through a torus on which the Hecke operator W_∞ (see Definition 5.1.5) acts as ϵ.

The general conjecture that Stark–Heegner points are defined over global fields, and certainly the full Conjecture 4.3.3 are very much open. However, theoretical evidence is provided in [2] and by the analogous constructions of part II. Computational evidence is provided in [7]. Theorem 4.2.1 is proven over the course of the next two chapters. We start by analyzing the Hecke structure of the torus T.

Chapter 5

Hecke actions

The homology groups we have already considered are endowed with natural Hecke actions, which we now describe.

5.1 Definitions

Let $\Delta_{\mathbf{Q}} = \mathbf{PGL}_2(\mathbf{Q})$, and let Δ denote one of the groups Γ, $\Gamma_0(N)$, or $\Gamma_0(M)$, considered as a subgroup of $\Delta_{\mathbf{Q}}$. For $\alpha \in \mathbf{PGL}_2(\mathbf{Q})$, define

$$\Delta\alpha\Delta = \bigsqcup_i \alpha_i\Delta, \quad \alpha_i \in \Delta_{\mathbf{Q}} \tag{5.1}$$

be a decomposition of the indicated double coset into left cosets.

Definition 5.1.1. Let A be a $\Delta_{\mathbf{Q}}$-module, and let $\alpha \in \Delta_{\mathbf{Q}}$. Define the Hecke operator $T(\alpha)$ on the group of Δ-co-invariants of A as follows. Let $m \in A$ represent the element $\widetilde{m} \in H_0(\Delta, A)$, and let

$$T(\alpha)\widetilde{m} := \sum_i \widetilde{\alpha_i^{-1}m} \in H_0(\Delta, A).$$

This definition is clearly independent of the choice of α_i. Also, for each $\gamma \in \Delta$ and each α_i, there exist unique j and $\gamma_i \in \Delta$ such that

$$\gamma^{-1}\alpha_i = \alpha_j\gamma_i^{-1}. \tag{5.2}$$

For δ fixed, the correspondence $i \mapsto j$ is a permutation. This implies that the definition of $T(\alpha)$ is independent of choice of representative m for $\tilde{m}$.

We define the Hecke operators on $H_1(\Delta, A)$ for $i > 0$ by dimension-shifting. Tensoring the exact sequence

$$0 \to \mathrm{Div}_0\,\Delta_{\mathbf{Q}} \to \mathrm{Div}\,\Delta_{\mathbf{Q}} \to \mathbf{Z} \to 0$$

with A and taking Δ-homology yields

$$H_1(\Delta, A) \;\cong\; \ker(H_0(\Delta, (\mathrm{Div}_0\,\Delta_{\mathbf{Q}}) \otimes A) \to H_0(\Delta, (\mathrm{Div}\,\Delta_{\mathbf{Q}}) \otimes A)), \qquad (5.3)$$

$$H_i(\Delta, A) \;\cong\; H_{i-1}(\Delta, (\mathrm{Div}_0\,\Delta_{\mathbf{Q}}) \otimes A) \text{ for } i > 1 \qquad (5.4)$$

since the induced modules $(\mathrm{Div}\,\Delta_{\mathbf{Q}}) \otimes A$ have trivial homology $H_i(\Delta, (\mathrm{Div}\,\Delta_{\mathbf{Q}}) \otimes A)$ for $i \geq 1$. The kernel in (5.3) is stable under $T(\alpha)$ for $\alpha \in \Delta_{\mathbf{Q}}$. Thus the isomorphisms (5.3) and (5.4) allow one to inductively define the Hecke operators $T(\alpha)$ on $H_i(\Delta, A)$ for any $\Delta_{\mathbf{Q}}$-module A and $i \geq 0$, with Definition 5.1.1 as the base case.

Remark 5.1.2. In (3.7) we described $H_1(\Delta, A)$ (with $\Delta = \Gamma$ and $A = \mathcal{M}$) by dimension-shifting with $\mathrm{Div}\,\Delta$ rather than $\mathrm{Div}\,\Delta_{\mathbf{Q}}$. We obtained

$$H_1(\Delta, A) \cong \ker(H_0(\Delta, (\mathrm{Div}_0\,\Delta) \otimes A) \to A). \qquad (5.5)$$

The purpose of using $\mathrm{Div}\,\Delta_{\mathbf{Q}}$ rather than $\mathrm{Div}\,\Delta$ in the present context is that the former group has the structure of a $\Delta_{\mathbf{Q}}$-module, whereas the latter does not. However, the Hecke operator $T(\alpha)$ on $H_1(\Delta, A)$ may be described in terms of the isomorphism (5.5) as follows. Let

$$k = \sum([\gamma] - [\delta]) \otimes m$$

represent an element of $H_1(\Delta, A)$ as in (5.5). Then

$$T(\alpha)k = \sum \sum_i ([\gamma_i] - [\delta_i]) \otimes \alpha_i^{-1} m,$$

where the α_i and γ_i are as in (5.1) and (5.2), and the δ_i are obtained from δ as in (5.2).

Lemma 5.1.3. *Given a short exact sequence of $\Delta_{\mathbf{Q}}$-modules*

$$0 \to A \to B \to C \to 0,$$

the associated long exact sequence in Δ-homology is equivariant for the Hecke operators.

Proof. This is immediate except for possibly the boundary map $\delta : H_i(\Delta, C) \rightarrow H_{i-1}(\Delta, A)$. By dimension-shifting, it suffices to consider the case $i = 1$. In this case, the map δ has the following explicit description. Let

$$d = \sum ([\gamma] - [\delta]) \otimes c \in H_1(\Delta, C)$$

as in (5.5). Lift each c to an element b of B. The element $\sum \gamma^{-1} b - \delta^{-1} b \in B$ maps to 0 in C and hence comes from an element of A; the image of this element in $H_0(\Delta, A)$ is independent of all choices and is the value $\delta(d)$. We find that

$$T(\alpha)(\delta(d)) = \sum \sum_i \alpha_i^{-1} \gamma^{-1} b - \alpha_i^{-1} \delta^{-1} b \tag{5.6}$$

while

$$\delta(T(\alpha)(d)) = \sum \sum_i \gamma_i^{-1} \alpha_i^{-1} b - \delta_i^{-1} \alpha_i^{-1} b. \tag{5.7}$$

In view of (5.2) and the comments following, these values are equal in $H_0(\Delta, A)$. $\qquad\square$

When $\Delta = \Gamma_0(M)$ or Γ and ℓ is prime, we write T_ℓ or U_ℓ for $T \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}$, according to whether ℓ divides M or not. Recall from (4.3) we have

$$\tilde{H} = \left(H_0(\Gamma_0(M), \mathcal{M}) / \operatorname{Tr}(H_0(\Gamma_0(N), \mathcal{M}) \oplus H_0(P^{-1}\Gamma_0(N)P, \mathcal{M})) \right)_T.$$

Define also

$$\tilde{H}' = \ker \partial : H_0(\Gamma_0(M), \mathcal{M}) \rightarrow H_0(\Gamma_0(N), \mathcal{M}) \oplus H_0(P^{-1}\Gamma_0(N)P, \mathcal{M})$$

with the boundary map ∂ as in Section 3.2. We wish to define Hecke operators on $\tilde{H}$ and $\tilde{H}'$.

Proposition 5.1.4. *The maps ∂ and Tr are equivariant with respect to the Hecke operators T_ℓ for $\ell \nmid M$ and U_ℓ for $\ell | N$.*

Proof. The map $m \mapsto Pm$ defines an isomorphism

$$H_0(P^{-1}\Gamma_0(N)P, \mathcal{M}) \cong H_0(\Gamma_0(N), \mathcal{M}). \tag{5.8}$$

One easily checks that this isomorphism is equivariant for all the operators T_ℓ.

Letting j run over the elements of $\mathbf{P}^1(\mathbf{F}_p)$, we may write

$$\Gamma_0(N) = \bigsqcup_j \gamma_j \Gamma_0(M), \quad P^{-1}\Gamma_0(N)P = \bigsqcup_j P^{-1}\beta_j P \Gamma_0(M)$$

where $\gamma_j = \begin{pmatrix} a_j & b_j \\ Nc_j & d_j \end{pmatrix}$ (resp. β_j) are elements of $\Gamma_0(N)$ such that $j = [a_j : c_j N] \in \mathbf{P}^1(\mathbf{F}_p)$ (resp. $j = [b_j : d_j]$). With the identification of (5.8), the maps ∂ and Tr become:

$$\partial : H_0(\Gamma_0(M), \mathcal{M}) \quad \to \quad H_0(\Gamma_0(N), \mathcal{M}) \oplus H_0(\Gamma_0(N), \mathcal{M}) \tag{5.9}$$

$$\tilde{m} \quad \mapsto \quad (\tilde{m}, -\widetilde{Pm}) \tag{5.10}$$

and

$$\mathrm{Tr} : H_0(\Gamma_0(N), \mathcal{M}) \oplus H_0(\Gamma_0(N), \mathcal{M}) \quad \to \quad H_0(\Gamma_0(M), \mathcal{M}) \tag{5.11}$$

$$(\tilde{m}, \tilde{n}) \quad \mapsto \quad \sum \widetilde{\gamma_j^{-1} m} - \sum \widetilde{P^{-1} \beta_j^{-1} n}. \tag{5.12}$$

Write $\partial = \partial_1 \oplus \partial_2$ and $\mathrm{Tr} = \mathrm{Tr}_1 \oplus \mathrm{Tr}_2$. It is clear that ∂_1 is T_ℓ-equivariant. As for ∂_2, one checks that for each α_i, there exists a unique α_j such that $\alpha_i P^{-1} \in P^{-1} \alpha_j \Gamma_0(M)$, and the correspondence $i \mapsto j$ is a permutation. This implies that ∂_2 is T_ℓ-equivariant.

For Tr_1 one similarly checks that for each α_i and γ_j, there exists a unique γ_k and α_h such that $\alpha_i \gamma_j \in \gamma_k \alpha_h \Gamma_0(M)$, and the map $(i, j) \mapsto (k, h)$ is a permutation. The same is true for the γ_j replaced by $\beta_j P$. $\qquad \square$

The proposition shows that the Hecke operators T_ℓ for $\ell \nmid M$ and U_ℓ for $\ell | N$ on $H_0(\Gamma_0(M), \mathcal{M})$ induce actions on $\tilde{H}$ and on $\tilde{H}'$. The situation for Hecke operators at p is more subtle. Let $\mathcal{N}$ denote the normalizer of $\Gamma_0(M)$ in

$$R^\times / U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PGL}_2(\mathbf{Z}[1/p]) \text{ such that } N \text{ divides } c \right\}, \tag{5.13}$$

where $U = \mathbf{Z}[1/p]^\times$ is embedded in $R^\times$ via scalar matrices. The determinant on $R^\times / U$ maps to U/U^2, which is a Klein 4-group. When restricted to $\mathcal{N}$, the determinant map induces an isomorphism

$$\mathcal{N} / \Gamma_0(M) \cong U/U^2 \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}.$$

Let α_p denote a matrix in $\mathcal{N}$ that maps to the image of p under the determinant map, and let α_∞ be a matrix that maps to the image of -1. To be explicit, we may take

$$\alpha_p = \begin{pmatrix} p & y \\ M & px \end{pmatrix}, \quad \alpha_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.14}$$

where $px - Ny = 1$.

Definition 5.1.5. Let A be a $\mathbf{PGL}_2(\mathbf{Q})$-module. The Atkin–Lehner involution at p acting on the homology groups of $\Gamma_0(M)$ is given by $W_p := T(\alpha_p)$. The Atkin–Lehner involution at infinity is defined by $W_\infty := T(\alpha_\infty)$.

Lemma 5.1.6. *The operator $U_p + W_p$ on $H_0(\Gamma_0(M), \mathcal{M})$ has image contained in the subgroup $\mathrm{Tr}_2(H_0(\Gamma_0(N), \mathcal{M}))$.*

Proof. The image of $\tilde{m}$ under $U_p + W_p$ is

$$(U_p + W_p)\tilde{m} = \sum_{u=0}^{p-1} \widetilde{\alpha_u^{-1} m} + \widetilde{\alpha_p^{-1} m},$$

where $\alpha_u = \begin{pmatrix} p & u \\ 0 & 1 \end{pmatrix}$ and α_p is as in (5.14). Now one checks that for each $\beta_i, i \in \mathbf{P}^1(\mathbf{F}_p)$, there exists a unique $j = 0, \ldots, p$ such that $\beta_i P \in \alpha_j \Gamma_0(M)$, and the map $i \mapsto j$ is a bijection. This implies that $(U_p + W_p)\tilde{m} = \mathrm{Tr}_2(\tilde{m})$. $\square$

Lemma 5.1.7. *The operator W_p interchanges the images of Tr_1 and Tr_2 in $H_0(\Gamma_0(M), \mathcal{M})$.*

Proof. For each $i \in \mathbf{P}^1(\mathbf{F}_p)$, there is a unique j such that $\beta_i P \in \gamma_j \alpha_p \Gamma_0(M)$, and the map $i \mapsto j$ is a permutation. $\square$

Lemmas (5.1.6) and (5.1.7) imply that the image of Tr is preserved under both U_p and W_p. Hence these operators have well defined induced actions on $\tilde{H}$, and their sum is the zero operator on $\tilde{H}$. A similar argument gives the same result for $\tilde{H}'$:

Lemma 5.1.8. *The operator $U_p + W_p$ is zero on the subspace $\ker \partial_1$ of $H_0(\Gamma_0(M), \mathcal{M})$. The involution W_p interchanges $\ker \partial_1$ and $\ker \partial_2$. Hence U_p and W_p have well defined induced actions on $\tilde{H}' = \ker \partial$, and $U_p + W_p = 0$ on $\tilde{H}'$.*

Definition 5.1.9. Write W for the involutions $U_p = -W_p$ on $\tilde{H}$ and $\tilde{H}'$. For $\Delta = \Gamma$, (5.1) implies that the operator $U_p = W_p$ is an involution; we write W for this involution as well.

5.2   Hecke equivariance of the integration map

In Section 4.2 we defined an integration map:

$$\fint\!\!\int \omega_{\mathrm{univ}} : ((\mathrm{Div}_0 \, \mathcal{H}_p) \otimes M)_\Gamma \to \tilde{T} = \mathbf{G}_m \otimes \tilde{H}. \tag{5.15}$$

In the previous section we defined a Hecke action on $((\mathrm{Div}_0\,\mathcal{H}_p) \otimes M)_\Gamma$ and also on $\tilde{T}$. For this section, we will write μ for μ^{univ}.

Proposition 5.2.1. *The integration map* (5.15) *is equivariant for T_ℓ, U_ℓ for $\ell \neq p$, and W_∞.*

Proof. The first observation is that since $\ell \neq p$, we may take the same set of α_i in defining the Hecke operators for Γ and $\tilde{H}$. Also, we have $\alpha_i e^* = e^*$ for the distinguished edge e^*. We now show that this implies

$$\sum_{i=0}^{\ell} \mu_{\alpha_i^{-1}m}(\alpha_i^{-1}U) = T_\ell(\mu_m(U)). \tag{5.16}$$

For $\gamma \in \Gamma$, write $\gamma^{-1}\alpha_i = \alpha_{\gamma(i)}\gamma_i^{-1}$ for some index $\gamma(i)$ and $\gamma_i \in \Gamma$. Then

$$\begin{aligned}
\sum_{i=0}^{\ell} \mu_{\alpha_i^{-1}m}(\alpha_i^{-1}U_{\gamma e^*}) &= \sum_{i=0}^{\ell} \mu_{\alpha_i^{-1}m}(\gamma_i U_e^*) \\
&= \sum_{i=0}^{\ell} \gamma_i^{-1}\alpha_i^{-1}m \\
&= \sum_{j=0}^{\ell} \alpha_j^{-1}\gamma^{-1}m
\end{aligned}$$

This proves equation (5.16).

We now calculate, for $k = ([\tau_2] - [\tau_1]) \otimes m \in ((\mathrm{Div}_0\,\mathcal{H}_p) \otimes M)_\Gamma$,

$$\begin{aligned}
\left(\fint\!\!\int \omega_\mu\right)(T_\ell k) &= \prod_i \fint_{\alpha_i^{-1}\tau_1}^{\alpha_i^{-1}\tau_2}\!\int_m \omega_\mu \tag{5.17}\\
&= \prod_i \lim_{||\mathcal{U}||\to 0} \prod_{U\in\mathcal{U}} \left(\frac{t_U - \alpha_i^{-1}\tau_2}{t_U - \alpha_i^{-1}\tau_1}\right) \otimes \mu_{\alpha_i^{-1}m}(U) \\
&= \prod_i \lim_{||\mathcal{U}||\to 0} \prod_{U\in\mathcal{U}} \left(\frac{\alpha_i^{-1}t_U - \alpha_i^{-1}\tau_2}{\alpha_i^{-1}t_U - \alpha_i^{-1}\tau_1}\right) \otimes \mu_{\alpha_i^{-1}m}(\alpha_i^{-1}U) \\
&= \prod_i \lim_{||\mathcal{U}||\to 0} \prod_{U\in\mathcal{U}} \left(\frac{t_U - \tau_2}{t_U - \tau_1}\right) \otimes \mu_{\alpha_i^{-1}m}(\alpha_i^{-1}U) \tag{5.18}\\
&= \lim_{||\mathcal{U}||\to 0} \prod_{U\in\mathcal{U}} \left(\frac{t_U - \tau_2}{t_U - \tau_1}\right) \otimes T_\ell(\mu_m(U)). \tag{5.19}
\end{aligned}$$

Equation (5.18) uses the fact that $\mu_m(\mathbf{P}^1(\mathbf{Q}_p)) = 0$, and (5.19) uses (5.16). The right-hand side of (5.19) is

$$T_\ell\left(\left(\fint\!\!\int \omega_\mu\right)(k)\right),$$

as desired. $\square$

Proposition 5.2.2. *The integration map* (5.15) *is W-equivariant.*

Proof. The key to this proposition is that the matrix α_p defining W_p on H interchanges the vertices v^* and w^*, and hence sends the edge e^* to its opposite. The action of W on the Γ-coinvariants of a $\mathbf{PGL}_2(\mathbf{Q})$-module is given by the action of a matrix in R of determinant p. Thus for each $\gamma \in \Gamma$, if we let $\beta = \alpha_p^{-1}\gamma\alpha_p \in \Gamma$, we find that

$$
\begin{aligned}
\mu_{\alpha_p^{-1}m}(\alpha_p^{-1}U_{\gamma e^*}) &= \mu_{\alpha_p^{-1}m}(\alpha_p^{-1}\gamma\mathbf{Z}_p) = \mu_{\alpha_p^{-1}m}(\beta\alpha_p^{-1}\mathbf{Z}_p) \\
&= -\mu_{\alpha_p^{-1}m}(\beta\mathbf{Z}_p) = -\beta^{-1}\alpha_p^{-1}m \\
&= -\alpha_p^{-1}\gamma^{-1}m = W(\mu_m(U_{\gamma e^*})).
\end{aligned}
$$

The proof of the proposition now follows along the proof of Proposition 5.2.1. $\square$

5.3 The homology of $\mathcal{M}$

In this section we study the Γ-module $\mathcal{M}$. Two fractions x and y in $\mathbf{P}^1(\mathbf{Q})$ are said to be *adjacent* if they can be written as $x = a/b$ and $y = c/d$ in $\mathbf{P}^1(\mathbf{Q})$ with a, b, c, d integral and $bc - ad = 1$. (This is a symmetric relation since we can write $y = c/d$ and $x = (-a)/(-b)$.)

Lemma 5.3.1. *For adjacent x and y in $\mathbf{P}^1(\mathbf{Q})$, let $D_{x,y} = [x] - [y] \in \mathcal{M}$. The $D_{x,y}$ generate $\mathcal{M}$, and the following relations generate all those satisfied by the $D_{x,y}$:*

$$
D_{x,y} + D_{y,x} = 0 \tag{5.20}
$$

$$
D_{\frac{a}{b}, \frac{a+c}{b+d}} + D_{\frac{a+c}{b+d}, \frac{c}{d}} + D_{\frac{c}{d}, \frac{a}{b}} = 0. \tag{5.21}
$$

Proof. The fact that the $D_{x,y}$ generate $\mathcal{M}$ is standard: between any two fractions $x_0 < y_0$ we may take a Farey sequence with large enough denominator as a sequence of adjacent fractions from x_0 to y_0. Thus the divisors of the form $[x_0] - [y_0]$, and hence all degree-zero divisors, are generated by the $D_{x,y}$.

Consider a relation among the $D_{x,y}$. Using relation (5.20), we may assume that all the coefficients in the relation are positive. It is clear that such a relation can be written as a sum of relations of the form

$$
r : \quad \sum_{i=0}^{n-1} D_{x_i, x_{i+1}} = 0
$$

where the indices x_i are taken modulo n. If $n = 2$ then r is evidently of the form (5.20);
assume that $n > 2$. We claim that if $x_0 = a/b$ and $x_{n-1} = c/d$ with $bc - ad = 1$, then either
$(a + c)/(b + d)$ or $(a - c)/(b - d)$ appears among the x_i. By applying the transformation
$x \mapsto (bx - a)/(-dx + c)$, which sends adjacent fractions to adjacent fractions, we may assume
that $x_0 = 0$ and $x_{n-1} = \infty$. Now 0 is adjacent to fractions of the form $1/d$, which lie in the
interval $[-1, 1]$, and ∞ is adjacent to the integers. Thus it suffices to show that adjacent
fractions in $\mathbf{Q}$ cannot contain an integer between them; for then 1 or -1 appears among
the x_i, which is equivalent to $(a + c)/(b + d)$ or $(a - c)/(b - d)$ appearing among the original
x_i. Hence suppose we have $a/b < m < c/d$ with $bc - ad = 1$ and m an integer. Multiplying
by bd we obtain

$$ad < bdm < bc \quad \text{or} \quad ad > bdm > bc.$$

Both possibilities contradict $bc - ad = 1$.

Thus we have shown that there is some x_j equal to $(a + c)/(b + d)$ or $(a - c)/(b - d)$.
We can then write our original relation as:

$$r : \quad \sum_{i=0}^{j-1} \left(D_{x_i, x_{i+1}} + D_{x_j, x_0} \right) + \left(D_{x_0, x_j} + D_{x_j, x_{n-1}} + D_{x_{n-1}, x_0} \right)$$

$$+ \sum_{i=j}^{n-2} \left(D_{x_i, x_{i+1}} + D_{x_{n-1}, x_j} \right) - \left(D_{x_j, x_0} + D_{x_0, x_j} \right) - \left(D_{x_j, x_{n-1}} + D_{x_{n-1}, x_j} \right). \quad (5.22)$$

The second term in (5.22) is of the form (5.20), while the last two are of the form (5.21).
The other two terms are relations that are shorter in length than r; continuing to decompose
the relations in this fashion gives the desired result. $\qquad\square$

In fact, the relations of Lemma 5.3.1 are independent. Write $\Theta := \mathbf{PSL}_2(\mathbf{Z})$, and
let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ be the standard generators of Θ with order 2 and 3,
respectively. Let $S \backslash \Theta$ and $T \backslash \Theta$ denote the sets of right cosets of the subgroups generated
by S and T. Define maps

$$\mathbf{Z}[S \backslash \Theta] \oplus \mathbf{Z}[T \backslash \Theta] \xrightarrow{\ s\ } \mathbf{Z}[\Theta] \xrightarrow{\ \epsilon\ } D,$$

where s is summation:

$$s(\{\theta, S\theta\}) := [\theta] + [S\theta] \quad \text{and} \quad s(\{\theta, T\theta, T^2\theta\}) := [\theta] + [T\theta] + [T^2\theta],$$

and ϵ is defined by $\epsilon(\theta) := \theta^{-1}(0) - \theta^{-1}(\infty)$.

Proposition 5.3.2. *The relations of Lemma 5.3.1 are independent. That is, the sequence of right Θ-modules*

$$0 \longrightarrow \mathbf{Z}[S\backslash\Theta] \oplus \mathbf{Z}[T\backslash\Theta] \xrightarrow{\ s\ } \mathbf{Z}[\Theta] \xrightarrow{\ \epsilon\ } \mathcal{M} \longrightarrow 0$$

is exact.

Proof. Lemma 5.3.1 implies exactness except for injectivity on the left. To prove the proposition, construct the undirected Cayley graph for Θ with respect to the subset consisting of $\{S, T, T^2\}$. This is the graph whose vertices are the elements of Θ and whose edges connect elements in the same right coset $S\backslash\Theta$ or $T\backslash\Theta$. Now suppose we have a dependence among the relations. Since each simple divisor $[\theta]$ appears in exactly two relations, namely one with the other member of its S-coset and one with the other two members of its T-coset, the coefficients of these two relations in our dependence must be negatives of each other. Label each vertex θ with the coefficient a_θ of its S-relation. By definition, the vertex corresponding to $S\theta$ has the same label: $a_{S\theta} = a_\theta$. Since the T-relation of θ has coefficient $-a_\theta$, the vertex corresponding to $T\theta$ must also be labelled a_θ. Continuing in this fashion, we see that the entire connected component of θ must have the same label. But S and T generate Θ, so the graph is connected; as Θ is infinite, we must have $a_\theta = 0$ for all θ. This proves the proposition. $\square$

Proposition 5.3.3. *The group $H_1(\Theta, \mathcal{M})$ is isomorphic to $\mathbf{Z}$. A generator for $H_1(\Theta, \mathcal{M})$ is given by $q_\Theta = ([ST] - [T^2] + [STS] - [T^2S]) \otimes ([0] - [\infty])$.*

Proof. Taking the Θ-homology of the short exact sequence of Proposition 5.3.2 yields

$$0 \longrightarrow H_1(\Theta, \mathcal{M}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\ s\ } \mathbf{Z} \longrightarrow 0.$$

This shows immediately that $H_1(\Theta, \mathcal{M}) \cong \mathbf{Z}$. Furthermore, the map s sends $(1,0)$ to 2 and $(0,1)$ to 3, so the kernel is generated by $(3, -2)$; in other words, $H_1(\Theta, \mathcal{M})$ is generated by each element of the form

$$\left(\sum_{i=1}^{3} ([\theta_i] + [S\theta_i]) - \sum_{i=1}^{2} ([\theta_i'] + [T\theta_i'] + [T^2\theta_i']) \right) \otimes ([0] - [\infty]).$$

for any θ_i and θ_i' in Θ. The expression for q_Θ given in the proposition involves choosing these matrices to obtain as much cancellation as possible. $\square$

Lemma 5.3.4. *The group $H_1(\Gamma_0(N), \mathcal{M})$ is isomorphic to $\mathbf{Z}$, and its image in $H_1(\Theta, \mathcal{M})$ corresponding to the inclusion $\Gamma_0(N) \subset \Theta$ is $c_N \cdot \delta_N \cdot H_1(\Theta, \mathcal{M})$, where*

$$c_N = \#\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z}) = N \prod_{\ell \mid N} \left(1 + \frac{1}{\ell}\right)$$

and

$$\delta_N = \left(\begin{cases} 1 & \textit{if } -1 \textit{ is a square in } \mathbf{Z}/N\mathbf{Z} \\ 1/2 & \textit{otherwise} \end{cases} \right) \cdot \left(\begin{cases} 1 & \textit{if } -3 \textit{ is a square in } \mathbf{Z}/N\mathbf{Z} \\ 1/3 & \textit{otherwise} \end{cases} \right).$$

Proof. The coset space $\Theta/\Gamma_0(N)$ is identified with $\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow [a : c].$$

Taking the homology sequence for $\Gamma_0(N)$ corresponding to the short exact sequence in Proposition 5.3.2 we obtain

$$0 \longrightarrow H_1(\Gamma_0(N), \mathcal{M}) \qquad\qquad (5.23)$$

$$\longrightarrow \mathbf{Z}[S\backslash\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})] \oplus \mathbf{Z}[T\backslash\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})] \xrightarrow{s} \mathbf{Z}[\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})] \longrightarrow H_0(\Gamma_0(N), \mathcal{M})$$

$$\longrightarrow 0.$$

Here $S\backslash\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$ represents the quotient of $\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$ by the left action of S; the elements of $S\backslash\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$ should be thought of as multisets (in particular, if $i \in \mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$ such that $Si = i$ then the image of $[i]$ under s is $2[i] \in \mathbf{Z}[\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})]$).

Consider an element r in the kernel of s. The graph theory argument given in the proof of Proposition 5.3.2 applies here again to show that the coefficients of all the T-orbits in r must be equal to a constant a (except for T-orbits that involve a fixed point of T, whose coefficients are $a/3$) and the coefficients of the S-orbits in r must equal $-a$ (except for S-fixed points, which have coefficient $-a/2$). Unlike the case of Proposition 5.3.2, this does not imply that $a = 0$ because here our graph is finite. Indeed, this shows that $H_1(\Gamma_0(N), \mathcal{M})$ is isomorphic to $\mathbf{Z}$ and gives its image in $H_1(\Theta, \mathcal{M})$. For example, if neither -1 nor -3 are squares in $\mathbf{Z}/N\mathbf{Z}$, then there are no fixed points for the action of S and T on $\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$. Then a generator for $H_1(\Gamma_0(N), \mathcal{M})$ is the sum of all $c_N/3$ of the T-orbits minus the sum of all $c_N/2$ of the S-orbits; hence the image of $H_1(\Gamma_0(N), \mathcal{M})$ in $H_1(\Theta, \mathcal{M})$ is $c_N \cdot H_1(\Theta, \mathcal{M})$. The other cases are similar. $\square$

Lemma 5.3.5. *The operator W_p acts as the identity on $H_1(\Gamma_0(p), \mathcal{M})$. The operator W_∞ acts as -1 on $H_1(\Gamma_0(p), \mathcal{M})$ and $H_1(\Theta, \mathcal{M})$.*

Proof. Taking the long exact sequence in homology associated to the sequence defining $\mathcal{M}$, we obtain

$$H_2(\Gamma_0(p), \mathbf{Z}) \longrightarrow H_1(\Gamma_0(p), \mathcal{M}) \xrightarrow{\ \nu\ } H_1(\Gamma_0(p), \operatorname{Div} \mathbf{P}^1(\mathbf{Q})) \xrightarrow{\ \iota\ } H_1(\Gamma_0(p), \mathbf{Z}).$$

$$(5.24)$$

Up to torsion, we can identify the ends of this sequence with the corresponding Betti homology groups of the open curve $Y_0(p)$:

$$H_*(\Gamma_0(p), \mathbf{Z})_T \cong H_*(Y_0(p)(\mathbf{C}), \mathbf{Z}).$$

The manifold $Y_0(p)(\mathbf{C})$ is a genus$(X_0(p))$-holed torus with two points removed.

There are two $\Gamma_0(p)$-equivalence classes of cusps, namely those of 0 and ∞. These two cusps have infinite cyclic stabilizers U_0 and U_∞, generated by

$$u_0 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \text{ and } u_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The $\Gamma_0(p)$-module $\operatorname{Div} \mathbf{P}^1(\mathbf{Q})$ is the direct sum of two induced trivial modules, one from U_0 and one from U_∞. Shapiro's lemma then yields

$$H_1(\Gamma_0(p), \operatorname{Div} \mathbf{P}^1(\mathbf{Q})) = H_1(U_0, \mathbf{Z}) \oplus H_1(U_\infty, \mathbf{Z}) = U_0 \oplus U_\infty \cong \mathbf{Z} \oplus \mathbf{Z}.$$

In terms of the topological interpretation, the map ι in (5.24) is the inclusion of simple loops around the two cusps (i.e., the removed points in the torus) in $H_1(Y_0(p)(\mathbf{C}), \mathbf{Z})$. As the difference of these two loops is clearly a boundary, they are equivalent in H_1. Thus the kernel of ι is generated by (u_0, u_∞^{-1}). [1]

The action of W_p is now easy to calculate. Unwinding the identifications in Shapiro's Lemma, we find that the element (u_0, u_∞^{-1}) corresponds to the element

$$([u_0] - [\operatorname{Id}]) \otimes [0] - ([u_\infty] - [\operatorname{Id}]) \otimes [\infty] \in H_1(\Gamma_0(p), \operatorname{Div} \mathbf{P}^1(\mathbf{Q})).$$

---

[1] One might object that the statement about the action of W_p on $H_1(\Gamma_0(p), \mathbf{Z})$ is clearly correct up to a sign because it is an involution, and we have not been careful in checking orientations and signs; but to show that our signs are correct in describing the kernel of ι, it suffices to check that there is a group homomorphism $\Gamma_0(p) \to \mathbf{Z}$ that takes the same non-zero value on u_0 and u_∞. Indeed such a homomorphism exists; Mazur [32, §II.2] calls it the *Dedekind–Rademacher homomorphism*.

Conjugation by α_p takes u_0 to u_∞^{-1} and u_∞ to u_0^{-1}. Hence on $U_0 \oplus U_\infty$, the involution W_p acts as $(a,b) \mapsto (-b,-a)$. Thus the action of W_p fixes elements of the form $(x,-x)$. This proves that W_p acts as the identity on $H_1(\Gamma_0(p),\mathcal{M})$. The proof that W_∞ acts as -1 on $H_1(\Gamma_0(p),\mathcal{M})$ and $H_1(\Theta,\mathcal{M})$ is much simpler and will be omitted. $\qquad \square$

Consider now the exact sequence (3.14) in the case where $N = 1$. We obtain

$$H_1(\Gamma_0(p),\mathcal{M}) \to H_1(\Theta,\mathcal{M})^2 \to H_1(\Gamma,\mathcal{M}).$$

Lemmas 5.3.4 and 5.3.5 imply that the first map sends a generator of $H_1(\Gamma_0(p),\mathcal{M})$ to

$$(-(p+1) \cdot \delta_p \cdot q_\Theta, (p+1) \cdot \delta_p \cdot q_\Theta).$$

In particular, the map $H_1(\Theta,\mathcal{M}) \to H_1(\Gamma,\mathcal{M})$ induced by the inclusion $\Theta \subset \Gamma$ is an injection.

Lemma 5.3.6. *The Hecke module $H_1(\Gamma_0(N),\mathcal{M})$ is Eisenstein; that is, T_ℓ acts as multiplication by $\ell + 1$ for $\ell \nmid N$.*

Proof. Since $H_1(\Gamma_0(N),\mathcal{M})$ injects into $H_1(\Theta,\mathcal{M})$ in a T_ℓ-equivariant manner for $\ell \nmid N$, it suffices to consider the case $N = 1$. Let α_i be as in (5.1) giving the action of T_ℓ. Write $q_\Theta = B \otimes ([0] - [\infty])$ with

$$B = [ST] - [T^2] + [STS] - [T^2 S],$$

and let

$$B_i = [(ST)_i] + [(STS)_i] - [(T^2)_i] - [(T^2 S)_i],$$

where the matrices with index i are as in (5.2). Since $H_1(\Theta,\mathcal{M}) \cong \mathbf{Z}$, we may write $T_\ell q_\Theta = a_\ell q_\Theta$ for some integer a_ℓ. Let $\Gamma_\ell = \mathbf{SL}_2(\mathbf{Z}[1/\ell])$, and denote by q_ℓ the image of q_Θ in $H_1(\Gamma_\ell,\mathcal{M})$. We have

$$
\begin{aligned}
a_\ell q_\ell \;&=\; \text{the image of } T_\ell q_\Theta \text{ in } H_1(\Gamma_\ell,\mathcal{M}) \\
&=\; \sum_{i=0}^{\ell} B_i \otimes \left([\alpha_i^{-1} 0] - [\alpha_i^{-1} \infty]\right) \\
&=\; \sum_{i=0}^{\ell} \alpha_\ell^{-1} \alpha_i B_i \otimes \left([\alpha_\ell^{-1} 0] - [\alpha_\ell^{-1} \infty]\right) \qquad\qquad (5.25)
\end{aligned}
$$

44

where $\alpha_\ell \in \Gamma_\ell$ is a fixed matrix of determinant ℓ, since then $\alpha_\ell^{-1}\alpha_i \in \Gamma_\ell$. Since α_ℓ gives the action of W_ℓ on $H_1(\Gamma_\ell, \mathcal{M})$, (5.25) equals

$$
\begin{aligned}
W_\ell \sum_{i=0}^{\ell} \alpha_i B_i \alpha_\ell^{-1} \otimes ([0] - [\infty]) &= W_\ell \sum_{i=0}^{\ell} B \alpha_i \alpha_\ell^{-1} \otimes ([0] - [\infty]) \\
&= W_\ell \sum_{i=0}^{\ell} B \otimes ([0] - [\infty]) \quad\quad (5.26) \\
&= (\ell+1) W_\ell(q_\ell) = (\ell+1) q_\ell. \quad\quad (5.27)
\end{aligned}
$$

Here (5.27) uses Lemmas 5.3.4 and 5.3.5, along with the W_ℓ-equivariance of

$$ H_1(\Gamma_0(\ell), \mathcal{M}) \to H_1(\Gamma_\ell, \mathcal{M}), $$

which is immediate from the definitions. Equation (5.26) results from Remark 3.2.2. Since $H_1(\Theta, \mathcal{M})$ injects into $H_1(\Gamma_\ell, \mathcal{M})$, we see that $a_\ell = \ell + 1$ as desired. $\square$

5.4 The lattice L

From (3.14) we have the exact sequence

$$ H_1(\Gamma_0(N), \mathcal{M})^2 \to H_1(\Gamma, \mathcal{M}) \to H_0(\Gamma_0(M), \mathcal{M}) \to H_0(\Gamma_0(N), \mathcal{M})^2. \quad (5.28) $$

The map $H_1(\Gamma, \mathcal{M}) \to H_0(\Gamma_0(M), \mathcal{M})$ is the map ϕ in (3.12). Hence (5.28) combined with Proposition 3.3.1 implies that the image of $H_1(\Gamma_0(N), \mathcal{M})^2$ in $H_1(\Gamma, \mathcal{M})$ under the integration map Φ_1 has trivial p-adic valuation. Thus, for the image of the integration map to be discrete in T, it must be the case that the image of $H_1(\Gamma_0(N), \mathcal{M})^2$ in T is finite. We will prove this by exploiting Hecke actions.

Define a modified Eisenstein ideal $\mathcal{I}$ of the (abstract) Hecke algebra by letting $\mathcal{I}$ be generated by $T_\ell - (\ell+1)$ for $\ell \nmid M$, $(p+1)W - (p+1)$, and $W_\infty + 1$.

Lemma 5.4.1. *The ideal $\mathcal{I}$ annihilates the image of $H_1(\Gamma_0(N), \mathcal{M})^2$ under the integration map Φ_1.*

Proof. The map $H_1(\Gamma_0(N), \mathcal{M})^2 \to H_1(\Gamma, \mathcal{M})$ is Hecke-equivariant for T_ℓ, $\ell \nmid M$, and W_∞ by basic computation; from Lemma 5.1.3 and Section 5.2 it follows that Φ_1 is Hecke-equivariant. Also, the action of T_p on $H_1(\Gamma_0(N), \mathcal{M})$ is given by $p+1$ matrices α_i of determinant p, hence the action of T_p on $H_1(\Gamma_0(N), \mathcal{M})$ becomes that of $(p+1)W$ on its image in $H_1(\Gamma, \mathcal{M})$. The result now follows from Lemmas 5.3.5 and 5.3.6. $\square$

Let $H^* = \mathrm{Hom}(H, \mathbf{Z})$ denote the dual of H, so $T(\mathbf{Q}_p) = \mathrm{Hom}(H^*, \mathbf{Q}_p^\times)$. It is a standard fact that $H^*/\mathcal{I}H^*$ is finite: after tensoring with $\mathbf{C}$, one obtains the space of holomorphic Eisenstein series that are cusp forms (of which there are none).[2] Thus Lemma 5.4.1 implies that the image of $H_1(\Gamma_0(N), \mathcal{M})^2$ under the integration map Φ_1 is finite.

Proposition 5.4.2. *The group of periods $L \subset T(\mathbf{Q}_p)$ is a Hecke-stable lattice of rank $2g$.*

Proof. Lemma 5.1.3 implies the Hecke-equivariance of the boundary maps

$$H_2(\Gamma, \mathbf{Z}) \to H_1(\Gamma, \mathcal{M}) \quad \text{and} \quad H_1(\Gamma, \mathcal{M}) \to H_0(\Gamma, (\mathrm{Div}_0\, \mathcal{H}_p) \otimes \mathcal{M}).$$

Hence the propositions of Section 5.2 imply that Φ_2 is Hecke-equivariant, and thus that L is Hecke-stable.

From (5.28), the kernel of the map

$$H_1(\Gamma, \mathcal{M}) \to \tilde{H}' = \ker \partial \cong \ker f_*^c$$

is the image of $H_1(\Gamma_0(N), \mathcal{M})^2$, which has finite image under the integration map. Thus it remains to show that the image of $H_2(\Gamma, \mathbf{Z})$ in $\tilde{H}'$ injects into H and has rank $2g$. The group $H_2(\Gamma, \mathbf{Z})$ may be understood using the sequence (3.3) again:

$$H_2(\Gamma_0(N), \mathbf{Z})^2 \to H_2(\Gamma, \mathbf{Z}) \to H_1(\Gamma_0(M), \mathbf{Z}) \to H_1(\Gamma_0(N), \mathbf{Z})^2. \qquad (5.29)$$

As in the proof of Proposition 4.3.1, the homology of $\Gamma_0(N)$ may be identified with that of $Y_0(N)$ outside of 2 and 3-torsion. Since $Y_0(N)$ is a non-compact Riemann surface, the group $H_2(Y_0(N), \mathbf{Z})$ vanishes. And the right-hand arrow of (5.29) may be identified with

$$f_*^Y : H_1(Y_0(M), \mathbf{Z}) \to H_1(Y_0(N), \mathbf{Z})^2.$$

Thus the image of $H_2(\Gamma, \mathbf{Z})$ in $\tilde{H}' = \ker f_*^c$ is precisely $H' := \ker f_*$. We need to show that H' injects into $H = \mathrm{coker}\, f^*$ and has finite cokernel. Yet the endomorphism $f_* \circ f^*$ of $H_1(X_0(N), \mathbf{Z})^2$ can be given explicitly by the matrix

$$f_* \circ f^* = \begin{pmatrix} p+1 & T_p \\ T_p & p+1 \end{pmatrix}.$$

Since the eigenvalues of T_p are bounded by $2\sqrt{p}$, this endomorphism is injective and has finite cokernel; the result follows. $\qquad\square$

---

[2] In the case where $N = 1$, Mazur [31] has conducted a detailed analysis of the group $H^*/\mathcal{I}H^*$. When $\mathcal{I}$ includes the element $W - 1$ rather than just $(p+1)(W-1)$ as in our setting, Mazur finds that $H^*/\mathcal{I}H^*$ is cyclic of size n, where n is the numerator of the fraction $(p-1)/12$.

Remark 5.4.3. The finite group $H/\ker f^*$ reflects congruences between modular forms of level N and M. In [3], the images of Stark–Heegner points in this finite Hecke module under the p-adic valuation map is explored.

Chapter 6

Proof of the uniformization

Let $\mathscr{T}$ denote the Hecke algebra of H (that is, the subring of the ring of endo-morphisms of the group H generated over $\mathbf{Z}$ by T_ℓ for $\ell \nmid M$, U_ℓ for $\ell | N$, and W). In Proposition 3.3.1 we gave a combinatorial description of the map $\mathrm{ord}_p : L \to H \otimes \mathbf{Z}_p$:

$$L \subset T(\mathbf{Q}_p) \xrightarrow{\ \mathrm{ord}_p\ } H \otimes \mathbf{Z} \longrightarrow H \otimes \mathbf{Z}_p.$$

Consider now the logarithm $\log_p : L \to H \otimes \mathbf{Z}_p$ given by

$$L \subset T(\mathbf{Q}_p) \xrightarrow{\ \log_p\ } H \otimes \mathbf{Z}_p,$$

where now and throughout this thesis we choose the branch of the logarithm for which $\log_p(p) = 0$. We will define an element $\mathscr{L}_p \in \mathscr{T} \otimes \mathbf{Z}_p$, called the $\mathscr{L}$-invariant of T/L, such that

$$\mathscr{L}_p \, \mathrm{ord}_p \, \lambda = \log_p \lambda \quad \text{for all } \lambda \in L. \tag{6.1}$$

Remark 6.0.4. Let $H_\pm = H/(W_\infty \mp 1)$ be the maximal quotients of H on which complex conjugation acts as a scalar ± 1. The fact that there is an element $\mathscr{L}_p$ in $\mathscr{T} \otimes \mathbf{Q}_p$ satisfying (6.1) for each of the factors $H_- \otimes \mathbf{Q}_p$ and $H_+ \otimes \mathbf{Q}_p$ follows from the fact that each of these modules is free of rank one over $\mathscr{T} \otimes \mathbf{Q}_p$. The fact that the same $\mathscr{L}_p$ works on each factor, and that this element is integral, follows from our specific construction and proof.

Our goal is to connect T/L with the abelian variety $J = J_0(M)^{p\text{-new}}$. This abelian variety has purely toric reduction at p, and its p-adic uniformization can be described as follows. Let S denote the set of supersingular points in characteristic p on $X_0(N)$, and let $X := \mathrm{Div}_0 \, S$ denote the group of degree-zero divisors on S. The group X has a natural

Hecke action: by T_ℓ for $\ell \nmid M$ (and U_ℓ for $\ell | M$, including $\ell = p$) by sending a supersingular point on $X_0(N)$ to the formal sum of the $\ell + 1$ (resp. ℓ) ℓ-isogenous supersingular points, counted with multiplicity; the operator $W = U_p$ has order two and is also given by the action of $\mathrm{Gal}(\mathbf{F}_{p^2}/\mathbf{F}_p)$ on the supersingular points. It is well known that the Hecke algebra of X equals that of H; in other words, there is a ring homomorphism $\mathscr{T} \to \mathrm{End}(X)$ sending $T_\ell \mapsto T_\ell$, etc.[1] Thus we may consider X as a module for $\mathscr{T}$. Let $G_p = \mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ act on X (via the $\mathrm{Gal}(\mathbf{F}_{p^2}/\mathbf{F}_p)$-action on S) and on $\mathrm{Hom}(X, \bar{\mathbf{Q}}_p^\times)$ by $\sigma(h)(x) = \sigma(h(\sigma^{-1}(x)))$. There is a rigid analytic $\mathscr{T}[G_p]$-equivariant isomorphism

$$J(\bar{\mathbf{Q}}_p) \cong \mathrm{Hom}(X, \bar{\mathbf{Q}}_p^\times)/X,$$

where the inclusion

$$Q : X \to \mathrm{Hom}(X, \mathbf{Q}_p^\times) = X^* \otimes \mathbf{Q}_p^\times$$

is given by a symmetric pairing

$$X \times X \to \mathbf{Q}_p^\times. \tag{6.2}$$

(Here we have written $X^* = \mathrm{Hom}(X, \mathbf{Z})$.) The Hecke operators are self-adjoint for this pairing. Composing (6.2) with the p-adic valuation gives a $\mathbf{Z}$-valued pairing (the "monodromy pairing") on X which is non-degenerate, and hence yields an injection

$$\mathrm{ord}_p Q : X \to X^*.$$

Similarly, composing Q with $\log_p$ yields

$$\log_p Q : X \to X^* \otimes \mathbf{Z}_p.$$

We will then show, with $\mathscr{L}_p$ as in (6.1), the relation

$$\mathscr{L}_p \mathrm{ord}_p Q(x) = \log_p Q(x) \text{ for all } x \in X. \tag{6.3}$$

Equations (6.1) and (6.3) imply theorem 4.2.1 as follows. For a set of primes $\mathcal{P}$, we say that two analytic spaces are $\mathcal{P}$-isogenous if there is an isogeny between them whose degree is supported on the elements of $\mathcal{P}$. Let $\pi_\pm : H \to H_\pm$ denote the natural projections.

---

[1]This is because after tensoring with $\mathbf{C}$, the spectrum of T_ℓ on X consists of the ℓ-th Fourier coefficients of a basis of p-new forms of level M; the spectrum of T_ℓ on H consists of each these eigenvalues repeated twice.

Since Φ_2 is equivariant for W_∞, and $H \to H_- \oplus H_+$ has cokernel supported at 2, it follows that T/L is $\{2\}$-isogenous to

$$(T/L)_- \oplus (T/L)_+ := (\mathbf{G}_m \otimes H_-)/\pi_- L \oplus (\mathbf{G}_m \otimes H_+)/\pi_+ L.$$

We will show that each of $(T/L)_\pm$ is isogenous to J.

Recall the Hecke-equivariant map $\psi : \tilde{H} \to H$ from Section 4.2 used to define our modular symbol valued in H, and denote by $\psi_- : \tilde{H}_- \to H_-$ the induced map obtained by modding out by $W_\infty + 1$. Recall that $H' = \ker f_*$ and let H'_- be its corresponding quotient.

Since all of the groups below are free of rank 1 over $\mathscr{T} \otimes \mathbf{Q}$ after tensoring with $\mathbf{Q}$, it is possible to find Hecke-equivariant maps ξ_- and ξ'_- fitting into a commutative diagram

$$
\begin{array}{ccc}
H'_- \longrightarrow \tilde{H}_- \xrightarrow{\psi_-} H_- & & (6.4) \\
\quad\downarrow{\scriptstyle \xi'_-} \qquad\qquad\qquad\quad \downarrow{\scriptstyle \xi_-} & & \\
X \xrightarrow{\quad \mathrm{ord}_p Q \quad} X^*, & &
\end{array}
$$

where the horizontal arrow $H'_- \to \tilde{H}_-$ is the natural inclusion. Recall that

1. The map $\Phi_2 : H_2(\Gamma, \mathbf{Z}) \to T$ factors through $\Phi_1 : H_1(\Gamma, \mathcal{M}) \to T$.

2. The composite

$$H_1(\Gamma_0(N), \mathcal{M})^2 \longrightarrow H_1(\Gamma, \mathcal{M}) \xrightarrow{\ \Phi_1\ } T$$

   has finite image.

3. The image of $H_2(\Gamma, \mathbf{Z})$ in $H_1(\Gamma, \mathcal{M})/H_1(\Gamma_0(N), \mathcal{M})^2$ is canonically identified with H' (see the proof of Proposition 5.4.2).

The identification of (3) implies that ξ'_- induces a map $H_2(\Gamma, \mathbf{Z})_- \to X$, also denoted ξ'_-. Consider the diagram

$$
\begin{array}{ccc}
H_2(\Gamma, \mathbf{Z})_- \xrightarrow{\ \pi_- \Phi_2\ } \mathbf{Q}_p^\times \otimes H_- & & (6.5) \\
\quad\downarrow{\scriptstyle \xi'_-} \qquad\qquad\qquad\quad \downarrow{\scriptstyle \xi_-} & & \\
X \xrightarrow{\quad Q \quad} \mathbf{Q}_p^\times \otimes X^*. & &
\end{array}
$$

Proposition 3.3.1 implies that the composition of the top row of (6.5) with ord_p is equal to the composition of $H_2(\Gamma, \mathbf{Z})_- \to H'_-$ with the top row of (6.4). Thus the commutativity of (6.4) implies the commutativity of the p-adic valuation of (6.5). Since all the maps in (6.5) are Hecke-equivariant, equations (6.1) and (6.3) show that the commutativity of the ord_p

of (6.5) *automatically implies* the commutativity of the $\log_p$ of (6.5). Thus the diagram (6.5) itself commutes, up to elements in the kernel of both $\log_p$ and ord_p; these elements are torsion of order dividing $p - 1$ (or 2 if $p = 2$).

Hence the map

$$\mathrm{Id} \otimes \xi_- : \mathbf{G}_m \otimes H_- \to \mathbf{G}_m \otimes X^*$$

induces an isogeny $(T/\Lambda)_- \to J$. Furthermore, the kernel of this isogeny is identified with the cokernel of

$$\xi'_- : H'_- \to X.$$

A similar argument for H_+ then proves Theorem 4.2.1 and furthermore bounds the primes dividing the degree of the isogeny to lie in

$$\mathcal{P} = \{\ell : \ell \text{ divides } 2(p-1) \text{ or the size of either } \mathrm{coker}\, \xi'_\pm : H'_\pm \to X\}.$$

6.1 Connection with the Mazur–Tate–Teitelbaum conjecture

Let E be an elliptic curve with conductor Np and split multiplicative reduction at p. In this chapter, we show that Theorem 4.2.1 implies the Mazur–Tate–Teitelbaum conjecture for E.

Let $\mathcal{I}_E^+$ denote the ideal of the Hecke algebra of $\tilde{H}$ corresponding to E and the "plus" modular symbol (that is, the ideal generated by $T_\ell - a_\ell$ for $\ell \nmid Np$, $W_p - 1$, and $W_\infty - 1$, where $a_\ell = \ell + 1 - \#E(\mathbf{F}_\ell)$). The quotient $\tilde{H}/\mathcal{I}_E^+$ has rank 1 over $\mathbf{Z}$; the projection

$$\Psi_E^+ : \tilde{H} \to H_E^+ := (\tilde{H}/\mathcal{I}_E^+)_T \cong \mathbf{Z}$$

is the *plus modular symbol* attached to E. We retain the notation of [35] and write

$$\lambda_E(a, M) := \Psi_E^+ \left(\left[-\frac{a}{M} \right] - [\infty] \right).$$

After tensoring with $\mathbf{G}_m$, the projection Ψ_E^+ yields a map $\varphi : \tilde{T} \to \mathbf{G}_m$. According to Theorem 4.2.1, the quotient $\mathbf{G}_m/\varphi(L)$ is an analytic space isogenous to the elliptic curve E. This implies that every element of $\varphi(L)$ is commensurable with the Tate period q_E of E. By evaluating a particular element of $\varphi(L)$, we will deduce the MTT conjecture. In what follows, we denote

$$\varphi \left(\oint_{\tau_1}^{\tau_2} \int_x^y \omega_{\mathrm{univ}} \right)$$

by

$$\fint_{\tau_1}^{\tau_2} \int_x^y \omega_E^+ \in \mathbf{C}_p^\times.$$

Consider the class $c \in H_1(\Gamma, \mathcal{M})$ represented by the element

$$\left(\left[\begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix} \right] - [\mathrm{Id}] \right) \otimes ([0] - [\infty]).$$

The element c is in the image of the boundary map from $H_2(\Gamma, \mathbf{Z})$, since a simple calculation shows that the image of c in the next term of the exact sequence

$$H_2(\Gamma, \mathbf{Z}) \to H_1(\Gamma, \mathcal{M}) \to H_1(\Gamma, \mathrm{Div}\, \mathbf{P}^1(\mathbf{Q})) = \Gamma_\infty^{\mathrm{ab}} \tag{6.6}$$

vanishes (this is true for any class represented by $([\gamma] - [\mathrm{Id}]) \otimes ([x] - [y])$, for γ stabilizing x and y). Here Γ_∞ denotes the stabilizer of ∞ in Γ, and the equality of (6.6) follows from Shapiro's Lemma. Thus the double integral

$$q' := \fint_\tau^{p^2\tau} \int_0^\infty \omega_E^+ \in \mathbf{Q}_p^\times$$

lies in $\varphi(L)$, and in particular does not depend on the choice of τ. In fact, since $W_p = W_\infty = 1$ on H_E^+, it follows (see [5, Proposition 5.13]) that the multiplicative double integral is invariant under the full group $R^\times/U$ (defined in (5.13)). In particular, one finds that

$$q := \fint_{\frac{\tau}{p}}^\tau \int_0^\infty \omega_E^+ \in \mathbf{Q}_p^\times$$

is independent of τ and that $q' = q^2$. Thus q is commensurable with q_E as well, so

$$\frac{\log_p(q)}{\mathrm{ord}_p(q)} = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}. \tag{6.7}$$

To evaluate q, choose τ to reduce to the standard vertex v^* of the tree $\mathcal{T}$. Since the matrix P^{-1} sends v^* to w^*, Lemma 3.3.2 implies that

$$\mathrm{ord}_p(q) = \Psi_E^+([0] - [\infty]) = \lambda_E(0, 1). \tag{6.8}$$

Let $n \geq 1$; for $a = 0, \ldots, p^n - 1$ define $U_a := a + p^n \mathbf{Z}_p$. To evaluate $\log_p q \in \mathbf{Z}_p$ modulo p^n, it suffices to take a cover of $\mathbf{P}^1(\mathbf{Q})$ by the sets

$$U_\infty := \{t \in \mathbf{P}^1(\mathbf{Q}_p) : \mathrm{ord}_p(t) < -n\}, \tag{6.9}$$

$$\frac{1}{p^n} U_a, \frac{1}{p^{n-1}} U_a, \ldots, \frac{1}{p} U_a \text{ for } a \not\equiv 0 \pmod{p}, \tag{6.10}$$

and U_a for all $a = 0, \ldots, p^n - 1$. The contributions to the integral defining $\log_p q$ from each of these terms are as follows:

$$
\begin{aligned}
U_\infty : \quad & 0, \\
\frac{1}{p^k} U_a : \quad & \log_p \left(\frac{a/p^k - \tau}{a/p^k - \tau/p} \right) \lambda_E(a, p^n) = \log_p \left(\frac{a - p^k \tau}{a - p^{k-1} \tau} \right) \lambda_E(a, p^n), \qquad (6.11) \\
U_a : \quad & \log_p \left(\frac{a - \tau}{a - \tau/p} \right) \lambda_E(a, p^n) = \log_p \left(\frac{a - \tau}{pa - \tau} \right) \lambda_E(a, p^n). \qquad (6.12)
\end{aligned}
$$

Summing these values (the terms of (6.11) for $k = 1, \ldots, n$ telescope, and the distribution relation

$$
\lambda_E(a', p^{n-1}) = \sum_{\substack{a \bmod p^n \\ a \equiv a' \pmod{p^{n-1}}}} \lambda_E(a, p^n)
$$

allows one to cancel terms in the denominator of (6.12) for all a with the terms in the numerator for $p | a$) one obtains

$$
\log_p(q) \equiv \sum_{\substack{a=1 \\ (a,p)=1}}^{p^n - 1} \log_p(a) \lambda_E(a, p^n) \pmod{p^n}. \qquad (6.13)
$$

Equations (6.7), (6.8), and (6.13) yield Conjecture II.13.1 of [35] for the trivial character. The more general statement for a character of conductor $c > 1$ may be obtained by repeating our analysis above for each $(v, c) = 1$ with the element

$$
([\gamma] - [\text{Id}]) \otimes ([v/c] - [\infty]) \in H_1(\Gamma, \mathcal{M}),
$$

with $\gamma \in \Gamma$ stabilizing v/c and ∞. We omit the details (see [5, §2.3]).

6.2 Hida families and the definition of $\mathscr{L}_p$

In this chapter we define the element $\mathscr{L}_p \in \mathscr{T} \otimes \mathbf{Z}_p$ that we later show satisfies (6.1) and (6.3). Let $X_0 = X_0(Np)$, and define a tower of curves X_r above X_0 corresponding to the congruence subgroups $\Gamma_r := \Gamma_0(N) \cap \Gamma_1(p^r)$, for $r \geq 1$:

$$
X_r(\mathbf{C}) = \mathcal{H}^* / \Gamma_r.
$$

The points of X_r classify triples (E, C, P), where E is an elliptic curve, C a cyclic subgroup of size N, and P a point of order p^r. The natural maps $X_{r+1} \to X_r$ send $(E, C, P) \mapsto (E, C, pP)$ for $r \geq 1$, and the map $X_1 \to X_0$ sends $(E, C, P) \mapsto (E, \langle C, P \rangle)$. Composing $X_r \to X_0$ with

the two maps $X_0 \to X_0(N)$ from (4.4), we obtain two degeneracy maps $X_r \to X_0(N)$ for each $r \geq 0$ (there are actually $r + 1$ natural degeneracy maps for each $r \geq 1$, but we will be interested in only these two). Let f_r^* denote the pullback on homology from the two copies of $X_0(N)$ to X_r, and define

$$H_r = \left[H_1(X_r, \mathbf{Z}_p)/f_r^*(H_1(X_0(N), \mathbf{Z}_p)^2) \right]_T,$$

so that $H_0 = H \otimes \mathbf{Z}_p$. Let $\mathscr{T}_r$ denote the Hecke algebra of H_r, generated over $\mathbf{Z}_p$ by T_ℓ for $\ell \nmid M$, U_ℓ for $\ell | M$, and the diamond operators $\langle d \rangle$ for $d \in (\mathbf{Z}/p^r \mathbf{Z})^\times$. Here $\mathscr{T}_0 = \mathscr{T} \otimes \mathbf{Z}_p$. Define the Hida Hecke algebra

$$\mathbf{T} := \varprojlim \mathscr{T}_r,$$

which has the structure of $\Lambda := \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$-module, where the group elements of $\mathbf{Z}_p^\times$ act via the diamond operators. Let I denote the augmentation ideal of Λ. If $\mathbf{T}^o$ denotes the image of $\mathbf{T}$ under Hida's ordinary projector, Hida has shown:

Theorem 6.2.1 (Hida, [20], Corollary 3.2). *The Λ-module $\mathbf{T}^o$ is free of finite rank and*

$$\mathbf{T}^o/I\mathbf{T}^o \cong \mathscr{T}_0.$$

Remark 6.2.2. The map $\mathbf{T}^o \to \mathscr{T}_0$ of Theorem 6.2.1 is the natural projection. It is clear that $I\mathbf{T}^o$ lies in the kernel; Hida's "control theorem" 6.2.1 is that $I\mathbf{T}^o$ is the entire kernel.

The standard identification $\langle d \rangle - 1 \mapsto d$ yields an isomorphism

$$I/I^2 \cong \varprojlim \mathbf{Z}_p^\times/(\mathbf{Z}_p^\times)^{p^n},$$

since the group $\mathbf{Z}_p^\times$ is abelian. Composing with $\log_p : \mathbf{Z}_p^\times \to \mathbf{Z}_p$, we obtain a map also denoted $\log_p : I/I^2 \to \mathbf{Z}_p$.

Let t be an element of $\mathbf{T}$ whose image in $\mathscr{T}_0$ vanishes. By Hida's Theorem 6.2.1, t^o lies in $I\mathbf{T}^o$ (where t^o is the image of t under the ordinary projector). Consider the image of t^o in

$$I\mathbf{T}^o/I^2\mathbf{T}^o = I/I^2 \otimes_\Lambda \mathbf{T}^o.$$

Using the map

$$\log_p \otimes \mathrm{Id} : I/I^2 \otimes_\Lambda \mathbf{T}^o \to \mathbf{Z}_p \otimes_\Lambda \mathbf{T}^o$$

we further map our element t to

$$\mathbf{Z}_p \otimes_\Lambda \mathbf{T}^o = \Lambda/I \otimes_\Lambda \mathbf{T}^o = \mathbf{T}^o/I\mathbf{T}^o = \mathscr{T}_0.$$

The image of t under this series of maps is denoted $t' \in \mathscr{T}_0$, to reflect the intuition that it represents the derivative of t in the direction of the level (i.e., the fact that $t^o \in I\mathbf{T}^o$ means that the "value" of the "function" t is 0 at the base of the tower, so its image in $I\mathbf{T}^o/I^2\mathbf{T}^o$ is its "derivative").

Since $U_p = -W_p$ on H, and W_p is an involution, the element $1 - U_p^2$ vanishes in $\mathscr{T}$. We define:

$$\mathscr{L}_p := (1 - U_p^2)' \in \mathscr{T}_0.$$

6.3 Proof that $\mathscr{L}_p$ is the $\mathscr{L}$-invariant of T/L

Let $\tau \in \mathcal{H}_p$ lie in the quadratic unramified extension K_p of $\mathbf{Q}_p$, and assume further that τ reduces to the central vertex v^* of the tree $\mathcal{T}$. Consider the map

$$\beta_{\mathscr{L}_p} : K_p^\times \otimes H \to K_p \otimes H, \quad k \otimes h \mapsto \log_p(k) \otimes h - \mathscr{L}_p(\mathrm{ord}_p(k) \otimes h).$$

Composing the 2-cocycle $d_{\tau,x} \in Z^2(\Gamma, T(K_p))$ from Section 4.3 with $\beta_{\mathscr{L}_p}$ yields

$$d_{\tau,x}^{\mathscr{L}_p} \in Z^2(\Gamma, K_p \otimes H), \quad d_{\tau,x}^{\mathscr{L}_p}(\gamma_1, \gamma_2) := \beta_{\mathscr{L}_p}\left(\fint_\tau^{\gamma_1 \tau} \int_{\gamma_1 x}^{\gamma_1 \gamma_2 x} \omega_\psi \right).$$

As in Section 4.3, the lattice $\beta_{\mathscr{L}_p}(L)$ is the smallest subgroup of $K_p \otimes H$ such that the cocycle $d_{\tau,x}^{\mathscr{L}_p}$ splits in the quotient; thus to prove equation (6.1), it suffices to prove that $d_{\tau,x}^{\mathscr{L}_p}$ splits.

We will in fact show a stronger result. Define a 1-cocycle

$$c_\tau \in Z^1(\Gamma, \mathrm{Hom}(\mathcal{M}, T(K_p)))$$

by the rule

$$c_\tau(\gamma)(m) := \fint_\tau^{\gamma\tau} \int_m \omega_\psi.$$

Composing c_τ with $\beta_{\mathscr{L}_p}$, we obtain a 1-cocycle

$$c_\tau^{\mathscr{L}_p} \in Z^1(\Gamma, \mathrm{Hom}(\mathcal{M}, K_p \otimes H)).$$

It is a basic calculation[2] that the splitting of $c_\tau^{\mathscr{L}_p}$ implies the splitting of $d_{\tau,x}^{\mathscr{L}_p}$; the splitting of $c_\tau^{\mathscr{L}_p}$ is in fact what we will show.

---

[2]If the 0-chain ρ_τ splits $c_\tau^{\mathscr{L}_p}$, then the 1-chain $\eta_{\tau,x}$ defined by $\eta_{\tau,x}(\gamma) = \rho_\tau([x] - [\gamma x])$ splits $d_{\tau,x}^{\mathscr{L}_p}$.

The main idea for splitting the cocycle $c_{\tau,x}^{\mathscr{L}_p}$ is to lift the modular symbol μ of measures on $\mathbf{P}^1(\mathbf{Q}_p)$ to a modular symbol of measures $\tilde{\mu}$ on a $\mathbf{Z}_p^\times$-bundle $\mathbf{X}$ over $\mathbf{P}^1(\mathbf{Q}_p)$. The space $\mathbf{X} := (\mathbf{Z}_p \times \mathbf{Z}_p)'$ is defined to be the set of pairs $(a,b) \in \mathbf{Z}_p \times \mathbf{Z}_p$ such that a and b are not both divisible by p; this set of "primitive vectors" makes an appearance in the earlier work of Greenberg and Stevens [13]. The space $\mathbf{X}$ admits a map

$$\pi : \mathbf{X} \to \mathbf{P}^1(\mathbf{Q}_p), \quad (a,b) \mapsto a/b.$$

The fibers of π are principal homogeneous spaces for $\mathbf{Z}_p^\times$. If we consider the elements of $\mathbf{X}$ as column vectors and let $\mathbf{GL}_2(\mathbf{Z}_p)$ act on the left, the map π is $\mathbf{GL}_2(\mathbf{Z}_p)$-equivariant. In this chapter, we will consider the groups $\Gamma_0(N), \Gamma_r, \Gamma$, etc. as subgroups of $\mathbf{GL}_2$ (rather than $\mathbf{PGL}_2$ as in previous chapters).

Remark 6.3.1. If the function $f(t) = t-\tau$ were integrable on $\mathbf{P}^1(\mathbf{Q}_p)$, a formal calculation would show that

$$\rho_\tau(m) = \int_{\mathbf{P}^1(\mathbf{Q}_p)} \log_p(t-\tau)\, \mathrm{d}\mu_m(t)$$

is an explicit splitting of the cocycle c_τ, i.e. that $d\rho_\tau = c_\tau$. However, this is not the case since $f(t)$ has a pole at $t = \infty$. This explains the role of the space $\mathbf{X}$: the function $f(a,b) = a-b\tau$ is integrable on $\mathbf{X}$, has a zero along the fiber over τ, and no poles.

Proposition 6.3.2. *There exists a $\Gamma_0(N)$-invariant* $\mathrm{Meas}(\mathbf{X}, H \otimes \mathbf{Z}_p)$*-valued modular symbol $\tilde{\mu}$ such that*

$$\tilde{\mu}_m(\mathbf{Z}_p^\times \times p\mathbf{Z}_p) = \mu_m(\mathbf{P}^1(\mathbf{Q}_p) - \mathbf{Z}_p) = -\psi(m) \tag{6.14}$$

for all $m \in \mathcal{M}$.

(Recall that the map ψ was defined in Section 4.2 to create a modular symbol valued in H rather than $\tilde{H}$.)

Remark 6.3.3. Since the $\Gamma_0(N)$-translates of $\mathbf{P}^1(\mathbf{Q}_p) - \mathbf{Z}_p$ form a disjoint open cover of $\mathbf{P}^1(\mathbf{Q}_p)$, the $\Gamma_0(N)$-invariance of $\tilde{\mu}$ combined with (6.14) implies that $\tilde{\mu}$ has the same total measure as μ, namely 0:

$$\tilde{\mu}_m(\mathbf{X}) = 0 \text{ for all } m \in \mathcal{M}.$$

Proof. Our methods follow those of Greenberg and Stevens [13, pg. 203]. Let A denote a free $\mathbf{Z}_p$-module of finite rank, viewed as a trivial $\Gamma_0(N)$-module. A $\Gamma_0(N)$-invariant $\mathrm{Meas}(\mathbf{X}, A)$-valued modular symbol is an element of

$$\mathbf{M}(A) := H^0(\Gamma_0(N), \mathrm{Hom}(\mathcal{M}, \mathrm{Meas}(\mathbf{X}, A))). \tag{6.15}$$

For each $r \geq 1$, let $\Gamma_r = \Gamma_0(N) \cap \Gamma_1(p^r)$ as in Section 6.2. The $\Gamma_0(N)$-module $\mathrm{Meas}(\mathbf{X}, A)$ is isomorphic to an inverse limit of induced modules from the groups Γ_r as follows. Let $\mathbf{X}_r := (\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z})'$, the set of primitive vectors in $(\mathbf{Z}/p^r\mathbf{Z})^2$. Then we have

$$\mathrm{Meas}(\mathbf{X}, A) = \varprojlim \mathrm{Meas}(\mathbf{X}_r, A),$$

where the maps $\mathrm{Meas}(\mathbf{X}_{r+1}, A) \to \mathrm{Meas}(\mathbf{X}_r, A)$ are given by $\mu_{r+1} \mapsto \mu_r$, where

$$\mu_r(x) = \sum_{y \equiv x \ (\mathrm{mod} \ p^r)} \mu_{r+1}(y).$$

The map $\Gamma_0(N)/\Gamma_r \to \mathbf{X}_r$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}$$

is a bijection and hence induces an isomorphism

$$\mathrm{Meas}(\mathbf{X}_r, A) \cong \mathrm{Ind}_{\Gamma_r}^{\Gamma_0(N)}(A).$$

Thus by Shapiro's Lemma and the universal coefficient theorem, (6.15) can be identified with

$$\varprojlim \mathrm{Hom}(H_0(\Gamma_r, \mathcal{M}), A). \tag{6.16}$$

Concretely, an element of (6.16) is a sequence of maps $\varphi_r : H_0(\Gamma_r, \mathcal{M}) \to A$, compatible in the sense that

$$\varphi_r(m) = \sum_i \varphi_{r+1}(\gamma_i^{-1} m) \in A,$$

for all $m \in \mathcal{M}$, where the γ_i range over a set of coset representatives for Γ_r/Γ_{r+1}. The sequence $\{\varphi_r\}$ defines a $\Gamma_0(N)$-invariant $\mathrm{Meas}(\mathbf{X}, A)$-valued modular symbol by the rule:

$$\tilde{\mu}_m \left(\left\{ x \in \mathbf{X} : x \equiv \begin{pmatrix} a \\ c \end{pmatrix} \ (\mathrm{mod} \ p^r) \right\} \right) = \varphi_r(\gamma^{-1} m)$$

where γ is a matrix in $\Gamma_0(N)$ that is equivalent to $\begin{pmatrix} a & * \\ c & * \end{pmatrix}$ modulo p^r.

For an element $\tilde{\mu}$ of (6.16) representing an element of $\mathbf{M}(A)$, the measure of the compact open set $\mathbf{Z}_p^\times \times p\mathbf{Z}_p$ (the inverse image under π of $\mathbf{P}^1(\mathbf{Q}_p) - \mathbf{Z}_p$) is given by the image of $\tilde{\mu}$ in $\mathrm{Hom}(H_0(\Gamma_0(Np), \mathcal{M}), A)$.

Extending our maps via $\mathbf{Z}_p$-linearity, and identifying $H_0(\Gamma_r, \mathcal{M})_T$ geometrically with $H_1(X_r, \text{cusps}, \mathbf{Z})$, we may write

$$\mathbf{M}(A) = \varprojlim \text{Hom}_{\mathbf{Z}_p}(H_1(X_r, \text{cusps}, \mathbf{Z}_p), A).$$

In relating the module $\mathbf{M}(A)$ to the work of Hida, it will be convenient to dualize the description above. Denote by $\check{A}$ the $\mathbf{Z}_p$-dual $\text{Hom}_{\mathbf{Z}_p}(A, \mathbf{Z}_p)$; then for two finite free $\mathbf{Z}_p$-modules A and B, it is clear that $\text{Hom}_{\mathbf{Z}_p}(A, B) = \text{Hom}_{\mathbf{Z}_p}(\check{B}, \check{A})$; we write the map corresponding to $f \in \text{Hom}_{\mathbf{Z}_p}(A, B)$ as $\check{f} \in \text{Hom}_{\mathbf{Z}_p}(\check{B}, \check{A})$, so $\check{f}(g)(a) := g \circ f(a)$. Identifying the dual of $H_1(X_r, \text{cusps}, \mathbf{Z}_p)$ with $H_1(Y_r, \mathbf{Z}_p)$ (where $Y_r = X_r - \text{cusps}$) via Poincaré duality, we then have

$$\mathbf{M}(A) = \varprojlim \text{Hom}_{\mathbf{Z}_p}(\check{A}, H_1(Y_r, \mathbf{Z}_p)).$$

The statement of the proposition is that there exists an element of $\mathbf{M}(H \otimes \mathbf{Z}_p)$ such that its image in $\text{Hom}_{\mathbf{Z}_p}(\check{H}, H_1(Y_0, \mathbf{Z}_p))$ is precisely $\check{\psi}$. Such an element exists since the maps $H_1(Y_r, \mathbf{Z}_p) \to H_1(Y_{r-1}, \mathbf{Z}_p)$ are surjective. $\qquad\square$

In the course of the above proof, we showed $\mathbf{M}(A) = \text{Hom}_{\mathbf{Z}_p}(\mathbf{M}, A)$, where

$$\mathbf{M} = \varinjlim H_1(X_r, \text{cusps}, \mathbf{Z}_p).$$

As usual $\mathbf{M}$ has a Hecke algebra generated over $\mathbf{Z}_p$ by the diamond operators and the operators T_ℓ, etc.

Remark 6.3.4. A point of caution is in order: since the tower defining $\mathbf{M}$ is (essentially) dual to the tower of Section 6.2, one must correspondingly take the dual Hecke operators. In other words, T_ℓ is given by $T\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}$ in the notation of Chapter 5, since these are the operators that are compatible with the maps in the *direct* limit defining $\mathbf{M}$. In particular, the action of U_p is given by $T\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, for which the α_i in (5.1) can be chosen to be

$$\left\{ \begin{pmatrix} 1 & 0 \\ Np^r i & p \end{pmatrix} \right\}_{i=0}^{p-1}$$

for the group $\Delta = \Gamma_r$.

Define the ordinary part $\mathbf{M}(A)^o \subset \mathbf{M}(A)$ to be the set of homomorphisms that factor through the ordinary projector $\mathbf{M} \to \mathbf{M}^o \subset \mathbf{M}$. In order to have U_p invertible, we will always assume that the modular symbol $\tilde{\mu}$ arises from $\mathbf{M}^o$.

Let the modular symbol $\tilde{\mu}$ correspond to a $\mathbf{Z}_p$-module homomorphism $f : \mathbf{M}^o \to A$. For each element $t \in \mathrm{End}(\mathbf{M}^o)$ of the Hecke algebra of $\mathbf{M}^o$, the map $f \circ t : \mathbf{M}^o \to A$ yields another measure-valued modular symbol, which we denote by $\tilde{\mu}^t$.

We also extend all measures on $\mathbf{X}$ to the larger space $\mathbf{Y} := \mathbf{Q}_p^2 - 0$ by imposing invariance under multiplication by p:

$$\tilde{\mu}_m(pU) = \tilde{\mu}_m(U)$$

for all compact opens $U \subset \mathbf{Y}$; this extension is well defined because $\mathbf{X}$ forms a fundamental domain for the action of multiplication by p on $\mathbf{Y}$. The purpose of this extension is that $\mathbf{Y}$ (considered as column vectors) has a natural action of Γ by left multiplication, whereas $\mathbf{X}$ does not. Recall that P denotes the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Proposition 6.3.5. *Let U be an compact open subset of $\mathbf{X}$ and $\tilde{\mu}$ as above. We have*

$$\tilde{\mu}_{Pm}(PU) = \begin{cases} \tilde{\mu}_m^{U_p}(U) & \text{if } U \subset \mathbf{Z}_p^\times \times p\mathbf{Z}_p \\ \tilde{\mu}_m^{U_p^{-1}}(U) & \text{if } U \subset \mathbf{Z}_p \times \mathbf{Z}_p^\times \end{cases}.$$

Proof. Let $U(a, c; r, s)$ denote the basic compact open set

$$U(a, c; r, s) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{X} : x \equiv a \pmod{p^r}, y \equiv c \pmod{p^s} \right\}.$$

We will demonstrate the first case of the proposition by considering $U = U(a, c; r, r)$ with $Np|c$ and $(a, p) = 1$. Then

$$
\begin{aligned}
\tilde{\mu}_{Pm}(PU) &= \tilde{\mu}_{Pm}\left(U\left(a, \frac{c}{p}; r, r-1 \right) \right) \\
&= \sum_{i=0}^{p-1} \tilde{\mu}_{Pm}\left(U\left(a, \frac{c}{p} + Np^{r-1}i; r, r \right) \right) \\
&= \varphi_r\left(\sum_{i=0}^{p-1} \begin{pmatrix} a & * \\ \frac{c}{p} + Np^{r-1}i & * \end{pmatrix}^{-1} Pm \right) \\
&= \varphi_r\left(\sum_{i=0}^{p-1} \begin{pmatrix} 1 & 0 \\ Np^r i & p \end{pmatrix}^{-1} \begin{pmatrix} a & * \\ c & * \end{pmatrix}^{-1} m \right) \\
&= \tilde{\mu}_m^{U_p}(U),
\end{aligned}
$$

where the $*$'s are chosen so that the resulting matrices lie in $\Gamma_0(N)$. This proves the first case of the proposition. Similarly, one shows that

$$\tilde{\mu}_{P^{-1}m}(P^{-1}V) = \tilde{\mu}_m^{U_p}(V)$$

for $V \subset p\mathbf{Z}_p \times \mathbf{Z}_p^\times$; letting $U = P^{-1}V$ proves the second case of the proposition. $\qquad\square$

Corollary 6.3.6. *The push forward under π of the modular symbol $\tilde{\mu}$ is precisely μ, i.e.,*

$$\tilde{\mu}_m(\pi^{-1}(U)) = \mu_m(U)$$

for all $m \in \mathcal{M}$ and all compact open $U \subset \mathbf{P}^1(\mathbf{Q}_p)$.

Proof. The fact that $\tilde{\mu}$ is $\Gamma_0(N)$-invariant means that

$$\tilde{\mu}_{\gamma m}(\gamma U) = \tilde{\mu}_m(U) \tag{6.17}$$

for all $\gamma \in \Gamma_0(N)$. Furthermore, we showed in Proposition 6.3.5 that we can choose a Γ-stable basis of compact opens U of $\mathbf{X}$ satisfying

$$\tilde{\mu}_{Pm}(PU) = \tilde{\mu}_m^{U_p^{\pm 1}}(U). \tag{6.18}$$

Combining (6.17) and (6.18) we find

$$\tilde{\mu}_{P^{-1}\gamma Pm}(P^{-1}\gamma PU) = \tilde{\mu}_m^{U_p^e}(U) \tag{6.19}$$

for $\gamma \in \Gamma_0(N)$, where e denotes some even power depending on γ. Since Γ is generated by its subgroups $\Gamma_0(N)$ and $P^{-1}\Gamma_0(N)P$ (our description of the action of Γ on the tree $\mathcal{T}$ in Section 4.1 shows that Γ is the amalgam of these two subgroups with respect to their intersection $\Gamma_0(M)$, cf. [40, §II.1.4]), equations (6.17) and (6.19) imply that

$$\tilde{\mu}_{\gamma m}(\gamma U) = \tilde{\mu}_m^{U_p^e}(U) \tag{6.20}$$

for all $\gamma \in \Gamma$. Let us apply this rule with $U = \mathbf{Z}_p^\times \times p\mathbf{Z}_p$:

$$\begin{aligned}
\tilde{\mu}_m(\gamma U) &= \tilde{\mu}_{\gamma^{-1}m}^{U_p^e}(U) \\
&= -U_p^e(\psi(\gamma^{-1}m)) \tag{6.21} \\
&= \mu_m(\gamma(\mathbf{P}^1(\mathbf{Q}_p) - \mathbf{Z}_p)). \tag{6.22}
\end{aligned}$$

Here (6.21) follows from property (6.14) defining $\tilde{\mu}$; (6.22) uses the fact that $U_p^2 = 1$ on H, and the definition of μ. Since the Γ translates of $\mathbf{P}^1(\mathbf{Q}_p) - \mathbf{Z}_p$ and its complement form a basis of compact opens for $\mathbf{P}^1(\mathbf{Q}_p)$, the result follows. $\qquad\square$

The modular symbol $\tilde{\mu}$ can be used to split the cocycle $c_\tau^{\mathscr{L}_p}$ explicitly. Define a 0-chain $\rho_\tau \in C^0(\Gamma, \mathrm{Hom}(\mathcal{M}, K_p \otimes H))$ by the rule

$$
\begin{aligned}
\rho_\tau(m) &:= \int_{\mathbf{X}} \log_p(a - b\tau)\, \mathrm{d}\tilde{\mu}_m(a,b) \\
&:= \lim_{\|\mathcal{U}\| \to 0} \sum_{U \in \mathcal{U}} \log_p(a_U - b_U\tau) \otimes \tilde{\mu}_m(U),
\end{aligned}
$$

where the limit is over uniformly finer covers $\mathcal{U}$ of $\mathbf{X}$ by disjoint compact opens U, and (a_U, b_U) is an arbitrary point of U. We will show in stages that $d\rho_\tau = c_\tau^{\mathscr{L}_p}$.

Proposition 6.3.7. *If* $\gamma \in \Gamma_0(N)$, *then*

$$
\begin{aligned}
\rho_\tau(\gamma^{-1}m) - \rho_\tau(m) &= \log_p\left(\fint_\tau^{\gamma\tau} \int_m \omega_\psi \right) \\
&= c_\tau^{\mathscr{L}_p}(\gamma)(m).
\end{aligned}
$$

Proof. Recall our assumption that τ reduces to the distinguished vertex v^* of the Bruhat–Tits tree of $\mathbf{PGL}_2(\mathbf{Q}_p)$. Since $\Gamma_0(N)$ preserves this vertex, Lemma 3.3.2 shows that

$$
\mathrm{ord}_p\left(\fint_\tau^{\gamma\tau} \int_m \omega_\psi \right) = 0,
$$

and hence the first equality of the proposition implies the second. Write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Using the $\Gamma_0(N)$ invariance of $\tilde{\mu}$ we calculate $\rho_\tau(\gamma^{-1}m) - \rho_\tau(m)$:

$$
\begin{aligned}
&\int_{\mathbf{X}} \log_p(x - y\tau)\, \mathrm{d}\tilde{\mu}_m(\gamma(x,y)) - \int_{\mathbf{X}} \log_p(x - y\tau)\, \mathrm{d}\tilde{\mu}_m(x,y) \\
&= \int_{\mathbf{X}} \left[\log_p((dx - by) - (-cx + ay)\tau) - \log_p(x - y\tau) \right] \mathrm{d}\tilde{\mu}_m(x,y) \\
&= \int_{\mathbf{X}} \log_p\left(\frac{(dx/y - b) - (-cx/y + a)\tau}{x/y - \tau} \right) \mathrm{d}\tilde{\mu}_m(x,y).
\end{aligned}
$$

Since the integrand depends only on x/y and the push forward of $\tilde{\mu}$ is μ, the above expression equals

$$
\begin{aligned}
&\int_{\mathbf{P}^1(\mathbf{Q}_p)} \log_p\left(\frac{(dt - b) - (-t + a)\tau}{t - \tau} \right) \mathrm{d}\mu_m(t) \\
&= \int_{\mathbf{P}^1(\mathbf{Q}_p)} \log_p\left(\frac{t(d + c\tau) - (a\tau + b)}{t - \tau} \right) \mathrm{d}\mu_m(t) \\
&= \int_{\mathbf{P}^1(\mathbf{Q}_p)} \log_p\left(\frac{t - \gamma\tau}{t - \tau} \right) \mathrm{d}\mu_m(t) + \int_{\mathbf{P}^1(\mathbf{Q}_p)} \log_p(d + c\tau)\, \mathrm{d}\mu_m(t) \\
&= \log_p\left(\fint_\tau^{\gamma\tau} \int_m \omega_\psi \right),
\end{aligned}
$$

where the last equality follows since μ has total measure zero. $\qquad\square$

For the matrix $P \notin \Gamma_0(N)$, the situation is somewhat different.

Proposition 6.3.8.

$$\int_{\mathbf{X}} \log_p(x - y\tau) \, d\tilde{\mu}_{Pm}^{U_p}(x, y) \quad - \quad \int_{\mathbf{X}} \log_p(x - y\tau) \, d\tilde{\mu}_m(x, y) \tag{6.23}$$

$$= \quad (\log_p - \mathscr{L}_p \operatorname{ord}_p) \left(\oint_\tau^{P^{-1}\tau} \int_m \omega_\psi \right)$$

Proof. We use the change of variables $(x, y) \mapsto (px, y)$ and the decomposition

$$P^{-1}\mathbf{X} = (\mathbf{Z}_p \times \mathbf{Z}_p^\times) \bigsqcup \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} (\mathbf{Z}_p^\times \times p\mathbf{Z}_p)$$

to break up the first integral (note also $\log_p(px - y\tau) = \log_p(x - \frac{y\tau}{p})$):

$$\int_{\mathbf{X}} \log_p(x - y\tau) \, d\tilde{\mu}_{Pm}^{U_p}(x, y) \quad = \quad \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \log_p \left(x - \frac{y\tau}{p} \right) d\tilde{\mu}_{Pm}^{U_p}(px, y)$$

$$+ \int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p \left(x - \frac{y\tau}{p} \right) d\tilde{\mu}_{Pm}^{U_p}(x, y/p)$$

$$= \quad \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \log_p \left(x - \frac{y\tau}{p} \right) d\tilde{\mu}_m(x, y)$$

$$+ \int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p \left(x - \frac{y\tau}{p} \right) d\tilde{\mu}_m^{U_p^2}(x, y),$$

by Proposition 6.3.5. Thus the left-hand side of (6.23) becomes

$$\int_{\mathbf{X}} \log_p \left(\frac{x - \frac{y\tau}{p}}{x - y\tau} \right) d\tilde{\mu}_m(x, y) - \int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p \left(x - \frac{y\tau}{p} \right) d\tilde{\mu}_m^{1 - U_p^2}(x, y). \tag{6.24}$$

The first integral of (6.24) can be pushed forward to $\mathbf{P}^1(\mathbf{Q}_p)$ as in the proof of Proposition 6.3.7 and equals

$$\log_p \left(\oint_\tau^{P^{-1}\tau} \int_m \omega_\psi \right).$$

The second integral of (6.24) may be further decomposed:

$$\int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p \left(x - \frac{y\tau}{p} \right) d\tilde{\mu}_m^{1 - U_p^2}(x, y) \quad = \quad \int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p(x) \, d\tilde{\mu}_m^{1 - U_p^2}(x, y)$$

$$+ \int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p \left(1 - \frac{y\tau}{px} \right) d\tilde{\mu}_m^{1 - U_p^2}(x, y). \tag{6.25}$$

The second integral of (6.25) is again a push forward to $\mathbf{P}^1(\mathbf{Q}_p) - \mathbf{Z}_p$; since the push forward of $\tilde{\mu}_m^{1 - U_p^2}$ is evidently the zero measure, this integral vanishes. Thus the proposition results from the lemma below (and Lemma 3.3.2). $\qquad \square$

Recall the notation of Section 6.2.

Lemma 6.3.9. *Let t be an element of* $\mathbf{T}$ *whose image in* $\mathscr{T}_0$ *vanishes. We have*

$$\int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p(x)\, \mathrm{d}\tilde{\mu}_m^t(x,y) = -t' \cdot \psi(m).$$

Proof. By Hida's Theorem 6.2.1, we need only consider elements of the form $t = (\langle d \rangle - 1)h$, with $h \in \mathbf{T}$. Modulo p^r,

$$\int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p(x)\, \mathrm{d}\tilde{\mu}_m^t(x,y) \equiv \sum_{\gamma \in \Gamma_0(Np)/\Gamma_r} \log_p(a)\varphi_r(h(\langle d \rangle - 1)\gamma^{-1}m), \tag{6.26}$$

where a represents the upper left entry of the matrix γ. The action of $\langle d \rangle$ is given by a matrix $\gamma_d \in \Gamma_0(N)$ such that

$$\gamma_d \equiv \begin{pmatrix} d^{-1} & * \\ 0 & d \end{pmatrix} \pmod{p^r},$$

so (6.26) becomes

$$\sum_{\gamma \in \Gamma_0(Np)/\Gamma_r} \log(a)\varphi_r(h\gamma_d^{-1}\gamma^{-1}m) - \sum_{\gamma \in \Gamma_0(Np)/\Gamma_r} \log(a)\varphi_r(h\gamma^{-1}m). \tag{6.27}$$

As γ ranges through coset representatives for $\Gamma_0(Np)/\Gamma_r$, the matrices $\gamma\gamma_d$ do as well; the change of variables $\gamma \mapsto \gamma\gamma_d$ in the first sum of (6.27) simplifies the entire expression to

$$\begin{aligned}
\sum_{\gamma \in \Gamma_0(Np)/\Gamma_r} \left(\log_p(ad) - \log_p(a)\right)\varphi_r(h\gamma^{-1}m) &= \log_p(d)\sum_{\gamma \in \Gamma_0(Np)/\Gamma_r} \varphi_r(h\gamma^{-1}m) \\
&= \log_p(d)\tilde{\mu}_m^h(\mathbf{Z}_p^\times \times p\mathbf{Z}_p) \\
&= -\log_p(d) \cdot h \cdot \psi(m).
\end{aligned}$$

This proves the desired result. $\square$

Propositions 6.3.7 and 6.3.8 together imply that for for $\gamma \in \Gamma_0(N)$, we have

$$\rho_\tau(P^{-1}\gamma P m) - \rho_\tau(m) = c_\tau^{\mathscr{L}_p}(P^{-1}\gamma P)(m). \tag{6.28}$$

Since the groups $\Gamma_0(N)$ and $P^{-1}\Gamma_0(N)P$ generate Γ, Propositions 6.3.7 and equation (6.28) yield the following proposition, which implies equation (6.1):

Proposition 6.3.10. *The chain* ρ_τ *splits the 1-cocycle* $c_\tau^{\mathscr{L}_p}$ *for the group* Γ, *i.e.,* $d\rho_\tau = c_\tau^{\mathscr{L}_p}$.

6.4 Proof that $\mathscr{L}_p$ is the $\mathscr{L}$-invariant of J

The methods of this chapter follow very closely those of [12], but we include the argument for completeness. The map Q yields an exact sequence of $\mathscr{T}[G_p]$-modules:

$$0 \to X \to \operatorname{Hom}(X, \bar{\mathbf{Q}}_p^\times) \to J(\bar{\mathbf{Q}}_p) \to 0. \tag{6.29}$$

The image of the first nontrivial map above lies in $\operatorname{Hom}(X, \mathbf{Q}_p^\times)$.

Definition 6.4.1. The $\mathscr{L}$-invariant of J is the element of $\operatorname{End}(X \otimes \mathbf{Q}_p)$ such that

$$\log_p Q(x) = \operatorname{ord}_p Q(\mathscr{L}x) \text{ for all } x \in X,$$

where $\log_p Q$ and $\operatorname{ord}_p Q$ have been extended via $\mathbf{Q}_p$-linearity.

Following Greenberg and Stevens, we will interpret the $\mathscr{L}$-invariant of J as arising from the deformation theory of the Galois action on its Tate module $\operatorname{Ta}_p J$. Let $\operatorname{rec} : \mathbf{Q}_p^\times \to G_p^{\mathrm{ab}}$ be the Artin reciprocity map. Write Frob_p for $\operatorname{rec}(p)^{-1}$; this is a lifting to G_p^{ab} of the Frobenius map on the maximal unramified extension of $\mathbf{Q}_p$.

From (6.29) one finds (by connecting (6.29) with itself via the multiplication by p^r map, employing the snake lemma, and taking the inverse limit over all r):

$$0 \to \operatorname{Hom}(X, \operatorname{Ta}_p \bar{\mathbf{Q}}_p^\times) \to \operatorname{Ta}_p J \to X \otimes \mathbf{Z}_p \to 0.$$

Twist the above sequence by the unramified character $\varphi : G_p \to \mathscr{T}^\times$ that sends Frob_p to U_p (so the module $X(\varphi)$ has trivial G_p-action), and tensor with $\mathbf{Q}_p$. We then have

$$0 \to \operatorname{Hom}(X(\varphi), \mathrm{V}_p \bar{\mathbf{Q}}_p^\times) \to \mathrm{V}_p J(\varphi) \to X(\varphi) \otimes \mathbf{Q}_p \to 0, \tag{6.30}$$

where V_p denotes $\mathbf{Q}_p \otimes \operatorname{Ta}_p$. We will denote the three terms in this sequence by A, B, and C, respectively. As noted by Greenberg and Stevens, the $\mathscr{L}$-invariant of J can be deduced from knowledge about deformations of the sequence (6.30). A deformation of the module A is a $\mathbf{T}^o[G_p]$-module $\mathbf{A}$ such that $\mathbf{A}/I\mathbf{A} \cong A$ as $(\mathbf{T}^o/I\mathbf{T}^o)[G_p] = \mathscr{T}_0[G_p]$-modules, where I is the augmentation ideal of Λ as in Section 6.2. Suppose we have a deformation of the sequence (6.30), that is, a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I\mathbf{A} & \longrightarrow & I\mathbf{B} & \longrightarrow & I\mathbf{C} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{A} & \longrightarrow & \mathbf{B} & \longrightarrow & \mathbf{C} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0,
\end{array}
\tag{6.31}
$$

where we have omitted the 0 terms on both ends of the vertical short exact sequences. Suppose further that $\mathbf{A}$ is a trivial deformation, in the sense that the action of G_p on $\mathbf{A}$ is given by the cyclotomic character (as it is on A). Let $\Psi : G_p \to \mathrm{End}(\mathbf{C})$ define the Galois action on $\mathbf{C}$. Since G_p acts trivially on C, for each $\mathbf{c} \in \mathbf{C}$ we have $\sigma(\mathbf{c}) - \mathbf{c} \in I\mathbf{C}$. Consider the image of $\sigma(\mathbf{c}) - \mathbf{c}$ in

$$I\mathbf{C}/I^2\mathbf{C} = I/I^2 \otimes_\Lambda \mathbf{C}. \tag{6.32}$$

As in Section 6.2, we map this via $\log_p \otimes \mathrm{Id}$ to

$$\mathbf{Z}_p \otimes_\Lambda \mathbf{C} = \Lambda/I \otimes_\Lambda \mathbf{C} = \mathbf{C}/I\mathbf{C} = C. \tag{6.33}$$

Thus to each $\sigma \in G_p$ and $\mathbf{c} \in \mathbf{C}$, we have associated an element denoted $\Psi'(\sigma)(\mathbf{c})$. Furthermore $\Psi'(\sigma)(I\mathbf{C}) = 0$, so $\Psi'(\sigma)$ factors through the quotient $\mathbf{C}/I\mathbf{C} = C$, and may thus be viewed as an element of $\mathrm{End}(C)$. It is trivial to check that $\Psi'(\sigma)$ depends only on the image of $\sigma \in G_p^{\mathrm{ab}}$. We now relate the $\mathscr{L}$-invariant of J to Ψ'.

Proposition 6.4.2. *Let $u \in 1 + p\mathbf{Z}_p$ be a nontrivial unit. Then we have*

$$\Psi'(\mathrm{Frob}_p) = \mathscr{L} \circ \frac{1}{\log_p u} \Psi'(\mathrm{rec}(u)) \tag{6.34}$$

as elements of $\mathrm{End}(C) = \mathrm{End}(X \otimes \mathbf{Q}_p)$.

Proof. Denote by $\Delta_C : I\mathbf{C} \to C$ the composition of the maps in (6.32) and (6.33). Define Δ_A similarly, and use the same notation for the induced maps on cohomology. A basic calculation verifies the commutativity of the following diagram of G_p-cohomology groups:

$$\begin{array}{ccccc}
H^0(C) & \xrightarrow{\delta_C} & H^1(I\mathbf{C}) & \xrightarrow{\Delta_C} & H^1(C) \\
{\scriptstyle -\delta_1}\downarrow & & {\scriptstyle \delta_2}\downarrow & & {\scriptstyle \delta_3}\downarrow \\
H^1(A) & \xrightarrow{\delta_A} & H^2(I\mathbf{A}) & \xrightarrow{\Delta_A} & H^2(A)
\end{array} \tag{6.35}$$

All of the maps labelled δ arise from coboundary maps in (6.31). Now G_p acts trivially on C, so $H^0(C) = C$ and $H^1(C) = \mathrm{Hom}_{\mathrm{cont}}(G_p^{\mathrm{ab}}, C)$. Furthermore, for each nontrivial unit $u \in 1 + p\mathbf{Z}_p$, the map $H^1(C) \to C \oplus C$ given by

$$\pi : \xi \mapsto \left(\xi(\mathrm{Frob}_p), \frac{1}{\log_p u} \xi(\mathrm{rec}(u)) \right) \tag{6.36}$$

is an isomorphism independent of u.

By definition, $\Psi'(\sigma)(c) = \Delta_C \circ \delta_C(c)(\sigma)$. Also, since $\mathbf{A}$ is a trivial deformation, $\delta_A = 0$. Thus from the commutativity of (6.35), we have

$$\delta_3\left(\Psi'(\mathrm{Frob}_p)(c), \frac{1}{\log_p u}\Psi'(\mathrm{rec}(u))(c)\right) = 0 \qquad (6.37)$$

for all $c \in C$. We will determine the kernel of δ_3 via the perfect pairings of Tate duality:

$$H^i(C) \times H^{2-i}(A) \to H^2(\mathbf{Q}_p(1)) = \mathbf{Q}_p, \quad i = 0, 1, 2.$$

This pairing may be described explicitly for $i = 1$ as follows. For each element $\hat{c} \in \mathrm{Hom}(X, \mathbf{Q}_p^\times)$, we define an element $\gamma_{\hat{c}} \in H^1(A)$; choose a p^nth root $\hat{c}^{1/p^n}$ of $\hat{c}$ in $\mathrm{Hom}(X, \bar{\mathbf{Q}}_p^\times)$ for each n, compatible in the sense that $(\hat{c}^{1/p^{n+1}})^p = \hat{c}^{1/p^n}$. The assignment

$$\sigma \mapsto (\sigma - 1)(\hat{c}^{1/p^n}) \in \mathrm{Hom}(X, \mathrm{Ta}_p\, \bar{\mathbf{Q}}_p^\times)$$

is a cocycle representing a class denoted $\gamma_{\hat{c}} \in H^1(A)$. The definition of $\gamma_{\hat{c}}$ is independent of the choice of p^nth roots of $\hat{c}$. If

$$\hat{c} = \sum f \otimes q \in X^* \otimes \mathbf{Q}_p^\times = \mathrm{Hom}(X, \mathbf{Q}_p^\times)$$

and $\xi \in H^1(C)$, the Tate duality pairing is given by

$$\langle \xi, \gamma_{\hat{c}} \rangle = \sum f(\xi(\mathrm{rec}(q))) \in \mathbf{Q}_p.$$

From this description, it one verifies that the dual π^* of the isomorphism π from (6.36) satisfies

$$(\pi^*)^{-1} : H^1(A) \ \to \ C^* \oplus C^*$$
$$\gamma_{\hat{c}} \ \mapsto \ (-\mathrm{ord}_p(\hat{c}), \log_p(\hat{c})). \qquad (6.38)$$

By the self-duality of (6.30), the kernel of δ_3 is dual to the image of δ_1. Yet δ_1 is the map $c \mapsto \gamma_{Q(c)}$. Thus (6.38) implies that the kernel of δ_3 consists of elements of the form $(\mathscr{L}a, a)$; hence (6.37) yields the result. $\qquad\square$

The following theorem of Mazur and Wiles is the main arithmetic ingredient towards proving $\mathscr{L} = \mathscr{L}_p$.

Theorem 6.4.3 (Mazur-Wiles, [34]). *There exists a deformation sequence as in* (6.31) *with the Galois action on $\mathbf{C}$ given by $\Psi : G_p \to \mathbf{T}^o \to \mathrm{End}(\mathbf{C})$ satisfying*

$$\Psi(\mathrm{Frob}_p) = U_p^2 \in \mathbf{T}^o \ and \ \Psi(\mathrm{rec}(u)) = \langle u \rangle$$

for $u \in 1 + p\mathbf{Z}_p$.

Proof. This is Proposition 2 of Chapter 8 in [34]; see the comments at the end of that chapter for a description of the Galois action. Although [34] deals only with $N = 1$, the constructions of [33] from which the result is derived are carried out for higher level N as well. □

We may now verify equation (6.3):

Proposition 6.4.4. *The $\mathscr{L}$-invariant of J is equal to $\mathscr{L}_p$.*

Proof. With Ψ as in Theorem 6.4.3, it is clear that $\Psi'(\mathrm{Frob}_p) = \mathscr{L}_p$ and $\Psi'(\mathrm{rec}(u)) = \log_p u$. Thus the proposition follows from Proposition 6.4.2. □

Part II

Gross–Stark units

Chapter 7

Introduction

When the modular Jacobian $J_0(N)$ is replaced by the group of units $\mathcal{O}^\times$ in the ring of algebraic numbers, the role of Heegner points on $J_0(N)$ is played by elliptic units in $\mathcal{O}^\times$. Elliptic units are obtained as special values of modular units. A modular unit is a meromorphic function on $X_0(N)$ whose zeroes and poles lie only at the cusps. An example of such a function is

$$\alpha(z) = \prod_{d|N} \Delta(dz)^{n_d}, \text{ where } \sum_{d|N} n_d = 0.$$

For fixed N, the collection of modular units α of this form is a subgroup of finite index in the group of all modular units. Elliptic units are obtained by evaluating modular units at complex numbers $\tau \in \mathcal{H}^*/\Gamma_0(N)$ that lie in quadratic imaginary extensions K of $\mathbf{Q}$. The theory of complex multiplication shows that $\alpha(\tau)$ for such τ lies in a specific class field H of K, and describes the action of $\mathrm{Gal}(H/K)$ on the collection of $\alpha(\tau)$. The goal of part II of this thesis is to present a conjectural formulation of an analogous construction when K is a real quadratic field. See [6] for a description of the classical theory of elliptic units in the form that motivates our current presentation, and [27] for a detailed study of the group of modular units.

Let K be a real quadratic field and p a prime that is inert in K. The completion K_p is a quadratic unramified extension of $\mathbf{Q}_p$. We begin by applying the constructions of part I to associate to a modular unit α and $\tau \in K - \mathbf{Q}$ an element $u(\alpha, \tau) \in K_p^\times$. The construction of $u(\alpha, \tau)$ is obtained by using the same group

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}_2(\mathbf{Z}[1/p]) \text{ such that } N|c \right\} \tag{7.1}$$

70

as in Chapter 4, but choosing a different module $\mathcal{M}$ whose universal torus $\tilde{T}$ admits Eisenstein quotients in which complex conjugation acts as -1. The image of the lattice $L = \operatorname{Im}\Phi_2$ in such an Eisenstein quotient of $\tilde{T}$ is trivial. The elements $u(\alpha, \tau)$ are simply the images of Stark–Heegner points in the corresponding $\mathbf{G}_m$ quotient of $\tilde{T}/L$. Motivated by Conjecture 4.3.3, Conjecture 8.2.5 states that $u(\alpha, \tau)$ is a p-unit in a specific narrow ring class field of K depending on τ and denoted H_τ.

While this conjecture is still open, Chapter 9 relates the p-adic valuation of $u(\alpha, \tau)$ to the value at 0 of a certain zeta function $\zeta(\alpha, \tau, s)$. Gross has shown (as a consequence of the analytic class number formula) that there are unique p-units u_{GS} in H_τ satisfying this exact relation, and has given a precise conjecture (analogous to those of Stark in the archimedean setting) relating the p-adic logarithm of $\mathrm{N}_{K_p/\mathbf{Q}_p} u_{\mathrm{GS}}$ to the first derivative of a partial p-adic zeta function $\zeta_p(\alpha, \tau, s)$ [16]. This p-adic zeta function is obtained by interpolating $\zeta(\alpha, \tau, s)$ at negative integers s. The main result of part II is Theorem 10.0.1, which shows that the $u(\alpha, \tau)$ satisfy Gross's relationship. This result may be viewed as an analogue for real quadratic fields of the Kronecker limit formula. Conjecture 8.2.5, which implies that $u_{\mathrm{GS}} = u(\alpha, \tau)$, therefore implies Gross's conjecture for H_τ/K at the prime p.

It should be stressed that Conjecture 8.2.5 leads to a genuine strengthening of the Gross–Stark conjectures of [16] in the setting of ring class fields of real quadratic fields, and also of the refinement of these conjectures proposed in [17]. Indeed, the latter exploits the special values at $s = 0$ of abelian L-series attached to K, as well as derivatives of the corresponding p-adic zeta functions, to recover the images of Gross–Stark units in $K_p^\times/\bar{\mathcal{O}}_K^\times$, where $\bar{\mathcal{O}}_K^\times$ is the topological closure in $K_p^\times$ of the unit group of K. Conjecture 8.2.5 proposes an explicit formula for the Gross–Stark units themselves. It would be interesting to see whether other instances of the Stark conjectures (both classical, and p-adic) are susceptible to similar refinements[1].

---

[1]In a purely archimedean context, recent work of Ren and Sczech on the Stark conjectures for a complex cubic field suggests a positive answer to this question.

Chapter 8

Definition of the units

With Γ as in (7.1) and $\mathcal{M} = \mathrm{Div}_0 \, \mathbf{P}^1(\mathbf{Q})$, Proposition 4.1.1 implies that the maximal torsion-free quotient of $\tilde{H}/H$ has rank 1 over $\mathbf{Z}$. As a Hecke-module, this quotient is Eisenstein. Let $\psi : \tilde{H} \mapsto \mathbf{Z}$ denote the projection onto this quotient. Concretely, we have

$$\psi([x] - [y]) = \begin{cases} 1 & \text{if } x \sim 0 \text{ and } y \sim \infty \\ 0 & \text{if } x \sim y \\ -1 & \text{if } x \sim \infty \text{ and } y \sim 0, \end{cases}$$

where the equivalence relation $\sim$ has two classes, $\mathbf{Z}_p \cap \mathbf{Q}$ and its complement. A simple computation shows that in this case the multiplicative double integral becomes the cross-ratio:

$$\fint_{\tau_1}^{\tau_2} \int_x^y \omega_\psi = \frac{\tau_2 - y}{\tau_1 - y} \cdot \frac{\tau_1 - x}{\tau_2 - x} \in \mathbf{Q}_p(\tau_1, \tau_2)^\times.$$

The 2-cocycle d defined in (4.8) splits modulo $\psi(L) = \langle -1, p \rangle$, and one may define a Stark–Heegner point attached to τ as in (4.3.2). However, one checks that the point defined in this way is simply a fundamental unit of norm 1 in the ring $\mathcal{O}_\tau$. In other words, no non-trivial extensions of K arise from this construction. The main problem with this simple setting is that the modular symbol ψ has eigenvalue $+1$ for complex conjugation. Since $\mathbf{G}_m$ has a single period $2\pi i$ which is purely imaginary, one expects to construct interesting Stark–Heegner points only by using modular symbols with eigenvalue -1 for complex conjugation. To define such a modular symbol, we retain the group Γ but shrink the support of the module $\mathcal{M}$ to only contain one Γ-orbit of cusps. This creates Eisenstein quotients of $\tilde{H}$ in which complex conjugation acts as -1.

8.1 Eisenstein "minus" modular symbols

Consider the Γ-module

$$\mathcal{M} = \mathrm{Div}_0 \, \Gamma\infty = \mathrm{Div}_0 \left\{ \frac{a}{c} \text{ such that } N|c \right\}.$$

Since $\mathcal{M}$ only consists of divisors supported on one Γ-orbit of cusps, homomorphisms from $\mathcal{M}$ are called *partial modular symbols.* Chapter 2 produces a universal multiplicative double integral with values in the torus $\mathbf{G}_m \otimes \tilde{H}$, where $\tilde{H}$ is the maximal torsion-free quotient of

$$H_0(\Gamma_0(Np), \mathcal{M})/f^*(H_0(\Gamma_0(N), \mathcal{M})^2).$$

We will be interested in projections of $\tilde{H}$ onto certain Eisenstein factors, as follows. Let

$$\alpha(z) = \prod_{d|N} \Delta(dz)^{n_d} \tag{8.1}$$

be a modular unit of level N with neither pole nor zero at the cusp ∞ of $\Gamma_0(N)$. Then we have

$$\sum_{d|N} n_d = 0, \quad \sum_{d|N} n_d \cdot d = 0. \tag{8.2}$$

From α we produce a modular unit

$$\alpha^*(z) := \alpha(z)/\alpha(pz)$$

of level Np, which again has no zeroes or poles on $\Gamma\infty$.

The logarithmic derivative of α defines a meromorphic differential on $X_0(N)$ that is regular on the Γ-orbit of ∞. The differentials $\mathrm{dlog}\,\alpha$ and $\mathrm{dlog}\,\alpha^*$ can be written as

$$\mathrm{dlog}\,\alpha(z) = 2\pi i F_2(z)\,\mathrm{d}z, \qquad \mathrm{dlog}\,\alpha^*(z) = 2\pi i F_2^*(z)\,\mathrm{d}z, \tag{8.3}$$

where $F_2(z)$ and $F_2^*(z)$ are the weight two Eisenstein series on $\Gamma_0(N)$ and $\Gamma_0(Np)$, respectively, given by the formulae

$$F_2(z) = -24 \sum_{d|N} dn_d E_2(dz), \qquad F_2^*(z) = F_2(z) - pF_2(pz), \tag{8.4}$$

and $E_2(z)$ is the standard Eisenstein series of weight 2:

$$E_2(z) = \frac{2}{(2\pi i)^2} \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^2} = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n, \quad q = e^{2\pi i\tau}. \tag{8.5}$$

(We remark that the above series is not absolutely convergent and the resulting expression is not invariant under $\mathbf{SL}_2(\mathbf{Z})$.)

Define a homomorphism $\psi : \mathcal{M} \to \mathbf{Z}$ by the complex line integral

$$\psi([x] - [y]) := \frac{1}{2\pi i} \int_x^y \mathrm{dlog}\,\alpha \in \mathbf{Z}. \tag{8.6}$$

The homomorphism ψ is clearly $\Gamma_0(N)$-invariant. It can be viewed as giving the winding number around the other cusps of the image in $X_0(N)$ of a path in $\mathcal{H}^*$ from x to y, with each cusp weighted by the corresponding order of zero or pole of α. Similarly, integrating $\mathrm{dlog}\,\alpha^*$ yields a $\Gamma_0(Np)$-invariant homomorphism $\psi^* : \mathcal{M} \to \mathbf{Z}$ satisfying

$$\psi^*(m) = \psi(m) - \psi(P \cdot m), \tag{8.7}$$

where as usual P is the matrix representing multiplication by p.

The homomorphism ψ^* factors through $\tilde{H}$. Indeed, to check this for the image under f^* of the first copy of $H_0(\Gamma_0(N), \mathcal{M})$, let $\{\gamma_i\}_{i=1}^{p+1}$ be a set of coset representatives for $\Gamma_0(N)/\Gamma_0(Np)$. Then

$$
\begin{aligned}
\psi^*(f^*(m)) &= \sum_i \psi(\gamma_i^{-1} m) - \psi(P\gamma_i^{-1} m) \\
&= (p+1)\psi(m) - (T_p\psi)(m) = 0,
\end{aligned}
$$

as the q-expansion of the Eisenstein series F_2 shows that ψ has eigenvalue $p+1$ for T_p. A similar calculation holds for the image of the other copy of $H_0(\Gamma_0(N), \mathcal{M})$ under f^*. Thus the homomorphism $\psi^* : \tilde{H} \to \mathbf{Z}$ allows us to define a Γ-invariant $\mathrm{Meas}(\mathbf{P}^1(\mathbf{Q}_p), \mathbf{Z})$-valued partial modular symbol $\mu = \psi^* \circ \mu^{\mathrm{univ}}$ and yields a multiplicative double integral, written formally as the integral of $\mathrm{dlog}\,\alpha$:

$$\fint_{\tau_1}^{\tau_2}\!\!\int_m \mathrm{dlog}\,\alpha := \fint_{\tau_1}^{\tau_2}\!\!\int_m \omega_\mu = (\mathrm{Id} \otimes \psi^*)\left(\fint_{\tau_1}^{\tau_2}\!\!\int_m \omega_{\mathrm{univ}}\right) \in \mathbf{G}_m.$$

8.2 The unit $u(\alpha, \tau)$

Let $\tau \in \mathcal{H}_p$ lie in the quadratic unramified extension K_p of $\mathbf{Q}_p$. As in Section 6.3, define a 1-cocycle

$$c_\tau \in Z^1(\Gamma, \mathrm{Hom}(\mathcal{M}, K_p^\times))$$

by the rule

$$c_\tau(\gamma)(m) := \fint_\tau^{\gamma\tau}\!\!\int_m \mathrm{dlog}\,\alpha.$$

Proposition 8.2.1. *The cocycle c_τ splits, i.e., there is a 0-chain $\rho_\tau : \mathcal{M} \to K_p^\times$ such that*

$$d\rho_\tau(\gamma)(m) = \rho_\tau(\gamma^{-1}m) - \rho_\tau(m) = c_\tau(\gamma)(m)$$

for all $\gamma \in \Gamma$.

Proposition 8.2.1 will be proved over the course of the next three chapters.

Remark 8.2.2. A chain splitting the cocycle c_τ is unique only up to an element of the group of co-invariants $\mathcal{M}_\Gamma$. This group is finite, but not necessarily trivial. However, we will be constructing a particular splitting ρ_τ, and this choice of ρ_τ will remain fixed.

Following [5], we denote the chain ρ_τ with an "indefinite integral":

$$\fint^\tau \int_m \mathrm{dlog}\,\alpha := \rho_\tau(m).$$

The indefinite integral is Γ-invariant. This notation indicates the purpose of splitting the cocycle c_τ—it allows us to define an integral that depends only on one variable τ and not two endpoints of integration.

Recall the notation of Section 4.3, and Definition 4.3.2 in particular.

Definition 8.2.3. The unit $u(\alpha, \tau)$ is defined by

$$u(\alpha, \tau) := \fint^\tau \int_x^{\gamma_\tau x} \mathrm{dlog}\,\alpha \in K_p^\times,$$

where $x \in \Gamma\infty$.

This definition is independent of choice of x.

Remark 8.2.4. In Chapter 4 we defined Stark–Heegner points via the splitting of a 2-cocycle $d_{\tau,x}$. However, as remarked in Section 6.3, the splitting of c_τ implies the splitting of $d_{\tau,x}$, and the corresponding points coincide.

With the notation as in Conjecture 4.3.3, we propose

Conjecture 8.2.5. *If τ belongs to $\mathcal{H}_p^{\mathcal{O}}/\Gamma$, then $u(\alpha, \tau)$ belongs to $\mathcal{O}_H[1/p]^\times$, and in fact,*

$$u(\alpha, \mathfrak{a} \star \tau) = \mathrm{rec}(\mathfrak{a})^{-1} u(\alpha, \tau), \tag{8.8}$$

for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$.

8.3 The action of complex conjugation and of U_p

The partial modular symbol ψ used to define $u(\alpha, \tau)$ is *odd* in the sense that it has eigenvalue -1 for complex conjugation W_∞:

$$\psi([-x] - [-y]) = -\psi([x] - [y]) \text{ for all } x, y \in \Gamma\infty$$

(cf. [32], Chapter II, §3). The parity of m_α implies the following behavior of the elements $u(\alpha, \tau)$ under the action of complex conjugation, denoted as in Remark 4.3.4 by $\mathfrak{a}_\infty$.

Proposition 8.3.1. *Assume Conjecture 8.2.5. For all $\tau \in \mathcal{H}_p^{\mathcal{O}}$,*

$$\mathfrak{a}_\infty u(\alpha, \tau) = u(\alpha, \tau)^{-1}.$$

Proof. The fact that the partial modular symbol ψ is odd implies that the sign denoted w_∞ in proposition 5.13 of [5] is equal to -1. The proof of Proposition 8.3.1 is then identical to the proof of Proposition 5.13 of [5]. $\square$

Remark 8.3.2. In the context of Stark–Heegner points from part I, we obtained two sets of points on J, having eigenvalue $+1$ or -1 for W_∞, corresponding to the choice of real or imaginary periods attached to J, respectively. In the situation treated here, where J is replaced by the multiplicative group, only the odd modular symbol ψ remains available, in harmony with the fact that the multiplicative group has a single period, $2\pi i$, which is purely imaginary.

Remark 8.3.3. Suppose that $\mathcal{O}$ has a fundamental unit of negative norm. Then equivalence of ideals in the strict and usual sense coincide, so the narrow ring class field H associated to $\mathcal{O}$ is equal to the ring class field taken in the non-strict sense, which is totally real. Conjecture 8.2.5 predicts that $\mathfrak{a}_\infty$ should act trivially on $u(\alpha, \tau)$ in this case, and that the p-units $u(\alpha, \tau)$ should be trivial. In fact it can be shown, independently of any conjectures, that

$$u(\alpha, \tau) = 1, \qquad \text{for all } \tau \in \mathcal{H}_p^{\mathcal{O}}.$$

This suggests that interesting elements of $H^\times$ are obtained only when H is a totally complex extension of K. This explains why it is essential to work with equivalence of ideals in the narrow sense and with narrow ring class fields to obtain useful invariants.

The q-expansion of the Eisenstein series $\text{dlog}\,\alpha^*$ shows that it has eigenvalue 1 for the U_p operator. Similarly to the proof of Proposition 8.3.1, this implies that the sign denoted w in Proposition 5.13 of [5] equals 1. Thus the indefinite integral is invariant under all of

$$\tilde{\Gamma} = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PGL}_2(\mathbf{Z}[1/p]) \text{ such that } \det(\gamma) > 0 \text{ and } N|c \right\} \supset \Gamma.$$

In particular, the element $u(\alpha, \tau)$ depends only on the $\tilde{\Gamma}$ orbit of τ. Since $\tilde{\Gamma}$ acts transitively on the vertices of the Bruhat–Tits tree $\mathcal{T}$ of $\mathbf{PGL}_2(\mathbf{Q}_p)$, we may assume from now on that

$$\tau \text{ reduces to the central vertex } v^* \text{ of } \mathcal{T}. \tag{8.9}$$

Chapter 9

Special values of zeta functions

Consider the decomposition

$$
\begin{aligned}
K_p^\times &\cong \mathbf{Z} \times \mathcal{O}_{K_p}^\times \\
k &\mapsto \left(\mathrm{ord}_p(k), u_p(k) := \frac{k}{p^{\mathrm{ord}_p(k)}} \right).
\end{aligned}
$$

We will define the splitting ρ_τ by defining its components $\mathrm{ord}_p \rho_\tau$ and $u_p \rho_\tau$. In this chapter we define $\mathrm{ord}_p \rho_\tau$ and study its connection with zeta functions.

9.1 The zeta function

Let $\tau \in \mathcal{H}_p^{\mathcal{O}}$ satisfy assumption (8.9). There is a unique primitive integral binary quadratic form Q_τ such that $Q_\tau(\tau, 1) = 0$ and

$$
Q_\tau(x, y) = Ax^2 + Bxy + Cy^2, \quad \text{with } A > 0. \tag{9.1}
$$

The quadratic form Q_τ is indefinite. We will assume that the discriminant D of the $\mathbf{Z}[1/p]$-order $\mathcal{O}$ is relatively prime to N (by convention, the integer D is taken to be prime to p); this implies $N|A$. Furthermore, by (8.9) it follows that $B^2 - 4AC = D$ (rather than D times some power of p). The assumption (8.9) also implies that the generator γ_τ of $\Gamma_\tau/\langle \pm 1 \rangle$ belongs to $\Gamma_0(N)$. The matrix γ_τ fixes the quadratic form Q_τ under the usual action of $\mathbf{SL}_2(\mathbf{Z})$ on the set of binary quadratic forms. Furthermore, the simplifying assumption that $\gcd(D, N) = 1$ implies that $\gamma_\tau = \tilde{\gamma}_\tau$, where the latter matrix is taken to be the stabilizer of the form Q_τ in $\mathbf{SL}_2(\mathbf{Z})$.

For an indefinite binary quadratic form Q, we let γ_Q be a generator of its stabilizer in $\mathbf{SL}_2(\mathbf{Z})$. Note that Q takes on both positive and negative integer values, and that each value in the range of Q is taken on infinitely often, since Q is constant on γ_Q-orbits in $\mathbf{Z}^2$. Thus we set

$$\mathcal{W} := (\mathbf{Z}^2 - \{0\})/\langle \gamma_Q \rangle, \tag{9.2}$$

and let

$$\zeta_Q(s) = \sum_{(m,n) \in \mathcal{W}} \text{sign}(Q(m,n))|Q(m,n)|^{-s}, \quad \text{Re}(s) > 1, \tag{9.3}$$

where $\text{sign}(x) = \pm 1$ denotes the sign of a non-zero real number x. The function $\zeta_Q(s)$ extends by analytic continuation to a meromorphic function on $\mathbf{C}$ that is holomorphic except for a simple pole at $s = 1$.

With α as in (8.1) and (8.2), define

$$\zeta_\tau(s) := \zeta_{Q_\tau}(s), \qquad \zeta(\alpha, \tau, s) := \sum_{d \mid N} n_d d^s \zeta_{d\tau}(s). \tag{9.4}$$

The zeta functions $\zeta_{d\tau}(s)$ appearing in the definition of $\zeta(\alpha, \tau, s)$ are attached to the integral quadratic forms $Q_{d\tau}$, each of which has the same discriminant D. Note that $\zeta(\alpha, \tau, s)$ depends only on the $\Gamma_0(N)$-orbit of the element τ.

The main formula of this chapter is:

Theorem 9.1.1. *Suppose that τ belongs to $\mathcal{H}_p^\mathcal{O}$, and is normalized by the action of $\tilde{\Gamma}$ to satisfy (8.9). Then*

$$\zeta(\alpha, \tau, 0) = \frac{1}{12} \cdot \text{ord}_p(u(\alpha, \tau)).$$

Remark 9.1.2. Let $\mathbf{A}_K$ denote the ring of adeles of K. A finite order idèle class character

$$\chi = \prod_v \chi_v : \mathbf{A}_K^\times / K^\times \longrightarrow \mathbf{C}^\times$$

is called a *ring class character* if it is trivial on $\mathbf{A}_\mathbf{Q}^\times$. If χ is such a character, then its two archimedean components χ_{∞_1} and χ_{∞_2} attached to the real places of K are either both trivial or both equal to the sign character. In the former case χ is called *even* and in the latter, it is said to be *odd*. A ring class character can be interpreted as a character on the narrow Picard group $G_\mathcal{O} := \text{Pic}^+(\mathcal{O})$ of narrow ideal classes attached to a fixed order $\mathcal{O}$ of K whose conductor is equal to the conductor of χ.

The zeta functions $\zeta_\tau(s)$ with $\tau \in \mathcal{H}_p^{\mathcal{O}}$ can be interpreted as partial zeta functions encoding the zeta function of K twisted by odd ring class characters of $G_{\mathcal{O}}$. Letting τ_0 be an element of $\mathcal{H}_p^{\mathcal{O}}$ that is equivalent to $\sqrt{D}$ under the action of $\mathbf{SL}_2(\mathbf{Z})$, we have:

$$\sum_{\sigma \in G_{\mathcal{O}}} \chi(\sigma)\zeta_{\sigma \star \tau_0}(s) = \begin{cases} 0 & \text{if } \chi \text{ is even,} \\ L(K,\chi,s) & \text{if } \chi \text{ is odd.} \end{cases} \tag{9.5}$$

9.2 Dedekind sums

The line integral in (8.6) defining the partial modular symbol ψ can be expressed in terms of classical *Dedekind sums*

$$D\left(\frac{a}{m}\right) := \sum_{x=1}^{m-1} B_1\left(\frac{x}{m}\right) B_1\left(\frac{ax}{m}\right), \qquad \text{for } \gcd(a,m) = 1, \quad m > 0,$$

where

$$B_1(x) = \{x\} - 1/2 = x - [x] - 1/2$$

is the first Bernouilli polynomial made periodic. Rademacher proved:

$$\psi\left([\infty] - \left[\frac{a}{Nc}\right]\right) = \frac{1}{2\pi i} \int_\infty^{a/Nc} \mathrm{dlog}\,\alpha = -12 \sum_{d|N} n_d \cdot D\left(\frac{a}{Nc/d}\right). \tag{9.6}$$

The equation $\sum n_d d = 0$ simplifies the behavior of the Dedekind–Rademacher homomorphism, eliminating the extra terms appearing in Equation (2.1) of [32].

Meyer has proven a formula expressing the special values of partial zeta functions attached to real quadratic fields at $s = 0$ in terms of Dedekind sums (Cf. [42], equation (4.1) for a statement of Meyer's formula in the case where D is fundamental; the general case can be derived from equation (18) in §5 of [4], for example.) This formula can be used to derive the identity

$$\zeta(\alpha, \tau, 0) = -\sum_{d|N} n_d \cdot D\left(\frac{a}{Nc/d}\right), \tag{9.7}$$

where $\gamma_\tau = \begin{pmatrix} a & * \\ Nc & * \end{pmatrix}$. Equations (9.6) and (9.7) yield

$$\zeta(\alpha, \tau, 0) = \frac{1}{12} \cdot \psi\left([\infty] - [\gamma_\tau \infty]\right). \tag{9.8}$$

9.3 The p-adic valuation

Define a homomorphism $\psi_v : \mathcal{M} \to \mathbf{Z}$ for each vertex v of the tree $\mathcal{T}$ by assigning $\psi_{v^*} := \psi$ for the central vertex and extending by $\tilde{\Gamma}$-invariance:

$$\psi_{\gamma v}(m) = \psi_v(\gamma^{-1}m).$$

This is well defined since ψ is invariant under the stabilizer of v in $\tilde{\Gamma}$, namely $\Gamma_0(N)$. From (8.7), the measure of the open set $U_{e^*} = \mathbf{Z}_p$ under μ_m is $\psi_{v^*}(m) - \psi_{w^*}(m)$, where $w^* = P^{-1}v^*$, and e^* is the oriented edge from w^* to v^*. Thus by $\tilde{\Gamma}$-invariance, we have that

$$\mu_m(U_e) = \psi_{t(e)}(m) - \psi_{s(e)}(m)$$

for all oriented edges e of $\mathcal{T}$. Hence by Lemma 3.3.2, we see that

$$\operatorname{ord}_p \left(\oint_\tau^{\gamma\tau} \int_m \mathrm{dlog}\, \alpha \right) \;=\; \psi_{\gamma v^*}(m) - \psi_{v^*}(m)$$
$$=\; \psi(\gamma^{-1}m) - \psi(m).$$

We have thus shown

Proposition 9.3.1. *The chain* $\operatorname{ord}_p \rho_\tau \in C^0(\Gamma, \operatorname{Hom}(\mathcal{M}, \mathbf{Z}))$ *defined by* $\operatorname{ord}_p \rho_\tau(m) := \psi(m)$ *splits the cocycle* $\operatorname{ord}_p c_\tau$.

Since $u(\alpha, \tau) = \rho_\tau([\infty] - [\gamma_\tau \infty])$, Proposition 9.3.1 and (9.8) complete the proof of Theorem 9.1.1.

9.4 Values at negative integers

In this section, we generalize formula (9.8) by expressing $\zeta(\alpha, \tau, 1 - r)$ in terms of periods of certain Eisenstein series of weight $2r$, for odd $r \geq 1$.

The Eisenstein series of (8.4) and (8.5) are part of a natural *family* of Eisenstein series of varying weights. For even $k \geq 2$, consider the standard Eisenstein series of weight k:

$$E_k(z) = \frac{2(k-1)!}{(2\pi i)^k} \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^k} = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n. \tag{9.9}$$

Define likewise, as a function of the coefficients n_d used to define the modular unit α, the higher weight Eisenstein series

$$
\begin{aligned}
F_k(z) &= -24 \sum_{d|N} n_d \cdot d \cdot E_k(dz) \\
&= -\frac{48(k-1)!}{(2\pi i)^k} \sum_{m,n=-\infty}^{\infty}{}' \left(\frac{1}{(mz+n)^k} \sum_{d|(N,m)} n_d d \right) \\
&= -24 \sum_{n=1}^{\infty} \sigma_{k-1}(n) \sum_{d|N} n_d d q^{nd}.
\end{aligned}
\tag{9.10}
$$

The F_k are modular forms of weight k on $\Gamma_0(N)$, holomorphic on the upper half plane. These Eisenstein series have no constant term and hence are holomorphic at the cusp ∞. We also define, for the purposes of p-adic interpolation, the function

$$
F_k^*(z) = F_k(z) - p^{k-1} F_k(pz).
$$

We extend the definition of $E_k(z)$ and $F_k(z)$ to all $k \geq 2$ by letting $E_k = F_k = 0$ for k odd.

Recall the standard right action of $\mathbf{GL}_2^+(\mathbf{R})$ on the space of modular forms of weight k, given by

$$
F|_\gamma(z) = \frac{\det(\gamma)}{(cz+d)^k} F(\gamma z) \text{ when } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
$$

The definition of F_k^* can be written

$$
F_k^* = F_k - p^{k-2} F_k|_P(z), \quad \text{where } P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.
$$

The following proposition expresses $\zeta(\alpha, \tau, 1-r)$ in terms of periods of F_{2r}.

Proposition 9.4.1. *For all odd integers $r > 0$,*

$$
12 \cdot \zeta(\alpha, \tau, 1-r) = \int_{\infty}^{\gamma_\tau \infty} Q_\tau(z,1)^{r-1} F_{2r}(z) \, \mathrm{d}z.
$$

Proof. Let $k \geq 2$ be a positive integer and let $\widetilde{E}_k$ denote the weight k Eisenstein series

$$
\widetilde{E}_k(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^k}.
$$

By Hilfsatz 1 of [39], letting $z_0 \in \mathcal{H}$ be an arbitrary base point, the following identity holds for all integers $r > 1$:

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} \widetilde{E}_{2r}(z)\, \mathrm{d}z = (-1)^{r-1} \frac{(r-1)!^2}{(2r-1)!} D^{r-\frac{1}{2}} \sum_{\mathcal{W}} Q_\tau(m,n)^{-r},$$

with $\mathcal{W}$ as in (9.2). Suppose that $r > 1$ is an odd integer. Then

$$\sum_{\mathcal{W}} Q_\tau(m,n)^{-r} = \sum_{\mathcal{W}} \mathrm{sign}(Q_\tau(m,n))|Q_\tau(m,n)|^{-r} = \zeta_\tau(r),$$

so

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} \widetilde{E}_{2r}(z)\, \mathrm{d}z = \frac{(r-1)!^2}{(2r-1)!} D^{r-\frac{1}{2}} \zeta_\tau(r). \tag{9.11}$$

On the other hand, it follows from the relation (9.5) and from the functional equation for $L(K, \chi, s)$ for odd characters (cf. [28], Corollary 1 after Theorem 14 of §8, Chapter XIV) that $\zeta_\tau(s)$ satisfies the functional equation

$$\zeta_\tau(1-s) = \frac{D^{s-\frac{1}{2}}}{\pi^{2s-1}} \Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{2-s}{2}\right)^{-2} \zeta_\tau(s).$$

Hence if $r \geq 2$ is an even positive integer, $\zeta_\tau(1-r) = 0$, while if $r \geq 1$ is odd,

$$\zeta_\tau(1-r) = \frac{4D^{r-\frac{1}{2}}}{(2\pi)^{2r}} (r-1)!^2 \zeta_\tau(r).$$

Combining this functional equation with (9.11), we obtain

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} \widetilde{E}_{2r}(z)\, \mathrm{d}z = \frac{(2\pi)^{2r}}{4(2r-1)!} \zeta_\tau(1-r).$$

Since

$$E_k(z) = \frac{2(k-1)!}{(2\pi i)^k} \widetilde{E}_k(z),$$

it follows that

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} E_{2r}(z)\, \mathrm{d}z = -\frac{1}{2}\zeta_\tau(1-r). \tag{9.12}$$

From the definition of F_k, we have

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} F_{2r}(z)\, \mathrm{d}z = -24 \sum_{d|N} n_d \cdot d \cdot \int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} E_{2r}(dz)\, \mathrm{d}z.$$

Making the change of variables $w = d \cdot z$, we obtain

$$d \cdot \int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} E_{2r}(dz)\, \mathrm{d}z = \int_{d \cdot z_0}^{d \cdot \gamma_\tau z_0} Q_\tau \left(\frac{w}{d}, 1\right)^{r-1} E_{2r}(w)\, \mathrm{d}w.$$

Recall that $\tilde{\gamma}_{d\tau}$ denotes the generator of the stabilizer of $d\tau$ in $\mathbf{SL}_2(\mathbf{Z})$, chosen in such a way that $\gamma_{d\tau}$ is a positive power of $\tilde{\gamma}_{d\tau}$. We have

$$d\gamma_\tau z_0 = \tilde{\gamma}_{d\tau}(dz_0), \qquad Q_\tau\left(\frac{w}{d}, 1\right) = \frac{1}{d}Q_{d\tau}(w, 1).$$

Hence

$$d \cdot \int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1}E_{2r}(dz)\,\mathrm{d}z = \int_{dz_0}^{\tilde{\gamma}_{d\tau}(dz_0)} \frac{1}{d^{r-1}}Q_{d\tau}(w)^{r-1}E_{2r}(w)\,\mathrm{d}w.$$

The expression on the right is equal to $-\frac{d^{1-r}}{2}\zeta_{d\tau}(1-r)$, by (9.12). It follows that, for all odd $r > 1$,

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1}F_{2r}(z)\,\mathrm{d}z = 12\sum_{d|N} n_d d^{1-r}\zeta_{d\tau}(1-r) = 12 \cdot \zeta(\alpha, \tau, 1-r).$$

The integrand in the left-hand expression involves an Eisenstein series that is holomorphic at ∞, hence we may replace the base point $z_0 \in \mathcal{H}$ by the cusp ∞. $\qquad\square$

In the next chapter we will generalize Meyer's formula (corresponding to the case $r = 1$) by expressing the periods of F_{2k} in terms of generalized Dedekind sums.

9.5 Connection with the Gross–Stark conjecture

A general result of Deligne and Ribet (cf. the discussion in [16], §2) implies the existence of a p-adic meromorphic function $\zeta_p(\alpha, \tau, s)$ of the variable $s \in \mathbf{Z}_p$ characterized by its values on a dense set of negative integers:

$$\zeta_p(\alpha, \tau, n) = (1 - p^{-2n})\zeta(\alpha, \tau, n), \qquad \text{for all } n \leq 0, \quad n \equiv 0 \pmod{2(p-1)}. \qquad (9.13)$$

Let $U_{H,p}$ denote the group of p-units of H defined as in [16, Proposition 3.8]:

$$U_{H,p} := \{\epsilon \in H^\times : ||\epsilon||_{\mathfrak{D}} = 1 \text{ for all places } \mathfrak{D} \text{ that do not divide } p\}.$$

Since the places $\mathfrak{D}$ involved in the definition of $U_{H,p}$ include all the archimedean ones, it follows that $U_{H,p}$ is infinite only when H has no real embeddings, and that images of the elements of $U_{H,p}$ under all the complex embeddings of H lie on the unit circle.

Proposition 8.3.1 implies that the p-unit $u(\alpha, \tau)$ belongs to $U_{H,p}$ (assuming, of course, Conjecture 8.2.5). Theorem 9.1.1 states that

$$\mathrm{ord}_p(u(\alpha, \tau)) = 12 \cdot \zeta(\alpha, \tau, 0).$$

Thus the Gross-Stark conjecture, stated in Conjecture 2.12 of [16] (cf. the formulation given in Proposition 3.8 of [16]), suggests that one should have

$$\log_p \mathrm{N}_{K_p/\mathbf{Q}_p}(u(\alpha, \tau)) = -12 \cdot \zeta_p'(\alpha, \tau, 0). \tag{9.14}$$

In fact, the relation (9.14) is essentially *equivalent* (by varying α appropriately) to the Gross–Stark conjecture for H/K, assuming Conjecture 8.2.5. The next chapter is devoted to the explicit construction of $\zeta_p(\alpha, \tau, s)$ and to a proof of (9.14).

Chapter 10

A Kronecker limit formula

The first three sections of this chapter give an explicit construction of the p-adic zeta function $\zeta_p(\alpha, \tau, s)$ satisfying the interpolation property (9.13). The remainder of the chapter proves the main theorem of part II:

Theorem 10.0.1. *Suppose that τ belongs to $\mathcal{H}_p^{\mathcal{O}}$, and is normalized by the action of $\tilde{\Gamma}$ to satisfy (8.9). Then*

$$\zeta_p'(\alpha, \tau, 0) = -\frac{1}{12} \cdot \log_p \mathrm{N}_{K_p/\mathbf{Q}_p}(u(\alpha, \tau)).$$

Note the clear analogy between this formula and the classical Kronecker limit formula stated in [6]. Theorem 10.0.1 allows us to deduce the Gross–Stark conjecture for H/K from Conjecture 8.2.5. It should be pointed out that Conjecture 8.2.5 is stronger and more precise than Gross's conjecture in this setting, since it gives a formula for the Gross–Stark unit $u(\alpha, \tau)$ itself, and not just its "absolute value."

10.1 Measures associated to Eisenstein series

We recall the definitions of Section 6.3. Once again the key to splitting the cocycle c_τ is lifting μ to a bundle over $\mathbf{P}^1(\mathbf{Q}_p)$. Let

$$\mathbf{X} := (\mathbf{Z}_p \times \mathbf{Z}_p)' \subset (\mathbf{Q}_p \times \mathbf{Q}_p - \{0\})$$

(considered as column vectors) be the set of vectors $(x, y) \in \mathbf{Z}_p^2$ such that x and y are not both divisible by p in $\mathbf{Z}_p$. The space $\mathbf{Q}_p^2 - \{0\}$ is endowed with a natural action of Γ by left multiplication. There is a $\mathbf{Z}_p^\times$-bundle map

$$\pi : \mathbf{X} \to \mathbf{P}^1(\mathbf{Q}_p), \quad (x, y) \mapsto x/y.$$

The crucial technical ingredient in the construction of $\zeta_p(\alpha, \tau, s)$ and in the proof of Theorem 10.0.1 is the following result, which can be viewed as an extension of the construction of μ from F_2 to the family of Eisenstein series introduced in the previous section.

Theorem 10.1.1. *There is a unique $\tilde{\Gamma}$-invariant* $\mathrm{Meas}(\mathbf{Q}_p^2 - \{0\}, \mathbf{Z})$-*valued partial modular symbol $\tilde{\mu}$ satisfying the following properties:*

(1) *For every homogeneous polynomial $h(x, y) \in \mathbf{Z}_p[x, y]$ of degree $k - 2$ and every pair of cusps $r, s \in \Gamma\infty$,*

$$\int_{\mathbf{X}} h(x, y) \, d\tilde{\mu}_{[r]-[s]}(x, y) = \mathrm{Re}\left((1 - p^{k-2}) \int_r^s h(z, 1) F_k(z) \, dz\right). \qquad (10.1)$$

(2) *The measures $\tilde{\mu}_m$ are invariant under multiplication by p:*

$$\tilde{\mu}_m(pU) = \tilde{\mu}_m(U).$$

Furthermore this measure satisfies:

(3) *For every homogeneous polynomial $h(x, y) \in \mathbf{Z}_p[x, y]$ of degree $k - 2$,*

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} h(x, y) \, d\tilde{\mu}_{[r]-[s]}(x, y) = \mathrm{Re}\left(\int_r^s h(z, 1) F_k^*(z) \, dz\right).$$

The proof of Theorem 10.1.1 will be given at the end of this chapter (beginning with Section 10.4).

The following lemma shows how the measures $\tilde{\mu}_m$ are related to the measures μ_m.

Lemma 10.1.2. *For all compact opens $U \subset \mathbf{P}^1(\mathbf{Q}_p)$,*

$$\tilde{\mu}_m(\pi^{-1}U) = \mu_m(U) \qquad (10.2)$$

Proof. Define a partial modular symbol of measures $\pi_* \tilde{\mu}$ on $\mathbf{P}^1(\mathbf{Q}_p)$ by the rule

$$\pi_* \tilde{\mu}_m(U) := \tilde{\mu}_m(\pi^{-1}U).$$

To see that $\pi_* \tilde{\mu}$ is $\tilde{\Gamma}$-invariant, we calculate

$$\pi_* \tilde{\mu}_{\gamma m}(\gamma U) = \tilde{\mu}_{\gamma m}(\pi^{-1}(\gamma U)) = \tilde{\mu}_{\gamma m}(\gamma \pi^{-1}(U)),$$

where the last equality follows from the fact that both $\pi^{-1}(\gamma U)$ and $\gamma \pi^{-1}(U)$ are fundamental regions for the action of $\langle p \rangle$ on the inverse image of γU in $\mathbf{Q}_p^2 - \{0\}$. Hence

$$\pi_* \tilde{\mu}_{\gamma m}(\gamma U) = \tilde{\mu}_m(\pi^{-1}(U)) = \pi_* \tilde{\mu}_m(U).$$

Now, property (3) of Theorem 10.1.1 with $h(x, y) = 1$ implies that $\pi_* \tilde{\mu}$ agrees with μ on the compact open set $\mathbf{Z}_p$. Since the $\tilde{\Gamma}$-translates of $\mathbf{Z}_p$ form a basis of compact opens for $\mathbf{P}^1(\mathbf{Q}_p)$, we have $\pi_* \tilde{\mu} = \mu$. □

10.2 Construction of the p-adic L-function

The special values of $\zeta(\alpha, \tau, s)$ at certain even negative integers can be expressed in terms of the measure $\tilde{\mu}$ described in Section 10.1. For notational convenience we will write $\tilde{\mu}\{r \to s\}$ for $\tilde{\mu}_{[r]-[s]}$.

Lemma 10.2.1. *For all odd integers $r > 0$*

$$12(1 - p^{2r-2}) \cdot \zeta(\alpha, \tau, 1 - r) = \int_{\mathbf{X}} Q_\tau(x, y)^{r-1} \, \mathrm{d}\tilde{\mu}\{\infty \to \gamma_\tau \infty\}(x, y).$$

Proof. This follows directly from Proposition 9.4.1 and property (1) of Theorem 10.1.1. □

Suppose that the odd integer r is congruent to 1 modulo $p - 1$. By Lemma 10.2.1,

$$12(1 - p^{2r-2}) \cdot \zeta(\alpha, \tau, 1 - r) = \int_{\mathbf{X}} \langle Q_\tau(x, y) \rangle^{r-1} \, \mathrm{d}\tilde{\mu}\{\infty \to \gamma_\tau \infty\}(x, y), \qquad (10.3)$$

where for $x \in \mathbf{Z}_p^\times$, the expression $\langle x \rangle$ denotes the unique element in $1 + p\mathbf{Z}_p$ that differs multiplicatively from x by a $(p-1)$st root of unity. The advantage of the expression (10.3) is that it interpolates p-adically, expressing $\zeta(\alpha, \tau, 1 - r)$ with its Euler factor at p removed as a function of the p-adic variable r. This leads us to define

$$\zeta_p(\alpha, \tau, s) = \frac{1}{12} \int_{\mathbf{X}} \langle Q_\tau(x, y) \rangle^{-s} \, \mathrm{d}\tilde{\mu}\{\infty \to \gamma_\tau \infty\}(x, y)$$

for all $s \in \mathbf{Z}_p$. One recovers the p-adic L-function introduced in Section 9.5 that is uniquely characterized by the interpolation property (9.13).

In terms of this explicit definition of $\zeta_p(\alpha, \tau, s)$, we have

Lemma 10.2.2. *The derivative $\zeta_p'(\alpha, \tau, s)$ at $s = 0$ is given by*

$$\zeta_p'(\alpha, \tau, 0) = -\frac{1}{12} \int_{\mathbf{X}} \log_p \left(Q_\tau(x, y) \right) \mathrm{d}\tilde{\mu}\{\infty \to \gamma_\tau \infty\}(x, y).$$

Proof. This is a direct consequence of the definition. □

88

10.3 An explicit splitting of the one-cocycle

We now turn to the definition of the indefinite integral.

Proposition 10.3.1. *Let $\tilde{\mu}$ be as in Theorem 10.1.1. Then*

$$u_p \rho_\tau(m) := \oint_{\mathbf{X}} u_p(x - y\tau) \, \mathrm{d}\tilde{\mu}_m(x, y)$$

provides an explicit splitting of $u_p c_\tau$.

Proof. The proof follows the arguments of Section 6.3, with great simplification since the partial modular symbol $\tilde{\mu}$ is now $\tilde{\Gamma}$-invariant. Let $\gamma \in \tilde{\Gamma}$. Since $\gamma\mathbf{X}$ and $\mathbf{X}$ are both fundamental domains for multiplication by p on $\mathbf{Q}_p^2 - \{0\}$, and u_p is constant on cosets of $p^{\mathbf{Z}}$ in $K_p^\times$, it follows that

$$\oint_{\mathbf{X}} u_p(a - b\tau) \, \mathrm{d}\tilde{\mu}_{\gamma^{-1}m}(a, b) = \oint_{\mathbf{X}} u_p(a - b\tau) \, \mathrm{d}\tilde{\mu}_m(x, y)$$

where $\binom{a}{b} = \gamma^{-1}\binom{x}{y}$. Hence

$$\frac{\rho_\tau(\gamma^{-1}m)}{\rho_\tau(m)} = \oint_{\mathbf{X}} u_p\left(\frac{a - b\tau}{x - y\tau}\right) \mathrm{d}\tilde{\mu}_m(x, y). \tag{10.4}$$

Since $\left(\frac{a - b\tau}{x - y\tau}\right)$ is a constant multiple of $\left(\frac{x - y\gamma\tau}{x - y\tau}\right)$ and $\tilde{\mu}_m$ has total measure 0, equation (10.4) becomes

$$\oint_{\mathbf{X}} u_p\left(\frac{x - y\gamma\tau}{x - y\tau}\right) \mathrm{d}\tilde{\mu}_m(x, y) = \oint_{\mathbf{P}^1(\mathbf{Q}_p)} u_p\left(\frac{t - \gamma\tau}{t - \tau}\right) \mathrm{d}\mu_m(t)$$
$$= u_p c_\tau(\gamma)(m),$$

since the push forward of $\tilde{\mu}$ to $\mathbf{P}^1(\mathbf{Q}_p)$ is μ. $\quad\square$

We can now prove theorem 10.0.1:

Proof of Theorem 10.0.1. Since our chosen branch of $\log_p$ factors through u_p, Proposition 10.3.1 imples that

$$\log_p \mathrm{N}_{K_p/\mathbf{Q}_p}(u(\alpha, \tau)) = \log_p\left(\oint_{\mathbf{X}} u_p((x - y\tau)(x - y\tau')) \, \mathrm{d}\tilde{\mu}\{\infty \to \gamma_\tau\infty\}(x, y)\right)$$
$$= \int_{\mathbf{X}} \log_p Q_\tau(x, y) \, \mathrm{d}\tilde{\mu}\{\infty \to \gamma_\tau\infty\}(x, y), \tag{10.5}$$

where (10.5) follows since $Q_\tau(x, y)$ is proportional to $(x - \tau y)(x - \tau'y)$ and $\tilde{\mu}(\mathbf{X}) = 0$. The result now follows from Lemma 10.2.2. $\quad\square$

The remainder of this chapter is devoted to the proof of Theorem 10.1.1.

10.4 Generalized Dedekind sums

In this section we evaluate the integrals appearing in the right of (10.1) in Theorem 10.1.1, which characterize the partial modular symbol $\tilde{\mu}$. The computations of this section are not new, but we include them for completeness and notational consistency. Let f denote a modular form and let $a_f(0)$ denote the constant term of its q-expansion at ∞. For relatively prime integers a and c with $c \geq 1$, the function

$$A_f(s; a, c) = e^{\pi i s/2} c^{s-1} \int_0^\infty \left(f(it + a/c) - a_f(0) \right) t^{s-1} \, \mathrm{d}t$$

is well defined for $\operatorname{Re}(s)$ large enough, and has a meromorphic continuation to all of $\mathbf{C}$ (see [10]). For the Eisenstein series E_{2k} with $k > 1$, this continuation is given by

$$A_{E_{2k}}(s; a, c) = e^{\pi i s/2} c^{2k-2} \frac{\Gamma(s)}{(2\pi)^s} \sum_{h=1}^{c} \left[\zeta(s+1-2k, h/c) \sum_{m=1}^{\infty} \frac{1}{m^s} e^{2\pi i m h a/c} \right], \qquad (10.6)$$

where $\zeta(s, b)$ denotes the Hurwitz zeta function. This is a relatively standard computation, carried out for example in Proposition 3.1 of [10]. Let us calculate the real part of this expression for s an integer, $1 \leq s \leq 2k - 1$. When $s = 1$, the term for $h = c$ in (10.6) is taken to be

$$\lim_{s \to 1} \zeta(s + 1 - 2k)\zeta(s) \in \mathbf{R}.$$

The Hurwitz zeta function has the well known value $\zeta(1 - n, b) = -B_n(b)/n$. Furthermore, for a real number x,

$$\operatorname{Re}\left(\frac{i^s(s-1)!}{(2\pi)^s} \sum_{m=1}^{\infty} \frac{1}{m^s} e^{2\pi i m x} \right) = \frac{i^s(s-1)!}{2(2\pi)^s} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m^s} e^{2\pi i m x}$$

$$= \frac{(-1)^{s+1}}{2} \cdot \frac{\tilde{B}_s(x)}{s},$$

where

$$\tilde{B}_s(x) := \begin{cases} 0 & \text{if } s = 1 \text{ and } x \in \mathbf{Z} \\ B_s(\{x\}) = B_s(x - [x]) & \text{otherwise,} \end{cases}$$

and $B_s(x)$ is the standard Bernoulli polynomial of degree s (see [19, Section II]). We obtain

$$\operatorname{Re}\left(A_{E_{2k}}(s; a, c) \right) = c^{2k-2} \frac{(-1)^s}{2} \sum_{h=1}^{c} \frac{B_{2k-s}(h/c)}{2k-s} \cdot \frac{\tilde{B}_s(ha/c)}{s}. \qquad (10.7)$$

We would like to replace the term $B_{2k-s}(h/c)$ by $\tilde{B}_{2k-s}(h/c)$ in the sum above. Only the term for $h = c$, which we now consider, may cause difficulty. If s is even, then $B_{2k-s}(1) = B_{2k-s}(0)$ since in general one has

$$B_n(1 - x) = (-1)^n B_n(x).$$

If s is odd then the other term in the product is $\tilde{B}_s(a) = 0$. Thus in either case we may replace the term $B_{2k-s}(h/c)$ by $\tilde{B}_{2k-s}(h/c)$. This motivates the following definition.

Definition 10.4.1. Let $s, t \geq 0$. For a and c relatively prime and $c \geq 1$, the *generalized Dedekind sum* $\tilde{D}_{s,t}(a/c)$ is defined by

$$\tilde{D}_{s,t}(a/c) := c^{s-1} \sum_{h=1}^{c} \tilde{B}_s(h/c)\tilde{B}_t(ha/c).$$

The sum defining $\tilde{D}$ may be taken over any complete set of representatives $h \bmod c$. For $s, t \geq 1$, define

$$D_{s,t}(a/c) := \frac{\tilde{D}_{s,t}(a/c)}{st}.$$

Remark 10.4.2. When $s = t = 1$, we have

$$D_{1,1}(a/c) = \tilde{D}_{1,1}(a/c) = D(a/c),$$

where $D(a/c)$ is as in Section 9.2.

Equation (10.7) may be written in terms of generalized Dedekind sums as

$$\operatorname{Re}\left(A_{E_k}(s; a, c)\right) = c^{s-1}\frac{(-1)^s}{2}D_{k-s,s}(a/c). \tag{10.8}$$

This formula continues to hold when k is odd, since then the Dedekind sum $D_{k-s,s}(a/c)$ vanishes (using the relation $\tilde{B}_s(-x) = (-1)^s\tilde{B}_s(x)$.)

From the definition of F_k, we find

$$A_{F_k}(s; a, Nc) = -24\sum_{d|N} n_d A_{E_k}(s; a, Nc/d). \tag{10.9}$$

We are now ready to evaluate the integrals appearing in (10.1). Let $0 \leq n \leq k - 2$. Using the change of variables $z = it + a/(Nc)$, we have

$$\int_{a/Nc}^{i\infty} z^n F_k(z)\,\mathrm{d}z = \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc}\right)^{n-\ell} (Nc)^{-\ell} A_{F_k}(\ell+1; a, Nc)$$

$$= -24\sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc}\right)^{n-\ell} (Nc)^{-\ell} \sum_{d|N} n_d A_{E_k}\left(\ell+1; a, \frac{Nc}{d}\right). \tag{10.10}$$

In view of (10.8), the real part of (10.10) is equal to

$$12 \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc}\right)^{n-\ell} (-1)^{\ell} \sum_{d|N} n_d d^{-\ell} D_{k-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right). \tag{10.11}$$

As we now check, formula (10.11) remains valid for $k = 2$. In this case $n = 0$ and the desired formula simplifies to

$$\operatorname{Re}\left(\int_{a/Nc}^{i\infty} F_2(z)\,\mathrm{d}z\right) = 12 \sum_{d|N} n_d D_{1,1}\left(\frac{a}{Nc/d}\right),$$

which is nothing but equation (9.6) in light of Remark 10.4.2 and (8.3).

10.5   Measures on $\mathbf{Z}_p \times \mathbf{Z}_p$

Let $\xi = \frac{a}{Nc} \in \Gamma\infty$, and assume that p does not divide c. In this section we prove the following crucial lemma.

Lemma 10.5.1. *Let $\xi \in \Gamma\infty$ have denominator not divisible by p. There exists a unique $\mathbf{Z}_p$-valued measure ν_ξ on $\mathbf{Z}_p \times \mathbf{Z}_p$ such that for all $k \geq 2$,*

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p} h(x,y)\,\mathrm{d}\nu_\xi(x,y) = \operatorname{Re}\left((1 - p^{k-2}) \int_\xi^{i\infty} h(z,1) F_k(z)\,\mathrm{d}z\right) \tag{10.12}$$

for every homogeneous polynomial $h(x,y) \in \mathbf{Z}[x,y]$ of degree $k - 2$.

Equation (10.12) is equivalent to the statement that

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p} x^n y^m\,\mathrm{d}\nu_\xi(x,y) = \operatorname{Re}\left((1 - p^{n+m}) \int_{a/Nc}^{i\infty} z^n F_{n+m+2}(z)\,\mathrm{d}z\right)$$

$$= 12(1 - p^{n+m}) \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc}\right)^{n-\ell} (-1)^{\ell} \sum_{d|N} n_d d^{-\ell} D_{n+m-\ell+1,\ell+1}\left(\frac{a}{Nc/d}\right) \tag{10.13}$$

for all integers $n, m \geq 0$. Denote the last expression appearing in equation (10.13) by $I_{n,m} \in \mathbf{Q}$. The key tool in showing the existence and uniqueness of ν_ξ is the following result, which is the two-variable version of a classical theorem of Mahler (see Theorem 3.3.1 of [21]).

Lemma 10.5.2. *Let $b_{n,m} \in \mathbf{Z}_p$ be constants indexed by integers $n, m \geq 0$. There exists a unique $\mathbf{Z}_p$-valued measure ν on $\mathbf{Z}_p \times \mathbf{Z}_p$ such that*

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p} \binom{x}{n}\binom{y}{m}\,\mathrm{d}\nu(x,y) = b_{n,m}.$$

Thus to prove Lemma 10.5.1, we must show that the rational numbers

$$J_{n,m} := \sum_{i=0}^{n}\sum_{j=0}^{m} c_{n,i} c_{m,j} I_{i,j}$$

lie in $\mathbf{Z}_p$, where the integers $c_{n,i}$ are defined by the equation

$$\binom{x}{n} = \sum_{i=0}^{n} c_{n,i} x^{i}.$$

Our proof of this fact will follow the proof of the existence of p-adic Dirichlet L-functions, as in Section 3.4 of [21].

Consider the rightmost term appearing in the definition (10.13) of $I_{n,m}$ (here $k = n+m+2$):

$$
\begin{aligned}
d^{-\ell} D_{k-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right) &= d^{-\ell}\left(\frac{Nc}{d}\right)^{k-\ell-2}\sum_{h=1}^{Nc/d}\frac{\tilde{B}_{k-\ell-1}(\frac{h}{Nc/d})}{k-\ell-1}\frac{\tilde{B}_{\ell+1}(\frac{ha}{Nc/d})}{\ell+1}\\
&= \left(\frac{Nc}{d}\right)^{k-\ell-2}\sum_{h=1}^{Nc}\frac{\tilde{B}_{k-\ell-1}(\frac{h}{Nc/d})}{k-\ell-1}\frac{\tilde{B}_{\ell+1}(\frac{ha}{Nc})}{\ell+1}, \quad (10.14)
\end{aligned}
$$

where (10.14) follows from the distribution relation for Bernoulli polynomials. For each $h = 1,\ldots,Nc$, write $\theta = \{ha/Nc\}$. Let x be a formal variable and write $u = e^{x}$. Then the Bernoulli polynomials are given by the power series

$$\frac{u^{\theta}}{u-1} - \frac{1}{x} + F_h = \sum_{s=0}^{\infty}\frac{\tilde{B}_{s+1}(\frac{ha}{Nc})}{(s+1)!}x^{s}, \quad (10.15)$$

where $F_h = 1/2$ when $h = Nc$ and $F_h = 0$ otherwise (the error term F_h deals with the discrepancy between $\tilde{B}_1(0)$ and $B_1(0)$). Similarly, write $\beta_d = \{hd/Nc\}$, let y be a formal variable and write $v = e^{y}$; we then have

$$\sum_{d|N} n_d \frac{v^{\beta_d/d}}{v^{1/d}-1} + G_h = \sum_{t=0}^{\infty}\sum_{d|N} n_d \frac{\tilde{B}_{t+1}(\frac{hd}{Nc})}{(t+1)!}\left(\frac{y}{d}\right)^{t}, \quad (10.16)$$

where G_h is a constant in $\frac{1}{2}\mathbf{Z}$. Multiplying (10.15) and (10.16), and summing over all h, we obtain

$$
\begin{aligned}
H(u,v) &:= \sum_{h=1}^{Nc}\left(\sum_{d|N} n_d \frac{v^{\beta_d/d}}{v^{1/d}-1} + G_h\right)\left(\frac{u^{\theta}}{u-1} + F_h\right) \quad (10.17)\\
&= \sum_{s,t=0}^{\infty}\sum_{h=1}^{Nc}\sum_{d|N} n_d \frac{\tilde{B}_{s+1}(\frac{ha}{Nc})}{(s+1)!}\frac{\tilde{B}_{t+1}(\frac{hd}{Nc})}{(t+1)!}x^{s}\left(\frac{y}{d}\right)^{t}. \quad (10.18)
\end{aligned}
$$

The $-1/x$ terms from (10.15) have dropped out in (10.17) since summing (10.16) over all h gives the value 0. By the same reasoning, we may replace $\frac{u^\theta}{u-1}$ in equation (10.17) defining $H(u,v)$ by $\frac{u^\theta-1}{u-1}$ (this will be useful in later computations). Recalling that $u = e^x$ and $v = e^y$, we define the commuting differential operators

$$D_u = u\frac{\partial}{\partial u} = \frac{\partial}{\partial x} \text{ and } D_v = v\frac{\partial}{\partial v} = \frac{\partial}{\partial y}.$$

Using (10.14) and (10.18), we then have

$$(1 - p^{n+m})\sum_{d|N} n_d d^{-\ell}\frac{D_{n+m-\ell+1,\ell+1}\left(\frac{a}{Nc/d}\right)}{(n+m-\ell+1)(\ell+1)} =$$
$$(Nc)^{n+m-\ell}(D_u^\ell D_v^{n+m-\ell}H^*(u,v))|_{(u,v)=(1,1)},$$

where

$$H^*(u,v) := H(u,v) - H(u^p, v^p).$$

We thus find that

$$\begin{aligned} I_{n,m} &= 12\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\frac{a}{Nc}\right)^{n-\ell}(-1)^\ell(Nc)^{n+m-\ell}(D_u^\ell D_v^{n+m-\ell}H^*(u,v))|_{(1,1)} \\ &= (NcD_v)^m(aD_v - D_u)^n(12H^*(u,v))|_{(1,1)}. \end{aligned} \tag{10.19}$$

If we define a change of variables $(u,v) = (z^{-1}, w^{Nc}z^a)$, then $D_w = NcD_u$ and $D_z = aD_u - D_v$. Hence we obtain

$$J_{n,m} = \binom{D_w}{m}\binom{D_z}{n}(12H^*(u,v))|_{(1,1)}.$$

The following lemma will allow us to prove that these rational numbers lie in $\mathbf{Z}_p$.

Lemma 10.5.3. *Consider the subset R of $\mathbf{Z}_p(u^{1/Nc}, v^{1/Nc})$ defined by*

$$R := \left\{\frac{P}{Q} \text{ where } P, Q \in \mathbf{Z}_p[u^{1/Nc}, v^{1/Nc}] \text{ and } Q(1,1) \in \mathbf{Z}_p^\times\right\}.$$

Then R is a ring stable under the operators $\binom{D_w}{m}$ and $\binom{D_z}{n}$.

Proof. The proof of this proposition follows exactly as in Lemma 3.4.2 of [21], except for the subtlety that we must check that $\mathbf{Z}_p[u^{1/Nc}, v^{1/Nc}]$ is stable under the given differential operators; for this it suffices to check that for example

$$\binom{D_z}{n}(v^{1/Nc}) = \frac{z^n}{n!}\frac{\partial^n}{\partial z^n}(wz^{a/Nc}) = \binom{a/Nc}{n}wz^{a/Nc},$$

which lies in $\mathbf{Z}_p[u^{1/Nc}, v^{1/Nc}]$ because p does not divide Nc; similarly for the other cases. $\square$

Thus to prove that $J_{n,m} \in \mathbf{Z}_p$, it suffices to prove that $H^*(u,v)$ is an element of R, and for this it suffices to prove that $H(u,v) \in R$. Writing

$$\Psi_d(v) := 1 + v^{1/d} + \cdots + v^{(d-1)/d},$$

we have

$$
\begin{aligned}
\sum_{d|N} n_d \frac{v^{\beta_d/d}}{v^{1/d}-1} &= \frac{1}{v-1} \sum_{d|N} n_d v^{\beta_d/d} \Psi_d(v) \\
&= \frac{1}{\Psi_{Nc}(v)} \cdot \frac{\sum_{d|N} n_d v^{\beta_d/d} \Psi_d(v)}{v^{1/Nc}-1}.
\end{aligned}
\tag{10.20}
$$

Since the numerator of the rightmost term in (10.20) is a polynomial in $v^{1/Nc}$ that vanishes when $v^{1/Nc} = 1$, the rightmost term itself is a polynomial in $v^{1/Nc}$. Since we are assuming that p does not divide Nc, equation (10.20) then implies that

$$\sum_{d|N} n_d \frac{v^{\beta_d/d}}{v^{1/d}-1} \in R.$$

Similarly one shows that $\frac{u^\theta-1}{u-1} \in R$, and it follows that $H(u,v) \in R$. This concludes the proof of Lemma 10.5.1.

10.6   A partial modular symbol of measures on $\mathbf{Z}_p \times \mathbf{Z}_p$

In this section, we use the measures ν_ξ to construct a partial modular symbol of measures on $\mathbf{Z}_p \times \mathbf{Z}_p$ encoding the periods of F_k. Note that $\mathbf{Z}_p \times \mathbf{Z}_p$ is stable under the action of $\Gamma_0(N)$.

Lemma 10.6.1. *There exists a unique $\Gamma_0(N)$-invariant partial modular symbol ν of $\mathbf{Z}_p$-valued measures on $\mathbf{Z}_p \times \mathbf{Z}_p$ such that*

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p} h(x,y)\, d\nu\{r \to s\}(x,y) = \mathrm{Re}\left((1-p^{k-2}) \int_r^s h(z,1) F_k(z)\, dz \right) \tag{10.21}$$

for $r, s \in \Gamma\infty$, and every homogeneous polynomial $h(x,y) \in \mathbf{Z}[x,y]$ of degree $k-2$.

Proof. Uniqueness follows from Lemma 10.5.2; we must show existence. Let M denote the Γ-module of degree-zero divisors on the set $\Gamma\infty$. Let $M' \subset M$ be the set of divisors m for which there exists a $\mathbf{Z}_p$-valued measure ν_m on $\mathbf{Z}_p \times \mathbf{Z}_p$ such that

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p} h(x,y)\, d\nu_m(x,y) = \mathrm{Re}\left((1-p^{k-2}) \int_m h(z,1) F_k(z)\, dz \right).$$

We must show that $M' = M$.

It is clear that M' is a subgroup of M. We will show that M' is a $\Gamma_0(N)$-stable submodule. Let $m \in M'$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(N)$; for compact open $U \subset \mathbf{Z}_p \times \mathbf{Z}_p$ define

$$\nu_{\gamma m}(U) := \nu_m(\gamma^{-1}U).$$

Define a right action of $\Gamma_0(N)$ on the space of polynomials in two variables by

$$h|_\gamma(x, y) = h(Ax + By, Cx + Dy).$$

We calculate

$$
\begin{aligned}
\int_{\mathbf{Z}_p \times \mathbf{Z}_p} h(u, v)\, \mathrm{d}\nu_{\gamma m}(u, v) &= \int_{\mathbf{Z}_p \times \mathbf{Z}_p} h(u, v)\, \mathrm{d}\nu_m(\gamma^{-1}(u, v)) \\
&= \int_{\mathbf{Z}_p \times \mathbf{Z}_p} h|_\gamma(x, y)\, \mathrm{d}\nu_m(x, y) \\
&= \mathrm{Re}\left((1 - p^{k-2}) \int_m h|_\gamma(z, 1) F_k(z)\, \mathrm{d}z \right) \\
&= \mathrm{Re}\left((1 - p^{k-2}) \int_{\gamma m} h(u, 1) F_k(u)\, \mathrm{d}u \right), \qquad (10.22)
\end{aligned}
$$

where equation (10.22) uses the change of variables $u = \gamma z$ and the fact that $F_k|_{\gamma^{-1}} = F_k$. Therefore, M' is a $\Gamma_0(N)$-stable submodule of M. Lemma 10.5.1 shows that $[a/Nc] - [\infty]$ lies in M' when p does not divide c. Since the $\Gamma_0(N)$-module generated by these elements is all of M, we indeed have $M' = M$. Furthermore, the $\Gamma_0(N)$-invariance of ν follows from uniqueness and the calculation of (10.22) above. $\qquad \square$

10.7   From $\mathbf{Z}_p \times \mathbf{Z}_p$ to X

In this section we show that the measures $\nu\{x \to y\}$ of Lemma 10.6.1 are supported on the set $\mathbf{X} \subset \mathbf{Z}_p \times \mathbf{Z}_p$ of primitive vectors.

Lemma 10.7.1. *Let $r, s \in \Gamma\infty$ and let $k \geq 2$. We have*

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} h(x, y)\, \mathrm{d}\nu\{r \to s\}(x, y) = \mathrm{Re}\left(\int_r^s h(z, 1) F_k^*(z)\, \mathrm{d}z \right)$$

for every homogeneous polynomial $h(x, y) \in \mathbf{Z}[x, y]$ of degree $k - 2$.

Proof. The characteristic function of the open set $\mathbf{Z}_p \times \mathbf{Z}_p^\times$ is $\lim_{j \to \infty} y^{(p-1)p^j}$. For notational simplicity, let $g = (p-1)p^j$ throughout the remainder of this section. Then for $n, m \geq 0$ and $k = n + m + 2$, we have

$$
\int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} x^n y^m \, d\nu \left\{ \frac{a}{Nc} \to \infty \right\} (x, y)
$$

$$
= \lim_{j \to \infty} \int_{\mathbf{Z}_p \times \mathbf{Z}_p} x^n y^{m+g} \, d\nu \left\{ \frac{a}{Nc} \to \infty \right\} (x, y)
$$

$$
= \lim_{j \to \infty} 12(1 - p^{k+g-2}) \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc} \right)^{n-\ell} (-1)^\ell \times
$$

$$
\sum_{d|N} \frac{n_d}{d^\ell} D_{k+g-\ell-1,\ell+1} \left(\frac{a}{Nc/d} \right)
$$

$$
= 12 \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc} \right)^{n-\ell} (-1)^\ell \sum_{d|N} \frac{n_d}{d^\ell} \lim_{j \to \infty} D_{k+g-\ell-1,\ell+1} \left(\frac{a}{Nc/d} \right). \tag{10.23}
$$

Meanwhile we calculate

$$
\mathrm{Re} \left(\int_{\frac{a}{Nc}}^{i\infty} z^n F_k^*(z) \, dz \right)
$$

$$
= \mathrm{Re} \left(\int_{\frac{a}{Nc}}^{i\infty} z^n F_k(z) \, dz - p^{k-n-2} \int_{\frac{pa}{Nc}}^{i\infty} z^n F_k(z) \, dz \right)
$$

$$
= 12 \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc} \right)^{n-\ell} (-1)^\ell \times
$$

$$
\sum_{d|N} \frac{n_d}{d^\ell} \left[D_{k-\ell-1,\ell+1} \left(\frac{a}{Nc/d} \right) - p^{k-\ell-2} D_{k-\ell-1,\ell+1} \left(\frac{pa}{Nc/d} \right) \right]. \tag{10.24}
$$

The following lemma implies that (10.23) and (10.24) are equal, and finishes the proof. $\square$

Lemma 10.7.2. *Let $s, t \geq 0$. For a rational number x, we have in $\mathbf{Q}_p$:*

$$
\lim_{j \to \infty} \tilde{D}_{s+g,t}(x) = \tilde{D}_{s,t}(x) - p^{s-1} \tilde{D}_{s,t}(px). \tag{10.25}
$$

Proof. This essentially follows from the generalized Kummer congruences for Bernoulli polynomials. Let $x = a/c$ and assume first that p does not divide c. Let b denote an integer such that $abp \equiv 1 \pmod{c}$. Then

$$
\tilde{D}_{s,t}(a/c) = c^{s-1} \sum_{\ell=1}^{c} \tilde{B}_s(\ell bp/c) \tilde{B}_t(\ell/c). \tag{10.26}
$$

Similarly

$$\tilde{D}_{s+g,t}(a/c) = c^{s+g-1} \sum_{\ell=1}^{c} \tilde{B}_{s+g}(\ell bp/c)\tilde{B}_t(\ell/c)$$

and

$$\tilde{D}_{s,t}(pa/c) = c^{s-1} \sum_{\ell=1}^{c} \tilde{B}_s(\ell b/c)\tilde{B}_t(\ell/c).$$

Write $y = \{\ell bp/c\}$ and $y' = \{\ell b/c\}$. Since $c^g \to 1$, it suffices to prove that

$$\lim_{j\to\infty} B_{s+g}(y) = B_s(y) - p^{s-1}B_s(y').$$

For $s > 0$, this follows from the proof of Theorem 3.2 of [41], which applies for our purposes even in the case $s \equiv 0 \pmod{p-1}$. For $s = 0$, the desired equality follows from the fact that the p-adic L-function $L_p(s,\chi)$ for a Dirichlet character χ is analytic at $s = 1$ unless $\chi = 1$, in which case L_p has a simple pole with residue $1 - 1/p$. This completes the proof for the case $x \in \mathbf{Z}_p$.

We now handle the case $x \notin \mathbf{Z}_p$. From equation (10.26), one sees that

$$\tilde{D}_{s,t}(a/c) = c^{s-t}\tilde{D}_{t,s}(bp/c).$$

Thus the result proved above is that

$$\lim_{j\to\infty} \tilde{D}_{t,s+g}(bp/c) = \tilde{D}_{t,s}(bp/c) - p^{s-1}\tilde{D}_{t,s}(b/c) \tag{10.27}$$

whenever $p \nmid c$. By switching indices in a similar fashion, equation (10.25) for $x = a/bp$ becomes

$$\lim_{j\to\infty} (bp)^{s+g-t}\tilde{D}_{t,s+g}(c/bp) = (bp)^{s-t}\tilde{D}_{t,s}(c/bp) - p^{s-1}b^{s-t}\tilde{D}_{t,s}(c/b) \tag{10.28}$$

where $ac \equiv 1 \pmod{bp}$. We will reduce equation (10.28) to equation (10.27) by means of the reciprocity law for these generalized Dedekind sums, given in Theorem 2 of [19]. Let $b > 0$; the reciprocity law then states

$$b^{s-t}\tilde{D}_{t,s}(c/b) = \text{sign}(c)\sum_{\ell=0}^{t} \frac{s}{s+\ell}\binom{t}{\ell}(-1)^{s+\ell}b^{-\ell}c^{s-t+\ell}\tilde{D}_{t-\ell,s+\ell}(b/c) \tag{10.29}$$

$$+ \sum_{\sigma=0}^{s+t} \frac{\binom{s+t-\sigma-1}{t-1}\binom{s+t}{\sigma}}{\binom{s+t}{t}}(-1)^\sigma b^{\sigma-t}c^{s-\sigma}\tilde{D}_{t+s-\sigma,\sigma}(0) \tag{10.30}$$

$$+ \begin{cases} -\text{sign}(c)/4 & \text{if } s = t = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The sum in (10.29) is taken to be 0 if $s = 0$. We will call the terms in the sum on line (10.29) "type I" terms and those on line (10.30) "type II" terms. Using the Dedekind reciprocity law on each of the terms in (10.28), one easily checks that the desired limit holds for the type I terms by (10.27). The same is true for each of the type II terms with $\sigma = 0, \dots, s+t$. To conclude the proof, one checks that each of the type II terms for $\sigma = s+t+1, \dots, s+t+g$ arising from the reciprocity law for $(bp)^{s+g-t}\tilde{D}_{t,s+g}(c/bp)$ has ord_p greater than $\mathrm{ord}_p(g)$ minus some constant depending only on s and t. Thus in the limit, the sum of these terms vanishes. $\qquad\square$

We can now prove:

Lemma 10.7.3. *The measures $\nu\{r \to s\}$ are supported on $\mathbf{X}$.*

Proof. Let $\gamma \in \Gamma_0(N)$. As in (10.22) above, we calculate for a homogeneous polynomial $h(x, y)$ of degree $k - 2$,

$$\int_{\gamma(\mathbf{Z}_p \times \mathbf{Z}_p^\times)} h(x, y) \, \mathrm{d}\nu\{r \to s\}(x, y) = \mathrm{Re}\left(\int_r^s h(z, 1) F_k^*|_{\gamma^{-1}}(z) \, \mathrm{d}z \right). \tag{10.31}$$

Let $\{\gamma_i\}_{i=1}^{p+1}$ be a set of left coset representatives for $\Gamma_0(N)/\Gamma_0(Np)$. Then

$$\bigcup_{i=1}^{p+1} \gamma_i(\mathbf{Z}_p \times \mathbf{Z}_p^\times)$$

is a degree p cover of $\mathbf{X}$. Hence from (10.31) we find that

$$p \int_{\mathbf{X}} h(x, y) \, \mathrm{d}\nu\{r \to s\}(x, y) = \sum_{i=1}^{p+1} \mathrm{Re}\left(\int_r^s h(z, 1) F_k^*|_{\gamma_i^{-1}}(z) \, \mathrm{d}z \right). \tag{10.32}$$

Now

$$\begin{aligned} \sum_{i=1}^{p+1} F_k^*|_{\gamma_i^{-1}} &= \sum_{i=1}^{p+1} \left(F_k|_{\gamma_i^{-1}} - p^{k-2} F_k|_{P\gamma_i^{-1}} \right) \\ &= (p+1)F_k - T_p F_k = (p - p^{k-1})F_k, \end{aligned}$$

since F_k is evidently an eigenform for T_p with eigenvalue $1 + p^{k-1}$. Thus (10.32) becomes

$$\int_{\mathbf{X}} h(x, y) \, \mathrm{d}\nu\{r \to s\}(x, y) = \mathrm{Re}\left((1 - p^{k-2}) \int_r^s h(z, 1) F_k(z) \, \mathrm{d}z \right).$$

Therefore, the integral on $\mathbf{X}$ of every polynomial $h(x, y)$ equals the integral on $\mathbf{Z}_p \times \mathbf{Z}_p$ of $h(x, y)$; this implies that the measure $\nu\{r \to s\}$ is supported on $\mathbf{X}$. $\qquad\square$

10.8 The measures $\tilde{\mu}$ and Γ-invariance

The compact open set $\mathbf{X}$ is a fundamental domain for the action of multiplication by p on $\mathbf{Q}_p^2 - \{0\}$. Hence if we define for compact open $U \subset \mathbf{X}$:

$$\tilde{\mu}\{r \to s\}(U) := \nu\{r \to s\}(U),$$

then $\tilde{\mu}$ extends uniquely to a $\Gamma_0(N)$-invariant partial modular symbol of $\mathbf{Z}_p$-valued measures on $\mathbf{Q}_p^2 - \{0\}$ that is invariant under the action of multiplication by p:

$$\tilde{\mu}\{r \to s\}(pU) = \tilde{\mu}\{r \to s\}(U)$$

for all compact open $U \subset \mathbf{Q}_p^2 - \{0\}$. Lemmas 10.6.1, 10.7.1, and 10.7.3 show that $\tilde{\mu}$ satisfies properties (1) and (3) of Theorem 10.1.1. Furthermore, property (2) is satisfied by construction. Thus to complete the proof of Theorem 10.1.1, it remains to show that the partial modular symbol of measures $\tilde{\mu}$ is $\tilde{\Gamma}$-invariant and $\mathbf{Z}$-valued.

Lemma 10.8.1. *The partial modular symbol $\tilde{\mu}$ is invariant under $\tilde{\Gamma}$.*

Proof. Since $\tilde{\Gamma}$ is generated by $\Gamma_0(N)$ and $P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, it suffices to show that $\tilde{\mu}$ is invariant for the action of P. For a homogeneous polynomial $h(x,y)$ of degree $k - 2$, we have

$$\int_{\mathbf{X}} h(x,y)\,\mathrm{d}\tilde{\mu}\{P^{-1}r \to P^{-1}s\}(P^{-1}(x,y))$$
$$= \int_{P^{-1}\mathbf{X}} h(pu, v)\,\mathrm{d}\tilde{\mu}\left\{\frac{r}{p} \to \frac{s}{p}\right\}(u, v). \tag{10.33}$$

Writing $P^{-1}\mathbf{X}$ as a disjoint union

$$P^{-1}\mathbf{X} = (\mathbf{Z}_p \times \mathbf{Z}_p^\times) \bigsqcup \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} (\mathbf{Z}_p^\times \times p\mathbf{Z}_p)$$

and using the invariance of $\tilde{\mu}$ under multiplication by p, (10.33) becomes

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} h(pu, v)\,\mathrm{d}\tilde{\mu}\left\{\frac{r}{p} \to \frac{s}{p}\right\}(u, v) + \int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} h(u, v/p)\,\mathrm{d}\tilde{\mu}\left\{\frac{r}{p} \to \frac{s}{p}\right\}(u, v).$$

By the homogeneity of h, one simplifies the above expression:

$$p^{2-k} \int_{\mathbf{X}} h(pu, v) \, d\tilde{\mu} \left\{ \frac{r}{p} \to \frac{s}{p} \right\} (u, v)$$

$$+ (1 - p^{2-k}) \int_{\mathbf{Z}_p \times \mathbf{Z}_p^{\times}} h(pu, v) \, d\tilde{\mu} \left\{ \frac{r}{p} \to \frac{s}{p} \right\} (u, v)$$

$$= \operatorname{Re} \left(p^{2-k} (1 - p^{k-2}) \int_{\frac{r}{p}}^{\frac{s}{p}} h(pz, 1) F_k(z) \, dz + \right.$$

$$\left. (1 - p^{2-k}) \int_{\frac{r}{p}}^{\frac{s}{p}} h(pz, 1) F_k^*(z) \, dz \right)$$

$$= \operatorname{Re} \left((p^{2-k} - 1) \int_{\frac{r}{p}}^{\frac{s}{p}} h(pz, 1) p^{k-1} F_k(pz) \, dz \right) \qquad (10.34)$$

$$= \operatorname{Re} \left((1 - p^{k-2}) \int_{r}^{s} h(u, 1) F_k(u) \, du \right), \qquad (10.35)$$

where (10.34) uses the definition of F_k^* and (10.35) uses the change of variables $u = pz$. Since this equals the integral over $\mathbf{X}$ of $h(x, y)$ against the measure $\tilde{\mu}\{r \to s\}$, we find that $\tilde{\mu}$ is indeed invariant for the action of P. $\qquad \square$

10.9 Integrality of the measures

In this section we prove that the measures $\tilde{\mu}_m$, which *a priori* are only $\mathbf{Z}_p$-valued, actually take on integer values. We begin by reviewing the single-variable measures arising from Bernoulli polynomials. Let $e \geq 1$ be a positive integer divisible by N but not by p, and let

$$Z = \varprojlim \mathbf{Z}/ep^n \mathbf{Z} \cong \mathbf{Z}/e\mathbf{Z} \times \mathbf{Z}_p.$$

For each integer $k \geq 1$, we define a $\mathbf{Z}_p$-valued measure $\mathcal{F}_k$ on Z corresponding to the Eisenstein series F_{2k} by the rule

$$\mathcal{F}_k(a + ep^n \cdot Z) := \sum_{d|N} n_d \left(\frac{ep^n}{d} \right)^{k-1} \cdot \frac{1}{k} \cdot \tilde{B}_k \left(\frac{a}{ep^n/d} \right)$$

for each integer a. The distribution relation for Bernoulli polynomials demonstrates that this is indeed a distribution for each $k \geq 1$. Furthermore, the proof of the following proposition shows that these measures are actually $\mathbf{Z}_p$-valued. For $x \in Z$, let x_p denote the projection of x onto $\mathbf{Z}_p$.

Proposition 10.9.1. *For every compact open set $U \subset Z$ we have*

$$\mathcal{F}_k(U) = \int_U x_p^{k-1} \, d\mathcal{F}_1(x).$$

Proof. It suffices to consider U of the form $U = a + ep^n Z$ for integers a. We will prove that

$$\mathcal{F}_k(U) \equiv a^{k-1} \mathcal{F}_1(U) \pmod{p^{n-\epsilon} \mathbf{Z}_p} \tag{10.36}$$

where ϵ depends only on k. The key fact is that the Bernoulli polynomial $B_k(x)$ begins $x^k - \frac{1}{2}kx^{k-1} + \cdots$. Therefore

$$\mathcal{F}_k(U) \equiv \sum_{d|N} n_d \left(\frac{ep^n}{d} \right)^{k-1} \frac{1}{k} \left(\left(\frac{da}{ep^n} - \left[\frac{da}{ep^n} \right] \right)^k - \frac{k}{2} \left(\frac{da}{ep^n} - \left[\frac{da}{ep^n} \right] \right)^{k-1} \right) \tag{10.37}$$

modulo $p^{n-\epsilon} \mathbf{Z}_p$, where ϵ is the largest power of p appearing in the denominators of the coefficients of $B_k(x)/k$. The congruence (10.37) yields:

$$\begin{aligned}
\mathcal{F}_k(U) &\equiv \sum_{d|N} \frac{n_d}{k} \left(\left(\frac{d}{ep^n} \right) a^k - ka^{k-1} \left[\frac{da}{ep^n} \right] - \frac{k}{2} a^{k-1} \right) \pmod{p^{n-\epsilon} \mathbf{Z}_p} \\
&\equiv -\sum_{d|N} n_d a^{k-1} \left[\frac{da}{ep^n} \right], \tag{10.38}
\end{aligned}$$

where (10.38) uses $\sum n_d = \sum n_d d = 0$. Meanwhile we find

$$a^{k-1} \mathcal{F}_1(U) = a^{k-1} \sum_{d|n} n_d \left(\frac{d}{ep^n} - \left[\frac{da}{ep^n} \right] \right) = -\sum_{d|N} n_d a^{k-1} \left[\frac{da}{ep^n} \right]. \tag{10.39}$$

Equations (10.38) and (10.39) yield (10.36), proving the proposition. $\qquad\square$

The measures $\mathcal{F}_k$ may be used to calculate the modular symbol of measures $\tilde{\mu}$. Let the fraction a/Nc be fixed; we will write ν for $\tilde{\mu}\{a/Nc \to \infty\}$.

Let V be a compact open subset of $\mathbf{Z}_p^\times$, and let $f_i = \sum_{n=0}^{d_i} c_n(i) y^n$ be a sequence of polynomials such that $\lim_{i \to \infty} f_i(x)$ is the characteristic function of V. Then equation (10.13) for the moments of ν yields

$$\nu(\mathbf{Z}_p \times V) = \lim_{i \to \infty} 12(1 - p^{d_i}) \sum_{d|N} n_d \sum_{n=0}^{d_i} c_n(i) \cdot D_{n+1,1} \left(\frac{a}{Nc/d} \right). \tag{10.40}$$

Now as in (10.14), we have

$$D_{n+1,1} \left(\frac{a}{Nc/d} \right) = \left(\frac{Nc}{d} \right)^n \sum_{h=1}^{Nc} \frac{\tilde{B}_{n+1}\left(\frac{h}{Nc/d} \right)}{n+1} \cdot \tilde{B}_1 \left(\frac{ha}{Nc} \right).$$

Hence (10.40) becomes

$$\nu(\mathbf{Z}_p \times V) = \lim_{i \to \infty} 12 \sum_{h=1}^{Nc} \tilde{B}_1 \left(\frac{ha}{Nc} \right) \sum_{n=0}^{d_i} \sum_{d|N} n_d \left(\frac{Nc}{d} \right)^n \frac{\tilde{B}_{n+1}\left(\frac{h}{Nc/d} \right)}{n+1} c_n(i). \qquad (10.41)$$

Write $Nc = ep^r$ with p not dividing e. Then N divides e, and in terms of the measure $\mathcal{F}_1$ above we have

$$\sum_{n=0}^{d_i} \sum_{d|N} n_d \left(\frac{Nc}{d} \right)^n \frac{\tilde{B}_{n+1}\left(\frac{h}{Nc/d} \right)}{n+1} c_n(i) = \int_{h+ep^r Z} f_i(x_p) \, d\mathcal{F}_1(x)$$

by Proposition 10.9.1. Therefore (10.41) becomes

$$\nu(\mathbf{Z}_p \times V) = 12 \sum_{h=1}^{Nc} \tilde{B}_1 \left(\frac{ha}{Nc} \right) \mathcal{F}_1(\{x \in h + ep^r Z : x_p \in V\}).$$

Let us now specify V of the form $V = b + p^s \mathbf{Z}_p$, with $s \geq r$ and $b \in \mathbf{Z}_p^\times$. Then

$$\nu(\mathbf{Z}_p \times V) = 12 \sum_{\substack{h=1 \\ h \in b + p^r \mathbf{Z}_p}}^{Nc} \tilde{B}_1 \left(\frac{ha}{ep^r} \right) \sum_{d|N} n_d \tilde{B}_1 \left(\frac{y}{ep^s/d} \right) \qquad (10.42)$$

where y is an integer such that $y \equiv h \pmod{e}$ and $y \equiv b \pmod{p^s}$. Fixing one such y for each h we obtain

$$\begin{aligned} \nu(\mathbf{Z}_p \times V) &= -12 \sum_{\substack{h=1 \\ h \in b + p^r \mathbf{Z}_p}}^{Nc} \tilde{B}_1 \left(\frac{ha}{ep^r} \right) \sum_{d|N} n_d \left[\frac{y}{ep^s/d} \right] \qquad (10.43) \\ &\equiv -12 \frac{a}{Nc} \sum_{\substack{h=1 \\ h \in b + p^r \mathbf{Z}_p}}^{Nc} h \sum_{d|N} n_d \left[\frac{y}{ep^s/d} \right] \pmod{\mathbf{Z}}, \end{aligned}$$

where (10.43) uses $\sum n_d = \sum n_d d = 0$. Hence to prove integrality, it suffices to consider the case $a = 1$. For this purpose, we return to (10.42) with $a = 1$ and rewrite the expression in terms of a generalized Dedekind sum:

$$\nu(\mathbf{Z}_p \times V) = 12 \sum_{d|N} n_d \sum_{\substack{h=1 \\ h \in b + p^r \mathbf{Z}_p}}^{ep^r/d} \tilde{B}_1 \left(\frac{h}{ep^r/d} \right) \tilde{B}_1 \left(\frac{y}{ep^s/d} \right). \qquad (10.44)$$

The inner sum is the generalized Dedekind Sum denoted $C(1, 1, p^{s-r}, e/d, \frac{ke/d}{p^s}, 0)$ in [19], where k is an integer chosen so that $ke/d \equiv b \pmod{p^s}$. The reciprocity law governing

these Dedekind sums ([19, Theorem 2]) shows that this value equals

$$\sum_{\substack{h=1 \\ h \in b + p^r \mathbf{Z}_p}}^{ep^r/d} \tilde{B}_1\left(\frac{h}{ep^r/d}\right) \tilde{B}_1\left(\frac{y}{ep^s/d}\right) = \frac{(e/d)^2}{2p^{s-r}} - \sum_{i=1}^{p^{s-r}} \tilde{B}_1\left(\frac{i}{p^{s-r}}\right) \tilde{B}_1\left(\frac{(ip^r - k)e/d}{p^{s-r}}\right)$$

$$+ \frac{p^{s-r}}{2e/d}\tilde{B}_2\left(-\frac{b}{p^r}\right) + \frac{p^{s-r}}{2e/d}\tilde{B}_2\left(\frac{b}{p^s}\right) \tag{10.45}$$

$$- \tilde{B}_1\left(\frac{b}{p^s}\right). \tag{10.46}$$

Using this expression in equation (10.44), the terms from lines (10.45) and (10.46) vanish since $\sum n_d d = 0$. The remaining line yields only terms in $\mathbf{Z}[1/p]$. Since we know that ν is $\mathbf{Z}_p$-valued, we thus conclude that $\nu(\mathbf{Z}_p \times V) \in \mathbf{Z}$. Since the $\tilde{\Gamma}$ translates of the sets $\mathbf{Z}_p \times V$ form a basis of compact opens for $\mathbf{Q}_p^2 - \{0\}/p^{\mathbf{Z}} \cong \mathbf{X}$, the $\tilde{\Gamma}$-invariance of $\tilde{\mu}$ therefore implies:

Proposition 10.9.2. *The modular symbol of measures $\tilde{\mu}$ is $\mathbf{Z}$-valued.*

This completes the proof of Theorem 10.1.1.

Chapter 11

Computations

The formulas of Chapter 10 may be used to calculate the units $u(\alpha, \tau)$ to a high p-adic accuracy. In this chapter we describe some computations that supply empirical evidence for conjecture 8.2.5.

11.1   Method

As in Chapter 10, let K denote a real quadratic field in which p is inert, so that the completion K_p is the quadratic unramified extension of $\mathbf{Q}_p$. Let β be a primitive $(p^2 - 1)$st root of unity in $K_p^\times$, and let $\log_\beta$ denote the discrete logarithm with base β:

$$\log_\beta : K_p^\times \to \mathbf{Z}/(p^2 - 1)\mathbf{Z}, \text{ where } \frac{x}{p^{\operatorname{ord}_p(x)}\beta^{\log_\beta(x)}} \in 1 + p\mathcal{O}_{K_p}.$$

We then have the decomposition

$$K_p^\times \cong \mathbf{Z} \times \mathbf{Z}/(p^2 - 1)\mathbf{Z} \times \mathcal{O}_{K_p} \text{ given by } x \mapsto (\operatorname{ord}_p(x), \log_\beta(x), \log_p(x)).$$

For $x = u(\alpha, \tau)$ and $\gamma_\tau = \begin{pmatrix} a & * \\ Nc & * \end{pmatrix}$, these three components are given by the formulas

$$\operatorname{ord}_p(u(\alpha, \tau)) \quad = \quad -12 \sum_{d \mid N} n_d \cdot D\left(\frac{a}{Nc/d}\right), \tag{11.1}$$

$$\log_\beta(u(\alpha, \tau)) \quad = \quad \int_{\mathbf{X}} \log_\beta(x - y\tau) \, \mathrm{d}\tilde{\mu}\{\infty \to a/Nc\}(x, y), \tag{11.2}$$

$$\log_p(u(\alpha, \tau)) \quad = \quad \int_{\mathbf{X}} \log_p(x - y\tau) \, \mathrm{d}\tilde{\mu}\{\infty \to a/Nc\}(x, y). \tag{11.3}$$

The computations of (11.1) and (11.2) are easy to execute in practice (note that for (11.2) it suffices to take a cover of $\mathbf{X}$ in which x and y are determined modulo p), so we only elaborate upon the computation of (11.3).

Suppose we are content to calculate (11.3) to an accuracy of M p-adic digits. Let $m = [\infty] - [a/Nc] \in \mathcal{M}$. Then $\log_p(u(\alpha, \tau))$ equals

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p^{\times}} \log_p(y)\, \mathrm{d}\tilde{\mu}_m(x, y) + \int_{\mathbf{Z}_p \times \mathbf{Z}_p^{\times}} \log_p\left(\frac{x}{y} - \tau\right) \mathrm{d}\tilde{\mu}_m(x, y) \tag{11.4}$$

$$+ \int_{\mathbf{Z}_p^{\times} \times p\mathbf{Z}_p} \log_p(x)\, \mathrm{d}\tilde{\mu}_m(x, y) + \int_{\mathbf{Z}_p^{\times} \times p\mathbf{Z}_p} \log_p\left(1 - \frac{y\tau}{x}\right) \mathrm{d}\tilde{\mu}_m(x, y). \tag{11.5}$$

The first term of (11.4) is independent of τ. To evaluate this term to an accuracy of p^M, one finds a polynomial $f(y)$ that is congruent to $\log_p(y)$ modulo p^M for all $y \in \mathbf{Z}_p^{\times}$. This can be done as follows. For each $i = 1, \dots, p - 1$, let

$$g_i(y) = \prod_{\substack{j=1 \\ j \neq i}}^{p-1} (y - j)^M,$$

and let $h_i(y)$ denote the power series of $\log_p(y)/g_i(y)$ on the residue disc $i + p\mathbf{Z}_p$, truncated after $M + \log M$ terms (the extra $\log M$ terms account for the denominators divisible by powers of p in the power series of $\log_p$). Then letting $f_i(y) = g_i(y)h_i(y)$ and $f(y) = \sum_{i=1}^{p-1} f_i(y)$ produces the desired polynomial; it has degree $p(M + \log M)$. The first term of (11.4) may then be evaluated be replacing $\log_p(y)$ by $f(y)$ and using (10.24) to evaluate the integral of y^n on $\mathbf{Z}_p \times \mathbf{Z}_p^{\times}$ against the measure $\tilde{\mu}_m$.

The second term of (11.4) may be recognized as a push forward from $\mathbf{X}$ to $\mathbf{P}^1(\mathbf{Q}_p)$, and equals

$$\int_{\mathbf{Z}_p} \log_p(t - \tau)\, \mathrm{d}\mu_m(t) = \sum_{i=0}^{p-1} \int_{i+p\mathbf{Z}_p} \log_p(t - i + (i - \tau))\, \mathrm{d}\mu_m(t)$$

$$= \sum_{i=0}^{p-1} \left[\log_p(\tau - i)\mu_m(i + p\mathbf{Z}_p) + \int_{i+p\mathbf{Z}_p} \log_p\left(1 - \frac{t - i}{\tau - i}\right) \mathrm{d}\mu_m(t) \right].$$

The last integrand may be expanded as a power series in the residue disc $i + p\mathbf{Z}_p$, and hence to calculate the integral to an accuracy of p^M it suffices to calculate the moments

$$\int_{i+p\mathbf{Z}_p} (t - i)^n\, \mathrm{d}\mu_m(t) = p^n \int_{\mathbf{Z}_p} u^n\, \mathrm{d}\mu_{P_i^{-1}m}(u) \tag{11.6}$$

for $n = 0, \ldots, M-1$, where $P_i = \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$, and (11.6) uses the invariance of μ under $P_i \in \tilde{\Gamma}$.

Writing $P_i^{-1} m = \tilde{w} = [\infty] - [w]$, we calculate (11.6) by pulling back to $\mathbf{X}$:

$$
\begin{aligned}
\int_{\mathbf{Z}_p} u^n \, d\mu_{\tilde{w}}(u) &= \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} x^n y^{-n} \, d\tilde{\mu}_{\tilde{w}}(x,y) \\
&= \lim_{\substack{j \to \infty \\ g=(p-1)p^j}} \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} x^n y^{g-n} \, d\tilde{\mu}_{\tilde{w}}(x,y) \\
&= -\lim_{\substack{j \to \infty \\ g=(p-1)p^j}} 12 \sum_{\ell=0}^{n} \binom{n}{l} w^{n-\ell} (-1)^\ell \sum_{d|N} n_d d^{-\ell} D_{g-\ell+1,\ell+1}(dw), \quad (11.7)
\end{aligned}
$$

using (10.24). Writing $w = b/ep^r$ (with $p \nmid e$ and $N|e$) and employing the distribution relation for Bernoulli polynomials as in (10.14), the expression (11.7) maybe expressed in terms of the single-variable measures of Section 10.9:

$$
\begin{aligned}
\lim_{\substack{j \to \infty \\ g=(p-1)p^j}} \sum_{d|N} n_d d^{-\ell} D_{g-\ell+1,\ell+1}\left(\frac{bd}{ep^r}\right) &= \sum_{h=1}^{ep^r} \frac{\tilde{B}_{\ell+1}\left(\frac{hb}{ep^r}\right)}{\ell+1} \lim_{\substack{j \to \infty \\ g=(p-1)p^j}} \mathcal{F}_{g-\ell+1}(h + ep^r \cdot Z) \\
&= \sum_{h=1}^{ep^r} \frac{\tilde{B}_{\ell+1}\left(\frac{hb}{ep^r}\right)}{\ell+1} \int_{h+ep^r Z} x^{-\ell} \, d\mathcal{F}_1(x), \quad (11.8)
\end{aligned}
$$

by Proposition 10.9.1. The integrals of (11.8) may be computed modulo p^M by expanding $x^{-\ell}$ as a power series, and using Proposition 10.9.1 to calculate the moments of $\mathcal{F}_1$.

The terms of (11.5) may be calculated similarly using the methods described above for (11.4). Our method has broken down the computation of (11.3) into two parts. The first is the calculation of the following integrals, independent of τ:

1. $\int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \log_p(y) \, d\tilde{\mu}_m(x,y)$,

2. $\int_{i+p\mathbf{Z}_p} (t-i)^n \, d\mu_m(t), i = 0, \ldots, p = 1, n = 0, \ldots, M-1$,

3. $\int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} \log_p(x) \, d\tilde{\mu}_m(x,y)$,

4. $\int_{\mathbf{P}^1(\mathbf{Q}_p)-\mathbf{Z}_p} t^{-n} \, d\mu_m(t), n = 0, \ldots M-1$.

(The last moment arises in the computation of (11.5).) Hence our algorithm is to execute one program, which calculates for a given α, p, and M, each of these integrals once and for all as m ranges over a $\Gamma_0(N)$-module basis for $\mathcal{M}$, to an accuracy of p^M. (Using the $\Gamma_0(N)$-invariance of the indefinite integral, it suffices to calculate (1)–(4) for a $\Gamma_0(N)$-module basis

of $\mathcal{M}$ in order to evaluate the indefinite integral for all $m \in \mathcal{M}$.) This program executes $O(pM^2)$ operations. The output is stored in a file.

A second program is then run, inputting the integrals (1)–(4) from the output file of the first program, and calculating $u(\alpha, \tau)$ to an accuracy of p^M as described above. This calculation executes $O(pM)$ operations, and hence is rather quick even when M is large. Thus to compute the p-units $u(\alpha, \tau)$ to a high accuracy for various real quadratic fields K, it suffices to execute the (much slower) first program only once.

11.2 Results

The methods of Section 11.1 were used with the modular unit

$$\alpha(z) = \Delta(z)^2 \Delta(2z)^{-3} \Delta(4z)$$

of level $N = 4$ and various p. A p-adic accuracy of $M = 50$ digits was used. In our calculations, we restricted to fields K whose maximal orders embedded into R (see 4.11), so that the resulting units $u(\alpha, \tau)$ would be (conjecturally) defined over the narrow Hilbert class field H^+ of K. Since R admits a ring homomorphism to $\mathbf{Z}/N\mathbf{Z}$, the restriction in this case is that the discriminant D of K is congruent to 1 modulo 8. For each p we considered all $D < 500$ for which p is inert in K and $\mathcal{O}_K$ contains no unit of norm -1.

The programming language MAGMA was used for the computations. In each case, representatives for each of the h classes of quadratic forms were used to produce a τ_i (see [5, §5.2]) and the corresponding $u(\alpha, \tau_i) \in K_p^\times$. Conjecture 8.2.5 predicts that the conjugate of $u(\alpha, \tau_i) \in H^+$ over H is $u(\alpha, \tau_i)^{-1}$. Thus the characteristic polynomial of the $u(\alpha, \tau_i) \in H^+$ over K will be

$$P(x) = \prod_{i=1}^{h} (x - u(\alpha, \tau_i))(x - u(\alpha, \tau_i)^{-1}).$$

The polynomial $P(x)$ is computed in $K_p[x]$ to an accuracy of 50 p-adic digits, and a simple algorithm involving shortest lattice vectors (see [8, §1.6]) is used to recognize the resulting p-adic numbers as elements of K.

Remark 11.2.1. The modular symbol ψ attached to α actually takes values in $3\mathbf{Z}$, since α is the cube of the modular function

$$\eta(z)^8 \eta(2z)^{-12} \eta(4z)^4$$

of level 4. In order to minimize the heights of the points $u(\alpha, \tau)$, it is preferable to replace ψ with $\psi/3$.

Furthermore, after executing our algorithm, it was clear that in most cases our p-units were still powers of smaller units. If the integers $\mathrm{ord}_p(u(\alpha, \tau_i))$ and

$$\tilde{\mu}\{\infty \to a/Nc\}((u + p\mathbf{Z}_p) \times (v + p\mathbf{Z}_p))$$

for $(u, v) \in \mathbf{X}$ are divisible by a common integer r relatively prime to p, then formulas (11.1)–(11.3) yield a canonical rth root of $u(\alpha, \tau)$ in $K_p^\times$, by replacing ψ by ψ/r. In each case where $\mathrm{ord}_p u(\alpha, \tau) \neq 0$, we calculated the largest r for which this was the case.

The tables below present our results; we list for each discriminant the class number h of $\mathcal{O}_K$ (so $[H^+ : K] = 2h$), the maximal value of r as described in Remark 11.2.1, the values $\frac{1}{r} \mathrm{ord}_p u(\alpha, \tau)$, and the polynomial $P(x)$ of the $u(\alpha, \tau)^{1/r}$ scaled to clear powers of p from the denominator. In each case the polynomials produced are indeed characteristic polynomials of p-units in H^+. In many cases, the units listed are powers of smaller p-units in H^+; in these cases, the polynomial $P(x)$ of the largest root lying in H^+ is listed in the table on the following line (with the root taken implied by the value of r). This root is not necessarily uniquely defined, depending on the presence of roots of unity in H^+.

Remark 11.2.2. Since the units we produce conjecturally have trivial valuation at each place not lying above p, they are determined uniquely by their valuations at the places above p. In particular, when the class number of K is 1 and $\zeta(\alpha, \tau, 0) = 0$, we expect $u(\alpha, \tau)$ to be a root of unity. To produce non-trivial units in this case, one must work with a different modular unit α to avoid the "accidental zero" caused by the linear combination of zeta functions weighted by n_d used to define $\zeta(\alpha, \tau, s)$.

Similarly, if K has class number 2 and the zeta values $\zeta(\alpha, \tau, 0)$ for the two equivalence classes of binary quadratic forms are equal, then we expect the corresponding units to be equal, and our polynomial $P(x)$ to factor as a square. A different modular unit must be used to generate the full narrow Hilbert class field. These features of the construction are evident in the tables.

Table 11.1: Characteristic Polynomial of $u(\alpha, \tau)$ for $p = 3$

D	h	r	$\operatorname{ord}_p u(\alpha,\tau)^{1/r}$	$P(x)$
161	1	6	± 0	$x^2 - 2x + 1$
209	1	6	± 6	$729x^2 + 1358x + 729$
		36	± 1	$3x^2 + 5x + 3$
305	2	6	$\pm 2, \pm 4$	$6561x^4 - \frac{675\sqrt{D}+3987}{2}x^3 + \frac{75\sqrt{D}+4607}{2}x^2 - \frac{675\sqrt{D}+3987}{2}x + 6561$
		12	$\pm 1, \pm 3$	$81x^4 - \frac{9\sqrt{D}+345}{2}x^3 + \frac{15\sqrt{D}+419}{2}x^2 - \frac{9\sqrt{D}+345}{2}x + 81$
329	1	6	± 0	$x^2 - 2x + 1$
377	2	12	$\pm 1, \pm 3$	$81x^4 - \frac{21\sqrt{D}+207}{2}x^3 + \frac{21\sqrt{D}+499}{2}x^2 - \frac{21\sqrt{D}+207}{2}x + 81$
473	3	6	$\pm 2, \pm 2, \pm 6$	$3^{10}x^6 + \frac{15795\sqrt{D}+101493}{2}x^5 + \frac{12285\sqrt{D}+620541}{2}x^4 + \frac{34905\sqrt{D}+336763}{2}x^3 + \frac{12285\sqrt{D}+620541}{2}x^2 + \frac{15795\sqrt{D}+101493}{2}x + 3^{10}$
		12	$\pm 1, \pm 1, \pm 3$	$243x^6 + \frac{-9\sqrt{D}+945}{2}x^5 + \frac{15\sqrt{D}+1167}{2}x^4 + \frac{21\sqrt{D}+815}{2}x^3 + \frac{15\sqrt{D}+1167}{2}x^2 + \frac{-9\sqrt{D}+945}{2}x + 243$
497	1	6	± 0	$x^2 - 2x + 1$

Table 11.2: Characteristic Polynomial of $u(\alpha, \tau)$ for $p = 5$

D	h	r	$\operatorname{ord}_p u(\alpha, \tau)^{1/r}$	$P(x)$
33	1	6	± 2	$25x^2 + \frac{3\sqrt{D}-49}{2}x + 25$
		12	± 1	$5x^2 + \frac{3\sqrt{D}-1}{2}x + 5$
57	1	12	± 1	$5x^2 + \frac{-\sqrt{D}+9}{2}x + 5$
177	1	6	± 6	$5^6 x^2 + \frac{4011\sqrt{D}+5231}{2}x + 5^6$
		12	± 3	$125x^2 + \frac{21\sqrt{D}-191}{2}x + 125$
217	1	6	± 0	$x^2 - 2x + 1$
273	2	6	$\pm 2, \pm 2$	$(25x^2 + \frac{3\sqrt{D}-41}{2}x + 25)^2$
		12	$\pm 1, \pm 1$	$(5x^2 + \frac{-\sqrt{D}+3}{2}x + 5)^2$
297	1	12	± 3	$125x^2 - 74x + 125$
		36	± 1	$5x^2 + x + 5$
377	2	6	$\pm 2, \pm 6$	$5^8 x^4 + \frac{30375\sqrt{D}+533925}{2}x^3 + \frac{76545\sqrt{D}+102167}{2}x^2 + \frac{30375\sqrt{D}+533925}{2}x + 5^8$
		12	$\pm 1, \pm 3$	$625x^4 + \frac{75\sqrt{D}-655}{2}x^3 + \frac{-15\sqrt{D}+1447}{2}x^2 + \frac{75\sqrt{D}-655}{2}x + 625$
393	1	6	± 10	$5^{10} x^2 + 2275534x + 5^{10}$
		12	± 5	$3125x^2 + 4154x + 3125$
417	1	6	± 6	$5^6 x^2 - \frac{2109\sqrt{D}+18929}{2}x + 5^6$
		12	± 3	$125x^2 + \frac{19\sqrt{D}+111}{2}x + 125$
473	3	6	$\pm 2, \pm 2, \pm 6$	$5^{10} x^6 + \frac{-253125\sqrt{D}-4501875}{2}x^5 + \frac{496125\sqrt{D}+5836125}{2}x^4 + \frac{-59535\sqrt{D}-13546883}{2}x^3 + \frac{496125\sqrt{D}+5836125}{2}x^2 + \frac{-253125\sqrt{D}-4501875}{2}x + 5^{10}$
		12	$\pm 1, \pm 1, \pm 3$	$3125x^6 + \frac{-1125\sqrt{D}-1475}{2}x^5 + \frac{225\sqrt{D}+47345}{2}x^4 + \frac{-2655\sqrt{D}-6797}{2}x^3 + \frac{225\sqrt{D}+47345}{2}x^2 + \frac{-1125\sqrt{D}-1475}{2}x + 3125$
497	1	6	± 0	$x^2 - 2x + 1$

Table 11.3: Characteristic Polynomial of $u(\alpha, \tau)$ for $p = 7$

D	h	r	$\operatorname{ord}_p u(\alpha, \tau)^{1/r}$	$P(x)$
33	1	6	± 2	$49x^2 + 94x + 49$
		12	± 1	$7x^2 + 2x + 7$
129	1	12	± 1	$7x^2 - 2x + 7$
201	1	6	± 2	$49x^2 + 94x + 7$
		12	± 1	$7x^2 + 2x + 7$
209	1	12	± 3	$343x^2 - 610x + 343$
		36	± 1	$7x^2 + 5x + 7$
297	1	6	± 6	$7^6 x^2 + 153502x + 7^6$
		36	± 1	$7x^2 + 2x + 7$
321	3	6	$\pm 2, \pm 2, \pm 6$	$7^{10}x^6 - \frac{1188495\sqrt{D}+567084987}{2}x^5 + $ $\frac{-557865\sqrt{D}+433702773}{2}x^4 + \frac{5083155\sqrt{D}-475485877}{2}x^3$ $+ \frac{-557865\sqrt{D}+433702773}{2}x^2 - \frac{1188495\sqrt{D}+567084987}{2}x + 7^{10}$
		12	$\pm 1, \pm 1, \pm 3$	$7^5 x^6 - \frac{2205\sqrt{D}+53361}{2}x^5 + $ $\frac{3465\sqrt{D}+48699}{2}x^4 - \frac{4455\sqrt{D}+21791}{2}x^3$ $+ \frac{3465\sqrt{D}+48699}{2}x^2 - \frac{2205\sqrt{D}+53361}{2}x + 7^5$
377	2	6	$\pm 2, \pm 6$	$7^8 x^4 + \frac{-1210545\sqrt{D}+3900253}{2}x^3 + \frac{-172935\sqrt{D}+31066815}{2}x^2$ $+ \frac{-1210545\sqrt{D}+3900253}{2}x + 7^8$
		12	$\pm 1, \pm 3$	$2401x^4 + \frac{315\sqrt{D}+10017}{2}x^3 + \frac{405\sqrt{D}+15155}{2}x^2$ $+ \frac{315\sqrt{D}+10017}{2}x + 2401$
465	2	6	$\pm 4 , \pm 4$	$(2401x^2 - 4034x + 2401)^2$
		24	$\pm 1 , \pm 1$	$(7x^2 + 2x + 7)^2$
489	1	6	± 2	$49x^2 + 94x + 49$
		12	± 1	$7x^2 + 2x + 7$

Table 11.4: Characteristic Polynomial of $u(\alpha,\tau)$ for $p=11$

D	h	r	$\operatorname{ord}_p u(\alpha,\tau)^{1/r}$	$P(x)$
57	1	6	± 2	$121x^2 - \frac{15\sqrt{D}+233}{2}x + 121$
		12	± 1	$11x^2 - \frac{5\sqrt{D}+3}{2}x + 11$
105	2	6	$\pm 2, \pm 2$	$(121x^2 - \frac{39\sqrt{D}+73}{2}x + 121)^2$
		12	$\pm 1, \pm 1$	$(11x^2 - \frac{3\sqrt{D}+13}{2}x + 11)^2$
129	1	6	± 2	$121x^2 + \frac{-21\sqrt{D}+199}{2}x + 121$
		12	± 1	$11x^2 + \frac{\sqrt{D}+21}{2}x + 11$
161	1	6	± 0	$x^2 - 2x + 1$
217	1	6	± 0	$x^2 - 2x + 1$
249	1	6	± 6	$11^6 - \frac{167295\sqrt{D}+3198553}{2}x + 11^6$
		12	± 3	$11^3 - \frac{285\sqrt{D}+587}{2}x + 11^3$
305	2	6	$\pm 2, \pm 6$	$11^8 x^4 + \frac{10372725\sqrt{D}+344443077}{2}x^3 + \frac{23917275\sqrt{D}-61466353}{2}x^2$ $+ \frac{10372725\sqrt{D}+344443077}{2}x + 11^8$
		12	$\pm 1, \pm 3$	$11^4 x^4 + \frac{-2475\sqrt{D}+6853}{2}x^3 + \frac{-225\sqrt{D}+44467}{2}x^2$ $+ \frac{-2475\sqrt{D}+6853}{2}x + 11^4$
321	3	6	$\pm 2, \pm 2, \pm 6$	$11^{10} x^6 - \frac{1967882169\sqrt{D}+60603418095}{2}x^5 +$ $\frac{10953497049\sqrt{D}+178199983335}{2}x^4$ $- \frac{13842699651\sqrt{D}+210615242059}{2}x^3$ $+ \frac{10953497049\sqrt{D}+178199983335}{2}x^2 -$ $\frac{1967882169\sqrt{D}+60603418095}{2}x + 11^{10}$
		12	$\pm 1, \pm 1, \pm 3$	$11^5 x^6 + \frac{-4719\sqrt{D}+427251}{2}x^5 +$ $\frac{-37257\sqrt{D}+801537}{2}x^4 + \frac{-55935\sqrt{D}+531929}{2}x^3$ $+ \frac{-37257\sqrt{D}+801537}{2}x^2 + \frac{-4719\sqrt{D}+427251}{2}x + 11^5$
329	1	6	± 0	$x^2 - 2x + 1$
393	1	6	± 10	$11^{10} x^2 - 50395911602x + 11^{10}$
		12	± 5	$11^5 x^2 - 319798x + 11^5$
417	1	6	± 6	$11^6 x^2 + \frac{174795\sqrt{D}-2882153}{2}x + 11^6$
		12	± 3	$11^3 x^2 + \frac{215\sqrt{D}-813}{2}x + 11^3$
497	1	6	± 0	$x^2 - 2x + 1$

Bibliography

[1] M. Bertolini, H. Darmon. *Heegner points, p-adic L-functions and the Cerednik-Drinfeld uniformization.* Inventiones Math. **131** (1998) 453-491.

[2] M. Bertolini, H. Darmon. *The rationality of Stark–Heegner points over genus fields of real quadratic fields*, in progress.

[3] M. Bertolini, H. Darmon, S. Dasgupta. *Stark–Heegner points and special values of L-series*, in progress.

[4] J. Coates, W. Sinnott. *On p-adic L-functions over real quadratic fields.* Invent. Math. **25** (1974), 253–279.

[5] H. Darmon. *Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications.* Ann. of Math. (2) **154** (2001), no. 3, 589–639.

[6] H. Darmon, S. Dasgupta. *Elliptic units for real quadratic fields,* in progress.

[7] H. Darmon, P. Green, *Elliptic curves and class fields of real quadratic fields: algorithms and verifications.* Experimental Mathematics, **11:1**, (2002), 37–55.

[8] H. Darmon, R. Pollack, *The efficient calculation of Stark–Heegner points via overconvergent modular symbols,* in progress.

[9] E. deShalit, *On the p-adic periods of $X_0(p)$.* Math. Ann. **303**, (1995), 457–472.

[10] S. Fukuhara. *Generalized Dedekind symbols associated with the Eisenstein Series.* Proc. Amer. Math. Soc. **127** (1999), no. 9, 2561-2568.

[11] L. Gerritzen, M. van der Put, *Schottky groups and Mumford curves.* Lecture Notes in Mathematics, 817. Springer, Berlin, 1980.

[12] R. Greenberg, G. Stevens, *p-adic L-functions and p-adic periods of modular forms.* Invent. Math. **111** (1993), no. 2, 407–447.

[13] R. Greenberg, G. Stevens, *On the conjecture of Mazur, Tate, and Teitelbaum.* **p-adic monodromy and the Birch and Swinnerton-Dyer conjecture** (Boston, MA, 1991), 183–211. Contemp. Math., **165**, Amer. Math. Soc., Providence, RI, (1994).

[14] P. Griffiths, J. Harris. *Principles of algebraic geometry.* Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[15] B. H. Gross. *Kolyvagin's work on modular elliptic curves.* **L-functions and arithmetic** (Durham, 1989), 235–256, London Math. Soc. Lecture Note Ser., 153, Cambridge Univ. Press, Cambridge, 1991.

[16] B. H. Gross. *p-adic L-series at $s = 0$.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 3, 979–994 (1982).

[17] B. H. Gross. *On the values of abelian L-functions at $s = 0$.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), no. 1, 177–197.

[18] B. H. Gross, D. B. Zagier. *Heegner points and derivatives of L-series.* Invent. Math. **84** (1986), no. 2, 225–320.

[19] U. Halbritter. *Some new reciprocity formulas for generalized Dedekind sums.* Results Math. **8** (1985), no. 1, 21–46.

[20] H. Hida, *Iwasawa modules attached to congruences of cusp forms.* Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 2, 231–273.

[21] H. Hida. *Elementary theory of L-functions and Eisenstein series.* London Math. Society Student Texts. **26**. Cambridge University Press, 1993.

[22] T. Ichikawa, *Schottky uniformization theory on Riemann surfaces and Mumford curves of infinite genus.* J. Reine Angew. Math. **486** (1997), 45–68.

[23] Y. Ihara, *On congruence monodromy problems.* Vols. 1 and 2. Lecture Notes, Nos. 1 and 2, Department of Mathematics, University of Tokyo, Tokyo 1968.

[24] P. Koebe, *Über die Uniformisierung der algebraischen Kurven IV*, Math. Ann. **75** (1914), 42–129.

[25] V. A. Kolyvagin, *Euler systems.* **The Grothendieck Festschrift, Vol. II,** 435–483, Progr. Math., 87, Birkhäuser Boston, Boston, MA, 1990.

[26] V. A. Kolyvagin, D. Y. Logachëv, *Finiteness of the Shafarevich-Tate group and the group of rational points for some modular abelian varieties.* (Russian) Algebra i Analiz 1 (1989), no. 5, 171–196; translation in Leningrad Math. J. 1 (1990), no. 5, 1229–1253.

[27] D. S. Kubert, S. Lang, *Modular units.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 244. Springer-Verlag, New York-Berlin, 1981.

[28] S. Lang. *Algebraic number theory.* Second edition. Graduate Texts in Mathematics, 110. Springer-Verlag, New York, 1994.

[29] J. I. Manin, *Parabolic Points and Zeta Functions of Modular Curves.* Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), no. 1, 19–66.

[30] Y. I. Manin, V. Drinfeld. *Periods of p-adic Schottky groups.* Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday. J. Reine Angew. Math. 262/263 (1973), 239–247.

[31] B. Mazur. *Modular curves and the Eisenstein ideal.* Inst. Hautes ?tudes Sci. Publ. Math. No. 47 (1977), 33–186 (1978).

[32] B. Mazur. *On the arithmetic of special values of L functions.* Invent. Math. **55** (1979), no. 3, 207–240.

[33] B. Mazur, A. Wiles. *Class fields of abelian extensions of Q.* Invent. Math. 76 (1984), no. 2, 179–330.

[34] B. Mazur, A. Wiles. *On p-adic analytic families of Galois representations.* Compositio Math. **59** (1986), no. 2, 231–264.

[35] B. Mazur, J. Tate, J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer.* Invent. Math. **84** (1986), no. 1, 1–48.

[36] D. Mumford, *An Analytic Construction of Degenerating Curves over Complete Local Rings.* Compositio Math. **24** (1972), no. 2, 129–174.

[37] K. Ribet, *Congruence relations between modular forms.* **Proceedings of the International Congress of Mathematicians, Vol. 1, 2** (Warsaw, 1983), 503–514, PWN, Warsaw, 1984.

[38] F. Schottky, *Über eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Arguments univerändert bleibt,* J. reine angew. Math. **101** (1887), 227–272.

[39] C. L. Siegel. *Bernoullische Polynome und quadratische Zahlkörper.* Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1968) 7–38.

[40] J. P. Serre. *Trees.* Translated from the French original by John Stillwell. Corrected 2nd printing of the 1980 English translation. **Springer Monographs in Mathematics.** Springer-Verlag, Berlin, (2003).

[41] P. T. Young, *Kummer congruences for values of Bernoulli and Euler polynomials.* Acta Arithmetica, **48**, no. 3 (2001), 277-288.

[42] D. B. Zagier. *A Kronecker limit formula for real quadratic fields.* Math. Ann. **213** (1975), 153–184.