Stark's Conjectures

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a thesis
presented to the Department of Mathematics
in partial fulfillment
of the requirements for the degree
of Bachelor of Arts with Honors

Harvard University
Cambridge, Massachusetts
April 5, 1999

Contents

Acknowledgments

There are many people without whom I could never have completed this endeavor. First and foremost, I wish to thank my parents, who raised my brother Sandip and me in a loving environment in which we could flourish. Any successes I may achieve in my life are directly traceable to their support of my interests and devotion to my development. In addition to my parents, there are many other individuals who have supported me as I worked on this project and throughout my time at Harvard. In particular, I would certainly be lost without the love and friendship of Lisa Zorn.

There are many individuals who have shaped my still growing view of mathematics. Primary among these are Darlyn Counihan, who introduced me to the beauty of mathematics, Professor Arnold Ross, who introduced me to number theory, and Eric Walstein and Professor Lawrence Washington, who supported my work in high school. I am also greatly indebted to Professors Benedict Gross and Noam Elkies, whose instruction has defined my current view of modern mathematics.

With regards to this thesis, Professor Brian Conrad deserves an indescribable amount of thanks for the incredible number of hours he spent advising me. From helping me learn the basics of class field theory, to giving me an intuition for Stark's Conjectures, to editing the final paper, Professor Conrad went far beyond his duty as an advisor. I am greatly indebted to him for his patience and enthusiasm. I would also like to thank Professor Gross for suggesting this thesis topic to me. Finally, I am grateful to Adam Logan, Daniel Biss, and Jacob Lurie for their assistance in my work.

I dedicate this thesis to my parents, who taught me by example the values of love, education, and love of education.

1 Introduction

One fascinating and mysterious aspect of modern number theory is the interaction between the analytic and the algebraic points of view. The most fundamental example of this interaction is the relationship between the Dedekind zeta-function $\zeta_k(s)$ of a number field k and certain algebraic invariants of the field k. The definition of the complex-valued function $\zeta_k(s)$ is

$$\zeta_k(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \mathrm{N}\mathfrak{p}^{-s}}$$

for $\mathrm{Re}(s) > 1$, where the product runs over all nonzero prime ideals $\mathfrak{p}$ of k. Through the process of analytic continuation, we obtain a meromorphic function $\zeta_k(s)$ defined on the complex plane.

Interestingly, the *analytic* behavior of the function ζ_k allows one to prove purely *algebraic* facts about the number field k. For example, Dirichlet was able to exploit the fact that the meromorphic function $\zeta_{\mathbf{Q}}(s)$ has a simple pole at $s = 1$ in order to prove that there are infinitely many primes in every arithmetic progression of the form $\{a, a+b, a+2b, \cdots\}$ where a and b have no common factors (e.g., there are infinitely many primes of the form $12n + 5$). To prove this theorem, Dirichlet also had to define a generalization of the function $\zeta_k(s)$, called a Dirichlet L-function. His arguments relied on the subtle analytic fact that certain such L-functions do not have a zero or a pole at $s = 1$.

Even more, Dirichlet proved the celebrated "class number formula," which gives an explicit formula for the residue of $\zeta_k(s)$ at $s = 1$:

$$\mathrm{Res}(\zeta_k; 1) = \frac{2^{r_1}(2\pi)^{r_2} h_k R_k}{\sqrt{|D_k|} e_k}, \tag{1}$$

where r_1 and r_2 are the number of real and complex places of k respectively, and h_k, R_k, D_k, and e_k are the class number, regulator, absolute discriminant, and number of roots of unity of k, respectively. This is a remarkable formula because the definition of $\zeta_k(s)$ uses only *local* information about k (i.e. the prime ideal structure) and analytic continuation, but the residue $\mathrm{Res}(\zeta_k; 1)$ at 1 involves *global* invariants of k, such as h_k and R_k.

Using the functional equation for $\zeta_k(s)$, we can reformulate equation (1) as saying that the first non-zero term in the Taylor series of $\zeta_k(s)$ at $s = 0$ is given by

$$-\frac{h_k R_k}{e_k} s^{r_1 + r_2 - 1}. \tag{2}$$

The term $-h_k R_k / e_k$ is the ratio between the (presumably) transcendental number R_k and the algebraic number $-e_k/h_k$. Furthermore, the regulator R_k is the determinant of $(r_1 + r_2 - 1)$-dimensional square matrix whose entries are logarithms of the archimedean valuations of units belonging to k.

In the first half of this century, Artin, Hecke, Tate, and others established a general theory of L-functions and functional equations, partially extending some of the results obtained by Dirichlet. However, despite all of this work, there was no satisfactory generalization of the above Dirichlet class number formula in the context of arbitrary Artin L-functions. In the 1970's, H. M. Stark attempted to find such a formula. He was guided by the following:

Motivating Question. If K/k is a finite Galois extension with Galois group G and χ is the character of an irreducible, finite-dimensional, complex representation of G, is there a formula, analogous to (2), for the first non-zero coefficient in the Taylor expansion of $L_{K/k}(s, \chi)$ at $s = 0$? More precisely, if $r(\chi)$ is the order of $L_{K/k}(s, \chi)$ at $s = 0$, can this coefficient be expressed as the product of an algebraic number and the determinant of an $r(\chi) \times r(\chi)$ matrix whose entries are linear forms in logarithms of archimedean valuations of units belonging to K?

Answering this question would provide a generalization of the Dirichlet class number formula in two different ways. First, for any Galois extension K/k with Galois group G, the L-function $L_{K/k}(s, 1_G)$ corresponding to the trivial character 1_G is precisely the Dedekind zeta-function $\zeta_k(s)$. More importantly, however, there is the formula

$$\zeta_K(s) = \prod_{\chi} L_{K/k}(s, \chi)^{\chi(1)}$$

where the product runs over all irreducible characters χ of G. Therefore, providing a formula for the first non-zero Taylor coefficient of $L_{K/k}(s, \chi)$ for all irreducible χ would strengthen the statement of the Dirichlet class number formula by showing how the leading term of $\zeta_K(s)$ factors into pieces, one for each irreducible representation of G.

Questions about leading coefficients of the Taylor expansions of L-functions arise in many aspects of number theory. For example, there are L-functions in the theory of elliptic curves and the still unproven Birch-Swinnerton-Dyer conjecture is an analogue of the Dirichlet class number formula in the setting of elliptic curve L-functions. In fact, there have been efforts to prove the Birch-Swinnerton-Dyer conjecture by splitting it up into pieces, as Stark's conjecture attempts to do with Artin L-functions. Therefore, gaining a deeper understanding of Stark's conjecture may shed light on other analogous problems in number theory.

In the case where K/k is an *abelian* extension, Stark has given a refined conjecture which essentially states there exists a unit of K such that specific linear combinations of its archimedean valuations give the values of the derivatives $L'_{K/k}(0, \chi)$. In certain cases, this "Stark unit" can be seen to generate K over k, and hence the refined conjecture implies that K can be obtained from k by adjoining the value of a certain analytic function at zero. As Stark observed himself [24, pg. 63], "a reference to Hilbert's 12th problem may not be completely inappropriate." In fact, in cases where a solution to Hilbert's 12th problem is known, namely when k is either $\mathbf{Q}$ or a quadratic imaginary field, Stark was able to prove his abelian conjecture. For these many reasons, Stark's conjectures remain among the central open problems in number theory.

Unfortunately, for base fields k other than $\mathbf{Q}$ and quadratic imaginary fields, it is not known how to directly construct Stark units. Thus, progress on Stark's Conjecture in the case of general number fields has been mostly limited to "numerical verifications." This numerical evidence is now overwhelming, but a general strategy for a proof is still lacking.

We begin this thesis by formulating the non-abelian Stark conjecture, which states roughly that an expression analogous to the Dirichlet class number formula exists for non-abelian Artin L-functions. We then analyze the conjecture in greater detail for the cases $r(\chi) = 0$ and $r(\chi) = 1$. The study of the case $r(\chi) = 1$ leads to the notion of a "Stark unit". As we noted above, Stark gave a refinement of his conjecture in the case where K/k is abelian by using these Stark units.

In addition to explaining the proof by Sands of the abelian Stark conjecture for certain Galois groups with exponent 2, we provide a numerical confirmation of the abelian conjecture for a specific cubic base field k with a complex place; this is the first time a numerical confirmation has been done in such a case. Our methods follow those of Dummit, Sands, and Tangedal [6], who treated many cases where k is a totally real cubic field, but there are some additional problems that do not arise in the totally real case.

We conclude the thesis with a consideration of two more advanced topics. First, we discuss the Brumer-Stark conjecture, which combines the ideas of Stark's abelian conjecture and the work of Stickelberger on annihilators of ideal class groups. Second, we carefully explain the difficult proof of the non-abelian Stark conjecture for characters χ which assume only rational values. This includes the cases in which $\mathrm{Gal}(K/k)$ is a symmetric group.

The basic motivation and many of the results of this thesis are due to Stark, but the greatest influence on this thesis was provided by Tate [27]. His book provides an elegant and sophisticated account of work that has been done on Stark's conjectures, much of it by Tate himself. This work involves giving proofs of special cases of the conjectures, as well as finding more conceptual formulations of what is to be proven. Such reformulations led to a function field analogue of Stark's conjectures, and this analogue was proven by Deligne, thereby providing further conceptual evidence for the original conjectures. As is often the case in mathematics, properly formulating what is to be proven is an essential step towards greater understanding.

2 Basic notation

We now present the basic notation to be used throughout this thesis.

The symbols **Z**, **Q**, **R**, and **C** denote the integers, rational numbers, real numbers, and complex numbers, respectively. We fix a choice of $i = \sqrt{-1} \in \mathbf{C}$. The symbol k will denote an algebraic number field; that is, a finite extension of **Q**. A choice of k will be fixed for most of the discussion. Finite extensions of k are denoted K/k. The group of roots of unity in k is denoted $\mu(k)$, and the number of roots of unity in k is denoted e_k.

The set S_∞ will denote the set of archimedean primes of k, and the set S will be any finite set of primes of k containing S_∞. The finite (i.e. non-archimedean) primes of k are usually denoted $\mathfrak{p}, \mathfrak{q}, \ldots$. General primes of k (archimedean or non-archimedean) are denoted $v, v', \ldots$. The set of primes of K lying above those in S is denoted S_K. We write $\mathfrak{P}$ for a prime of K lying above the finite prime $\mathfrak{p}$ of k, and we write $w, w', \ldots,$ for primes of K lying above the primes $v, v', \ldots,$ of k. We use the terms "place of k" and "prime of k" interchangeably. If K_w and k_v denote the completions of the fields K and k with respect to the valuations w and v, respectively, then we write $[w : v]$ for the degree of the local extension K_w/k_v.

The ring of integers of k will be denoted $\mathcal{O}_k$ or simply $\mathcal{O}$, and the S-integers are written $\mathcal{O}_S$. The fractional ideals of $\mathcal{O}_k$ form a group $I(\mathcal{O}_k) = I_k$. To each finite prime $\mathfrak{p}$ of k, there corresponds the "valuation function" $v_\mathfrak{p}$ on k^* and on the group I_k. This function gives the valuation on k corresponding to $\mathfrak{p}$:

$$|x|_\mathfrak{p} = (\mathrm{N}\mathfrak{p})^{-v_\mathfrak{p}(x)}.$$

With this normalization, a uniformizer π for the local field $k_\mathfrak{p}$ has valuation $(\mathrm{N}\mathfrak{p})^{-1}$. If v is an archimedean place of k, we define

$$|x|_v = \begin{cases} |x| = \pm x & \text{if } v \text{ is a real place} \\ x\overline{x} & \text{if } v \text{ is a complex place} \end{cases}.$$

With these normalizations, the product formula can be written simply as

$$\prod_v |x|_v = 1,$$

where the product runs over all (inequivalent) places v of k. Furthermore, if w is a place of K lying above v, and $u \in k_v$, our normalizations yield the equation

$$|u|_w = |u|_v^{[w:v]}.$$

Whenever we have a finite group G, all G-modules V will be left modules. We write the action of $z \in \mathbf{Z}[G]$ on $v \in V$ as v^z. The reader is warned that with this notation, $v^{zy} = (v^y)^z$.

When K/k is a Galois extension, the Galois group $G = \mathrm{Gal}(K/k)$ acts on the primes of K in a natural way. For $x \in K$ and $\sigma \in G$, we have

$$|x^\sigma|_{\sigma w} = |x|_w.$$

For a finite prime $w = \mathfrak{P}$, this agrees with the definition

$$\mathfrak{P}^\sigma = \{x^\sigma : x \in \mathfrak{P}\}.$$

The decomposition group of w in G is denoted G_w.

Let $\mathrm{N}\mathfrak{p}$ be the size of the finite field $\mathcal{O}_k/\mathfrak{p}$. The Galois group $\mathrm{Gal}((\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_k/\mathfrak{p}))$ is generated by the Frobenius automorphism $x \mapsto x^{\mathrm{N}\mathfrak{p}}$. The coset of $I_\mathfrak{P}$ in $G_\mathfrak{P}$ which maps to the Frobenius automorphism is denoted $\sigma_\mathfrak{P}$. When $\mathfrak{p}$ is unramified in K this becomes an element $\sigma_\mathfrak{P}$, called the *Frobenius element* at $\mathfrak{P}$. The $G_\mathfrak{P}$, $I_\mathfrak{P}$, and $\sigma_\mathfrak{P}$ for the various $\mathfrak{P}$ lying above a fixed $\mathfrak{p}$ are conjugate by elements of G, so, when K/k is abelian, we denote them by $G_\mathfrak{p}$, $I_\mathfrak{p}$, and $\sigma_\mathfrak{p}$, respectively.

For a finite set A, we write $|A|$ for the cardinality of A. There is no risk of confusion with our notation for absolute values.

Fix a Noetherian ring F (usually either $\mathbf{Z}$ or a field). Unless otherwise specified, all $F[G]$-modules are understood to be finitely generated over F. For $F[G]$-modules V and W, we write $\mathrm{Hom}_G(V, W)$ for the $F[G]$-module homomorphisms from V to W. When there is no subscript, $\mathrm{Hom}(V, W)$ represents the F-linear homomorphisms from V to W. An $F[G]$-module structure on $\mathrm{Hom}(V, W)$ is defined by having

$$(g\varphi)(v) = g\varphi(g^{-1}v)$$

for any $\varphi \in \mathrm{Hom}(V, W)$. In particular, we have an $F[G]$-structure on the dual

$$V^* = \mathrm{Hom}(V, F),$$

where F has trivial G-action.

If χ is a character of G over some field L containing a characteristic 0 field F, let $F(\chi)$ be the field obtained by adjoining the values $\chi(\sigma)$ to F for all $\sigma \in G$. Note that $F(\chi)/F$ is abelian since it is a sub-extension of a cyclotomic extension. If $\alpha\colon L \to K$ is a map of fields, then we write χ^α for the function $\alpha \circ \chi\colon G \to K$. If V is an $L[G]$-module realizing χ, then χ^α is the character of the $K[G]$-module $V^\alpha = K \otimes_L V$. When we wish to emphasize a particular choice of α, we sometimes write $K \otimes_{L,\alpha} V$ instead of $K \otimes_L V$.

Throughout this work, if $\mathbf{A}$ is a subring of $\mathbf{C}$ and B is a $\mathbf{Z}$-module, we will denote by $\mathbf{A}B$ the $\mathbf{A}$-module $\mathbf{A} \otimes_\mathbf{Z} B$. When B has a G-module structure, $\mathbf{A}B$ has an $\mathbf{A}[G]$-module structure, with G acting on the right factor B and $\mathbf{A}$ acting on the left factor.

3 The non-abelian Stark conjecture

In this section, we present the non-abelian Stark conjecture. We begin by recalling the Dirichlet class number formula, which gives an explicit formula for the first non-zero coefficient in the Taylor expansion for the Dedekind zeta-function. The Stark conjecture, in its abstract form, extends this to general Artin L-functions. The main difficulty in stating the Stark conjecture is defining an analogue of the regulator appearing in the Dirichlet class number formula. Once we define the regulator, we show that for the base field $k = \mathbf{Q}$, this regulator has the form suggested in the motivating question in the Introduction. This special case was one of Stark's early results that led him to his general conjectures.

We then state the non-abelian Stark conjecture. We also analyze naturality properties of the conjecture and verify the independence of the conjecture from certain choices that are made. This will enable us to show that if Stark's conjecture is true for the base field $k = \mathbf{Q}$, it is true in general.

3.1 The Dedekind zeta-function

Let k be a number field, and let S be a finite set of primes of k containing the set of infinite primes S_∞. The definitions of Dedekind zeta-function ζ_k and its generalization $\zeta_{k,S}$ are given in A.1.1. Dedekind was able to relate the residue of the simple pole of ζ_k at $s = 1$ to certain algebraic invariants of the field k. This generalized Dirichlet's work in the specific case where k is a quadratic field (see [3] and [4]).

Theorem 3.1.1 (Dirichlet Class Number Formula at $s = 1$). *The function $\zeta_k(s)$ has a simple pole at $s = 1$ with residue*

$$\frac{2^{r_1}(2\pi)^{r_2} h_k R_k}{\sqrt{|D_k|}e_k},$$

where r_1 and r_2 are the number of real and complex places of k, respectively, and h_k, R_k, D_k, and e_k are the class number, regulator, absolute discriminant, and number of roots of unity of k, respectively.

Using the functional equation for $\zeta_k(s)$, we can give a reformulation of this result at $s = 0$.

Theorem 3.1.2 (Dirichlet Class Number Formula at $s = 0$). *The Taylor series of $\zeta_k(s)$ at $s = 0$ is*

$$\zeta_k(s) = -\frac{h_k R_k}{e_k} s^{r_1+r_2-1} + O(s^{r_1+r_2}) = -\frac{h_k R_k}{e_k} s^{|S_\infty|-1} + O(s^{|S_\infty|}).$$

Proof. The functional equation for $\zeta_k(s) = L(s, 1)$ can be obtained from Theorem A.8.1. We find that $\Lambda_k(s) = \Lambda_k(1 - s)$, where

$$\Lambda_k(s) = |D_k|^{\frac{s}{2}}\,\Gamma_{\mathbf{C}}(s)^{r_2}\Gamma_{\mathbf{R}}(s)^{r_1}\zeta_k(s),$$

with $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Since $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1, by using the Dirichlet class number formula at $s = 1$ and the functional equation for Λ_k, we see that as $s \to 0$,

$$\frac{2^{r_1+r_2}}{s^{r_1+r_2}}\zeta_k(s) \sim -\frac{2^{r_1+r_2}}{s} \cdot \frac{h_k R_k}{e_k}.$$

Here the symbol $\sim$ means that the ratio of the two sides is 1 as $s \to 0$. The result follows. □

We can give an analogous statement for a general finite set of places $S \supset S_\infty$ by using the following lemma. The definition of the S-integers $\mathcal{O}_S$ and the corresponding class number $h_S = h_{k,S}$ and regulator R_S are given in A.5.1 and A.5.4.

Lemma 3.1.3. *Let $\mathfrak{p}$ be a place of k not lying in S, and let $S' = S \cup \{\mathfrak{p}\}$. If m is the order of $\mathfrak{p}$ in the ideal class group of the S-integers $\mathcal{O}_S$, then we have*

- $h_{S'} = \frac{h_S}{m}$,
- $R_{S'} = m(\log \mathrm{N}\mathfrak{p}) R_S$,
- $\zeta_{k,S'} \sim (\log \mathrm{N}\mathfrak{p}) s \cdot \zeta_{k,S}(s)$ *as* $s \to 0$.

Proof. For the first assertion, note that there is a natural surjection $I(\mathcal{O}_S) \to I(\mathcal{O}_{S'})$ given by $\mathfrak{U} \mapsto \mathfrak{U}\mathcal{O}_{S'}$; surjectivity holds by a consideration of prime ideals. Composing with the projection to the class group $I(\mathcal{O}_{S'}) \to \mathrm{Cl}(\mathcal{O}_{S'})$, we get a surjection $\phi \colon \mathrm{Cl}(\mathcal{O}_S) \to \mathrm{Cl}(\mathcal{O}_{S'})$. We will show that the sequence

$$0 \longrightarrow \langle \mathfrak{p} \rangle \longrightarrow \mathrm{Cl}(\mathcal{O}_S) \xrightarrow{\phi} \mathrm{Cl}(\mathcal{O}_{S'}) \longrightarrow 0$$

is exact, where $\langle \mathfrak{p} \rangle$ is the subgroup of $\mathrm{Cl}(\mathcal{O}_S)$ generated by the class of $\mathfrak{p}$. By definition, $m = |\langle \mathfrak{p} \rangle|$, so this exact sequence will give the first part of the lemma. The fact that $\langle \mathfrak{p} \rangle$ lies in the kernel of ϕ is clear. Conversely, given any $\mathfrak{U} \in I(\mathcal{O}_S)$ representing an element in the kernel of ϕ, we can write $\mathfrak{U}\mathcal{O}_{S'} = \beta\mathcal{O}_{S'}$ with $\beta \in K^*$. In particular, $v_{\mathfrak{q}}(\mathfrak{U}) = v_{\mathfrak{q}}(\beta\mathcal{O}_S)$ for all finite places $\mathfrak{q}$ of $\mathcal{O}_S$ distinct from $\mathfrak{p}$. We then find that $\mathfrak{U} = \mathfrak{p}^e \beta \mathcal{O}_S$ where $e = v_{\mathfrak{p}}(\mathfrak{U}) - v_{\mathfrak{p}}(\beta)$, since both sides are fractional ideals of $\mathcal{O}_S$ with the same valuation at all places of $\mathcal{O}_S$. This completes the proof of the first assertion.

For the second assertion, let $\{u_1, \ldots, u_r\}$ with $r = |S| - 1$ be fundamental units, i.e. representatives for a basis for the maximal torsion-free quotient of $\mathcal{O}_S^*$. We claim that if $\mathfrak{p}^m = \pi\mathcal{O}_S$, then $\{u_1, \ldots, u_r, \pi\}$ are fundamental units for $\mathcal{O}_{S'}$. First we demonstrate that this claim will give the desired result. Since $v_{\mathfrak{q}}(\pi) = 0$ for $\mathfrak{q} \neq \mathfrak{p}$, the matrix $M(S')$ of logarithms whose determinant defines the regulator $R_{S'}$ has a particularly simple form in terms of the analogous matrix $M(S)$ defining $R(S)$, where we choose arbitrarily the same place $v_0 \in S \subset S'$ to exclude when defining these matrices:

$$M(S') = \left(\begin{array}{c|c} M(S) & * \\ \hline 0 & \log |\pi|_{\mathfrak{p}} \end{array} \right).$$

Then we have

$$R_{S'} = |\log |\pi|_{\mathfrak{p}}| R_S = (m \log \mathrm{N}\mathfrak{p}) R_S$$

as desired.

It remains (for the second part of the lemma) to prove our claim that π generates $\mathcal{O}_{S'}^*/\mathcal{O}_S^*$. Let u be a unit of $\mathcal{O}_{S'}$. By scaling by an appropriate power of π, we may assume that $0 \le i = v_{\mathfrak{p}}(u) \le m-1$. Then $\mathfrak{p}^i = u\mathcal{O}_S$ since both sides have equal valuation at all the places of $\mathcal{O}_S$. Since the order of $\mathfrak{p}$ in $\mathrm{Cl}(\mathcal{O}_S)$ is m, we must have $i = 0$, implying that $u \in \mathcal{O}_S^*$. This proves the claim.

For the final assertion of the lemma, recall that

$$\zeta_{k,S'}(s) = \left(1 - \frac{1}{\mathrm{N}\mathfrak{p}^s}\right) \zeta_{k,S}(s)$$

from the Euler product representation of ζ_k. Taking limits as $s \to 0$ gives the desired result. $\square$

Corollary 3.1.4. *The Taylor series of $\zeta_{k,S}(s)$ at $s = 0$ is*

$$\zeta_{k,S}(s) = -\frac{h_S R_S}{e_k} s^{|S|-1} + O(s^{|S|}).$$

This corollary gives the first non-zero Taylor coefficient of $\zeta_{k,S}$ as the quotient of the transcendental number R_S by the rational number $-\frac{e_k}{h_S}$. Furthermore, this transcendental number is the absolute value of the determinant of an $(|S|-1)$-dimensional square matrix whose entries are linear forms in the logarithms of the valuations of units in $\mathcal{O}_S$. The non-abelian Stark conjecture essentially says that such a formula can be given for an arbitrary Artin L-function. Before we can reformulate this conjecture, it will be convenient to review some basic facts about Artin L-functions.

3.2 Artin L-functions

Notation 3.2.1. Let K/k be a finite Galois extension of number fields with Galois group G, and let S be a finite set of primes of k containing the archimedean primes. Let χ be the character of a finite-dimensional representation V of G over $\mathbf{C}$. The Artin L-function associated to χ, relative to S, is given by

$$L_S(s,\chi) = \prod_{\mathfrak{p}\notin S} \det\left((1-\sigma_{\mathfrak{P}}\,\mathrm{N}\mathfrak{p}^{-s})|_{V^{I_{\mathfrak{P}}}}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$ (see section A.8). Suppose the Taylor series for $L_S(s,\chi)$ in a neighborhood of 0 is

$$L_S(s,\chi) = c_S(\chi)s^{r_S(\chi)} + O(s^{r_S(\chi)+1}).$$

Since the set S will usually be fixed, we often drop the subscript S and write $r(\chi) = r_S(\chi)$ and $c(\chi) = c_S(\chi)$ for notational convenience. Our goal is to state a conjecture concerning $c(\chi)$, so we should first determine the value $r(\chi)$ of the order of L_S at $s = 0$.

Let S_K be the set of primes of K lying above those in S. Let $Y_{K,S}$ be the free abelian group generated by S_K, and let $X_{K,S}$ be the "hyperplane"

$$X_{K,S} = \left\{ \sum_{w\in S_K} n_w \cdot w \in Y_{K,S} : \sum_{w\in S_K} n_w = 0 \right\}.$$

We will drop the subscripts for $Y = Y_{K,S}$ and $X = X_{K,S}$ when the field K and set of primes S are clear. Note that Y has a G-module structure in which G acts by permuting the w's lying above a fixed place v of k. More precisely, if we choose for each $v \in S$ a fixed place $w \in S_K$ lying above v, we have an isomorphism of $\mathbf{Z}[G]$-modules

$$Y \cong \bigoplus_{v\in S} \mathrm{Ind}_{G_w}^G \mathbf{Z} = \bigoplus_{v\in S} \mathbf{Z}[G] \otimes_{\mathbf{Z}[G_w]} \mathbf{Z}, \tag{3}$$

where the decomposition group G_w acts trivially on $\mathbf{Z}$.

Clearly X is a G-submodule of Y and we have an exact sequence

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\ \epsilon\ } \mathbf{Z} \longrightarrow 0,$$

where $\epsilon\colon \sum a_w w \mapsto \sum a_w$ is the augmentation map.

Now suppose $k \subset L \subset K$ with L/k Galois. Let $H \subset G$ be the subgroup fixing L. We have a natural embedding $Y_{K'} \hookrightarrow Y_K$ given by

$$w_L \mapsto \sum_{w|w_L} [w : w_L]w = \sum_{h\in H} h \cdot w_0, \tag{4}$$

where $[w : w_L]$ is the degree of the local extension K_w/L_{w_L} and w_0 is an arbitrary fixed place of K above w_L. The coefficients $[w : w_L]$ in (4) force the inclusion $Y_{K'} \hookrightarrow Y_K$ to induce an inclusion

$X_{K'} \hookrightarrow X_K$. Note that $X_L = \mathrm{N}H \cdot X_K$, where $\mathrm{N}H = \sum_{h\in H} h \in \mathbf{Z}[G]$. We do *not* in general have $X_L = (X_K)^H$, but $\mathrm{N}H \cdot X_K$ has finite index in X_K and it is clear that $F \otimes_{\mathbf{Z}} X_L = (F \otimes_{\mathbf{Z}} X_K)^H$ for any field F of characteristic zero.

We write χ_X and χ_Y for the characters of the $\mathbf{C}[G]$-modules $\mathbf{C}X$ and $\mathbf{C}Y$, respectively (see section 2 for this notation). Note that

$$\chi_Y = \chi_X + 1_G, \tag{5}$$

where 1_G is the trivial character of G.

Proposition 3.2.2. *If χ is the character of a $\mathbf{C}[G]$-module V with finite $\mathbf{C}$-dimension, then*

$$r_S(\chi) = \sum_{v\in S} \dim_{\mathbf{C}} V^{G_w} - \dim_{\mathbf{C}} V^G = \langle \chi, \chi_X \rangle_G = \dim_{\mathbf{C}} \mathrm{Hom}_G(V^*, \mathbf{C}X).$$

Remark 3.2.3. In the above formula, G_w denotes a choice of decomposition group in G of some place w above v. Since all choices are conjugate, $\dim_{\mathbf{C}} V^{G_w}$ only depends on v. Also, an interesting consequence of the equality $r(\chi) = \langle \chi, \chi_X \rangle_G$ is that $r(\chi) = r(\chi^\alpha)$ for any automorphism α of $\mathbf{C}$. This is not obvious from the analytic definition of $r(\chi)$.

Proof. We have

$$\dim_{\mathbf{C}} \mathrm{Hom}_G(V^*, \mathbf{C}X) = \dim_{\mathbf{C}}(\mathrm{Hom}(V^*, \mathbf{C}X))^G = \langle \chi \cdot \chi_X, 1_G \rangle_G$$

since $\mathrm{Hom}(V^*, \mathbf{C}X) \cong V \otimes_{\mathbf{C}} \mathbf{C}X$ has character $\chi \cdot \chi_X$. Since $\langle \chi \cdot \chi_X, 1_G \rangle_G = \langle \chi, \overline{\chi}_X \rangle_G$ and χ_X takes on only rational (in fact integer) values, the last equality of the proposition is proven.

For the second equality, we use (3), (5), and Frobenius Reciprocity:

$$\begin{aligned} \langle \chi, \chi_X \rangle_G &= \sum_{v\in S} \left\langle \chi, \mathrm{Ind}_{G_w}^G 1_{G_w} \right\rangle_G - \langle \chi, 1_G \rangle_G \\ &= \sum_{v\in S} \langle \chi|_{G_w}, 1_{G_w} \rangle_{G_w} - \dim V^G \\ &= \sum_{v\in S} \dim V^{G_w} - \dim V^G. \end{aligned}$$

It remains to show that $r(\chi)$ is equal to any of the other three expressions in the proposition. Brauer's Theorem (Theorem A.2.3) states that χ can be written as a $\mathbf{Z}$-linear combination

$$\chi = \sum_\theta n_\theta \,\mathrm{Ind}_{H_\theta}^G \theta,$$

where the characters θ are 1-dimensional representations of various subgroups H_θ of G. By the naturality properties of L-functions (see Proposition A.7.2) we find that

$$r(\chi) = \sum n_\theta r(\theta). \tag{6}$$

Here $r(\theta)$ is the order at $s = 0$ of $L_{F_\theta/k, S_{F_\theta}}(s, \theta)$, where $F_\theta = K^{H_\theta}$. Since the inner product $\langle\ ,\ \rangle$ satisfies the analogous naturality properties, we obtain

$$\langle \chi, \chi_X \rangle_G = \sum n_\theta \langle \theta, (\chi_X)|_H \rangle_H. \tag{7}$$

Comparing (6) and (7), it suffices to study the θ's instead of χ, so we are reduced to proving the desired equation for χ a 1-dimensional character.

If $\chi = 1_G$, then $L_S(s,\chi) = \zeta_{k,S}(s)$ and we obtain from Corollary 3.1.4 that

$$r(\chi) = |S| - 1 = \sum_{v \in S} \dim V^{G_w} - \dim V^G.$$

If χ is 1-dimensional and $\chi \neq 1_G$, then $V^G = \{0\}$. Furthermore, $L_{S_\infty}(s,\chi) = L(s,\chi)$ has neither a zero nor a pole at $s = 1$ in this case. Recall that the functional equation for $L(s,\chi)$, as in Theorem A.8.1, is

$$\Lambda_0(s,\chi) = g(s)\Lambda_0(1-s,\overline{\chi}), \tag{8}$$

where $g(s)$ is a nowhere vanishing holomorphic function, and

$$\Lambda_0(s) = \Gamma_{\mathbf{C}}(s)^{r_2}\Gamma_{\mathbf{R}}(s)^{a_1}\Gamma_{\mathbf{R}}(s+1)^{a_2}L(s,\chi),$$

with

$$a_1 = \sum_{v \text{ real}} \dim V^{G_w}, \qquad a_2 = \sum_{v \text{ real}} \operatorname{codim} V^{G_w}.$$

Equating the orders at $s = 0$ in (8), we obtain

$$-r_2 - a_1 + r_{S_\infty}(\chi) = 0,$$

so

$$r_{S_\infty}(\chi) = r_2 + a_1 = \sum_{v \in S_\infty} \dim V^{G_w},$$

thanks to the fact that we are in the case $\dim_{\mathbf{C}} V = 1$.

Since

$$L_S(s,\chi) = \prod_{\substack{\mathfrak{p} \in S - S_\infty \\ \chi(I_\mathfrak{p}) = 1}} (1 - \chi(\sigma_\mathfrak{p}) \operatorname{N}\mathfrak{p}^{-s}) L(s,\chi)$$

and $G_\mathfrak{p}$ is generated by $I_\mathfrak{p}$ and a Frobenius element $\sigma_\mathfrak{p}$, we find that

$$\begin{aligned} r_S(\chi) &= |\{\mathfrak{p} \in S - S_\infty : \chi(G_\mathfrak{p}) = 1\}| + r_{S_\infty}(\chi) \\ &= \sum_{\mathfrak{p} \in S - S_\infty} \dim V^{G_\mathfrak{p}} + r_{S_\infty}(\chi) \\ &= \sum_{v \in S} \dim V^{G_w}, \end{aligned}$$

as desired. □

For future reference, we restate this result for the case where χ is a 1-dimensional character.

Proposition 3.2.4. *If χ is a 1-dimensional character of G then*

$$r_S(\chi) = \begin{cases} |S| - 1 & \text{if } \chi = 1_G \\ |\{v \in S : \chi(G_v) = 1\}| & \text{otherwise.} \end{cases}$$

3.3 The Stark regulator

The next step in stating Stark's conjecture is to define an analogue of the regulator term R_S appearing in Corollary 3.1.4, the "generalized" Dirichlet class number formula.

Notation 3.3.1. As in the case of the standard regulator, we consider the units

$$\mathcal{O}_S^* = U_{k,S} = \{x \in k^* : |x|_v = 1 \text{ for } v \notin S\}$$

of the Dedekind ring $\mathcal{O}_S$. We will usually write $U = U_K$ for the S_K-units U_{K,S_K} when K and S are fixed.

Due to the product formula, we have the "logarithmic embedding" $\lambda = \lambda_K \colon U \to \mathbf{R}X$ given by

$$\lambda(u) = \sum_{w \in S_K} \log |u|_w \cdot w.$$

This is a $\mathbf{Z}[G]$-module homomorphism, and the unit theorem [10, V.I] states:

Theorem 3.3.2 (Unit Theorem). *The kernel of λ is the set of roots of unity $\mu(K)$ in K, and the image is a lattice of full rank $|S| - 1$ in $\mathbf{R}X$. Thus $U/\mu(K)$ is a free abelian group on $|S| - 1$ generators and $1 \otimes \lambda \colon \mathbf{R}U \to \mathbf{R}X$ is an isomorphism of $\mathbf{R}[G]$-modules.*

Remark 3.3.3. Suppose that $k \subset L \subset K$ with L/k Galois. Let $H \subset G$ be the subgroup fixing L. Recall the embedding $X_L \hookrightarrow X_K$ induced by (4). This inclusion is compatible with the homomorphism λ, which is to say the diagram

$$\begin{array}{ccc} U_L & \xrightarrow{\lambda_L} & \mathbf{R}X_L \\ \downarrow & & \downarrow \\ U_K & \xrightarrow{\lambda_K} & \mathbf{R}X_K \end{array}$$

commutes. One checks this by noting that for $u \in U_L$ we have $|u|_w = |u|_{w_L}^{[w:w_L]}$, where w_L is a place of L lying under a place w of K (see section 2).

By tensoring with $\mathbf{C}$, we obtain an isomorphism of $\mathbf{C}[G]$-modules $\mathbf{C}U \to \mathbf{C}X$ also denoted λ. Therefore, the $\mathbf{C}[G]$-modules $\mathbf{C}U$ and $\mathbf{C}X$ have the same character. Since extension of scalars commutes with formation of characters, the characters of the $\mathbf{Q}[G]$-modules $\mathbf{Q}U$ and $\mathbf{Q}X$ are equal. Therefore, these two $\mathbf{Q}[G]$-modules are *isomorphic*, though *not canonically.* Let $f \colon \mathbf{Q}X \to \mathbf{Q}U$ be any such $\mathbf{Q}[G]$-module isomorphism. Complexifying gives an isomorphism $f \colon \mathbf{C}X \to \mathbf{C}U$. An isomorphism $\mathbf{C}X \to \mathbf{C}U$ which arises in such a manner is said to be *defined over* $\mathbf{Q}$. Composing with λ gives an automorphism $\lambda \circ f$ of $\mathbf{C}X$. With this notation, we can give a reformulation of the generalized Dirichlet class number formula which will serve as the motivation for the statement of Stark's Conjecture.

Proposition 3.3.4. *Let $K = k$, and let f be an injection of G-modules $X \hookrightarrow U$. Consider the isomorphism*

$$f \colon \mathbf{C}X \to \mathbf{C}U.$$

obtained by complexifying. Composing with $\lambda \colon \mathbf{C}U \to \mathbf{C}X$ gives an automorphism $\lambda \circ f$ of the $\mathbf{C}[G]$-module $\mathbf{C}X$. Let $c(1)$ be the first non-zero coefficient in the Taylor series of

$$\zeta_k(s) = L_{k/k}(s, 1_G)$$

at $s = 0$. Then we have

$$c(1) = \pm \det(\lambda \circ f) \cdot \frac{h_S}{[U : f(X)]},$$

where h_S is the class number of the ring $\mathcal{O}_S$ of S-integers of k.

Proof. If we fix a place $v_0 \in S$, then we have the decomposition

$$X = \bigoplus_{S-\{v_0\}} \mathbf{Z}(v - v_0).$$

Let $\epsilon_v = f(v - v_0) \in U$. We explicitly evaluate

$$\lambda \circ f(v - v_0) = \lambda(\epsilon_v) = \sum_{v' \in S} \log |\epsilon_v|_{v'} \cdot v' = \sum_{S-\{v_o\}} \log |\epsilon_v|_{v'} (v' - v_0).$$

Here we use that

$$\prod_{v' \in S} |\epsilon_v|_{v'} = 1,$$

due to the product formula and the fact that $\epsilon_v \in U = \mathcal{O}_S^*$.

The matrix for $\lambda \circ f$ with respect respect to our basis $\{v - v_0 : v \in S - \{v_0\}\}$ of $\mathbf{C}X$ is

$$(\log |\epsilon_v|_{v'})_{v,v' \in S-\{v_0\}}.$$

Up to a sign, the determinant of this matrix is the product of the standard regulator R_S of the S-units and the index of the subgroup of $U/\mu(k)$ generated by the ϵ_v. Since the ϵ_v generate $f(X)$, we obtain

$$\det(\lambda \circ f) = \pm R_S [U : f(X)\mu(k)] = \pm \frac{R_S [U : f(X)]}{e_k}.$$

In the last equality, we use the fact that $f(X) \cap \mu(k) = \{1\}$, since f is an injection and X is torsion-free. The Dirichlet class number formula states that

$$c(1) = -\frac{h_S R_S}{e_k}.$$

The desired formula follows. $\square$

With this reformulation of the Dirichlet class number formula as our motivation, we now define the Stark regulator.

Let $f \colon \mathbf{C}X \to \mathbf{C}U$ be defined over $\mathbf{Q}$. Let V be a finite-dimensional $\mathbf{C}[G]$-module with character χ. The automorphism $\lambda \circ f$ gives, by functorality, a $\mathbf{C}$-linear automorphism

$$\mathrm{Hom}_G(V^*, \mathbf{C}X) \xrightarrow{(\lambda \circ f)_V} \mathrm{Hom}_G(V^*, \mathbf{C}X)$$

$$\varphi \longmapsto \lambda \circ f \circ \varphi.$$

The *Stark regulator* associated to f is the determinant of this automorphism:

$$R(\chi, f) = \det((\lambda \circ f)_V).$$

Since $(\lambda \circ f)_V$ is defined by functorality, it is clear that the regulator $R(\chi, f)$ depends only on the character χ and not its realization V. In section 3.6, we will determine how changing the isomorphism f changes the Stark regulator. Note that by Proposition 3.2.2, $R(\chi, f)$ is the determinant of an automorphism of a complex vector space of dimension $r(\chi)$.

Consider the case where $K = k$ and $G = \mathrm{Gal}(K/k) = \{1\}$. If V is the trivial representation, then we have a canonical isomorphism

$$\mathrm{Hom}_G(V^*, \mathbf{C}X) \cong \mathbf{C}X.$$

Therefore, the determinant of the automorphism $(\lambda \circ f)_V$ is equal to the determinant of $\lambda \circ f$. We interpret Proposition 3.3.4 as saying that the Stark regulator $R(\chi, f)$ should play the part of the regulator R_S in any analogue of the Dirichlet class number formula for general Artin L-functions. Indeed, in the case studied in Proposition 3.3.4, the Stark regulator differs from R_S by a factor which is a *rational* number.

3.4 Stark's original example

The form of Stark's conjecture which we give in section 3.5 is the culmination of observations Stark made while considering specific cases. In this section, we show that, in certain cases, the Stark regulator $R(\chi, f)$ has the form described in the Introduction: it is the determinant of a matrix of linear forms in logarithms.

The following example was analyzed by Stark in [24]. Let $k = \mathbf{Q}$ and assume that χ is a character of $G = \mathrm{Gal}(K/\mathbf{Q})$ not containing the trivial character 1_G as a summand. Furthermore, let $S = S_\infty = \{\infty\}$, where ∞ is the unique archimedean prime of $\mathbf{Q}$. (We will see later that proving Stark's conjecture with these assumptions will prove it in general.) Our goal here will be to explicitly write down the regulator $R(\chi, f)$ for a particular choice of isomorphism f. This expression will be the determinant of an $r(\chi) \times r(\chi)$ matrix whose entries are linear forms in logarithms of valuations of units belonging to K.

Let V be a $\mathbf{C}[G]$-module realizing the character χ. Chose a basis $\{e_i\}$ for V. For $\sigma \in G$, write $A(\sigma) = (a_{ij}(\sigma))_{i,j}$ for the matrix representing the automorphism σ of V with respect to the basis $\{e_i\}$, so

$$e_j^\sigma = \sum_i a_{ij}(\sigma) e_i \text{ for all } \sigma \in G.$$

Now fix an archimedean place w of K (note that since K is Galois over $k = \mathbf{Q}$, the archimedean places of K are conjugate to one another and hence are all complex or all real). If w is complex, let $\tau \in G$ represent complex conjugation in w; if w is real, let $\tau = 1$. In either case, τ generates the decomposition group G_w. We choose our basis $\{e_i\}$ of V so that the matrix of τ in this basis is

$$A(\tau) = \left(\begin{array}{c|c} \mathrm{Id}_a & 0 \\ \hline 0 & -\mathrm{Id}_b \end{array}\right),$$

where $a + b = \chi(1)$. Since $\chi(\tau) = \mathrm{Tr}\, A(\tau) = a - b$, we deduce from Proposition 3.2.2 that

$$r(\chi) = \dim V^{G_w} = a = \frac{1}{2}(\chi(1) + \chi(\tau)), \tag{9}$$

thanks to our assumption that V contains no trivial subrepresentation.

In order make a convenient choice of isomorphism f, we will use a theorem of Minkowski [12] on the existence of a special unit in K.

Lemma 3.4.1 (Minkowski's Unit Theorem). *Let $K/\mathbf{Q}$ be a finite Galois extension. There is a unit ϵ of K fixed by τ such that there is exactly one multiplicative relation among the ϵ^σ for $\sigma \in G/G_w$, and this relation is*

$$\prod_{\sigma \in G/G_w} \epsilon^\sigma = \pm 1.$$

Choose a unit $\epsilon \in K$ as in Lemma 3.4.1. We define the isomorphism $f_\epsilon \colon \mathbf{Q}X \to \mathbf{Q}U$ of $\mathbf{Q}[G]$-modules to be the map induced by the G-homomorphism $f_\epsilon \colon Y \to U$ given by $\sigma w \mapsto \epsilon^\sigma$. The fact that $f_\epsilon \colon \mathbf{Q}X \to \mathbf{Q}U$ is an isomorphism follows from the special property of ϵ in Lemma 3.4.1. To calculate the regulator $R(\chi, f_\epsilon)$, we identify $\mathrm{Hom}_G(V^*, \mathbf{C}X)$ with $\mathrm{Hom}_G(V^*, \mathbf{C}Y)$; this is possible because $\mathbf{C}Y = \mathbf{C}X \oplus \mathbf{C}$ and $\mathrm{Hom}_G(V^*, \mathbf{C}) = 0$ by our assumption that V does not contain the trivial representation. We further identify $\mathrm{Hom}_G(V^*, \mathbf{C}Y)$ with $(V \otimes \mathbf{C}Y)^G$ by using the canonical $\mathbf{C}[G]$-linear isomorphism

$$\iota \colon V \otimes \mathbf{C}Y \to \mathrm{Hom}(V^*, \mathbf{C}Y).$$

With these identifications, the regulator $R(\chi, f_\epsilon)$ is the determinant of the automorphism $1 \otimes (\lambda \circ f_\epsilon)$ of $(V \otimes \mathbf{C}Y)^G$. Indeed, if $\iota(\nu \otimes \mu) = \theta$, then $(\lambda \circ f_\epsilon)_V(\theta) = \lambda \circ f_\epsilon \circ \theta$ and

$$\lambda \circ f_\epsilon \circ \theta(\varphi) = \varphi(\nu)\lambda \circ f_\epsilon(\mu) = \iota\big((1 \otimes (\lambda \circ f_\epsilon))(\nu \otimes \mu)\big)(\varphi).$$

In order to compute the determinant of $1 \otimes (\lambda \circ f_\epsilon)$ on $(V \otimes \mathbf{C}Y)^G$, note that any element of $(V \otimes \mathbf{C}Y)^G$ can be written uniquely as

$$x = \sum_{\sigma \in G/G_w} x_\sigma \otimes \sigma w$$

with $x_1 \in V^{G_w}$ and $x_\sigma = \sigma x_1$ for all $\sigma \in G/G_w$. We thus have a $\mathbf{C}$-linear isomorphism

$$\Phi \colon (V \otimes \mathbf{C}Y)^G \to V^{G_w}$$

given by $x \mapsto x_1$. Recall that we have chosen our basis of V so that $e_1, \ldots, e_a$ form a basis for V^{G_w}. We may calculate

$$\begin{aligned}
\Phi \circ (1 \otimes (\lambda \circ f_\epsilon)) \circ \Phi^{-1}(e_j) &= \Phi \circ (1 \otimes (\lambda \circ f_\epsilon)) \left(\sum_\sigma \sigma e_j \otimes \sigma w \right) \\
&= \Phi \left(\sum_\sigma \sigma e_j \otimes \lambda \circ f_\epsilon(\sigma w) \right) \\
&= \Phi \left(\sum_{\sigma, \sigma'} \log |\epsilon^\sigma|_{\sigma' w} (\sigma e_j \otimes \sigma' w) \right) \\
&= \Phi \left(\sum_{\sigma, \sigma', i} \log |\epsilon^\sigma|_{\sigma' w} a_{ij}(\sigma)(e_i \otimes \sigma' w) \right) \\
&= \sum_{\sigma, i} \log |\epsilon^\sigma|_w a_{ij}(\sigma) e_i, \qquad (10)
\end{aligned}$$

where all sums for σ and σ' range over representatives of the elements of G/G_w. Since the right side of (10) is an element of V^{G_w}, the coefficient $\sum_\sigma \log |\epsilon^\sigma|_w |a_{ij}(\sigma)$ of e_i for $i > a$ must be zero. We thus see that our regulator is

$$R(\chi, f_\epsilon) = \det \left(\sum_{\sigma \in G/G_w} a_{ij}(\sigma) \log |\epsilon^\sigma|_w \right)_{1 \le i,j \le a}.$$

Since we saw that $a = r(\chi)$ in equation (9), the regulator $R(\chi, f_\epsilon)$ is the determinant of an $r(\chi) \times r(\chi)$ matrix whose entries are linear forms in logarithms of valuations of units belonging to K.

Remark 3.4.2. $R(\chi, f_\epsilon)$ is not quite the same as Stark's $R(\chi, \epsilon)$ in [24]. Stark lets G act on V on the right, rather than on the left as we do here. Thus our matrices $A(\sigma) = (a_{ij}(\sigma))$ must be replaced by their transposes $A'(\sigma) = (a'_{ij}(\sigma))$ in order to agree with Stark's notation. Also, Stark's $R(\chi, \epsilon)$ has extra factor of $|G_w|^a$, since his definition is

$$R(\chi, \epsilon) = \det\left(\sum_{\sigma\in G} a'_{ij}(\sigma)\log|\epsilon^\sigma|_w\right)_{1\le i,j\le a} = |G_w|^a \det\left(\sum_{\sigma\in G/G_w} a'_{ij}(\sigma)\log|\epsilon^\sigma|_w\right)_{1\le i,j\le a}.$$

3.5 The non-abelian Stark conjecture

Having seen that the Stark regulator $R(\chi, f)$ is a reasonable analogue of the regulator R_S appearing in the Dirichlet class number formula (at least for the base field $k = \mathbf{Q}$), we now state Stark's conjecture. With Proposition 3.3.4 as our motivation, we propose:

Conjecture 3.5.1 (Stark). *Let K/k be a finite Galois extension of number fields with $G = \mathrm{Gal}(K/k)$. Let χ be the character of a finite-dimensional representation of G over $\mathbf{C}$, and let $f\colon \mathbf{Q}X \to \mathbf{Q}U$ be a $\mathbf{Q}[G]$-module homomorphism. Define*

$$A(\chi, f) = \frac{R(\chi, f)}{c(\chi)} \in \mathbf{C}.$$

Then

$$\begin{cases} A(\chi, f) \in \mathbf{Q}(\chi) \text{ and} \\ A(\chi, f)^\alpha = A(\chi^\alpha, f) \text{ for all } \alpha \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q}). \end{cases} \tag{11}$$

Equivalently,

$$A(\chi^\alpha, f) = A(\chi, f)^\alpha \tag{12}$$

for all field automorphisms α of $\mathbf{C}$.

The equivalence of the two statements of the conjecture follows from basic field theory, as we now explain. The fact that (11) implies (12) is clear. Conversely, suppose we have (12). If $\alpha \in \mathrm{Aut}\,\mathbf{C}$ fixes $\mathbf{Q}(\chi)$, then $A(\chi, f) = A(\chi^\alpha, f) = A(\chi, f)^\alpha$. Proposition A.11.3 implies that $A(\chi, f) \in \mathbf{Q}(\chi)$. The statement that $A(\chi, f)^\alpha = A(\chi^\alpha, f)$ for all elements $\alpha \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$ follows from the fact that $\mathrm{Aut}\,\mathbf{C} \to \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$ is surjective (Proposition A.11.1).

Observe that when $k = \mathbf{Q}$, $S = \{\infty\}$, $\langle \chi, 1_G\rangle_G = 0$, and $f = f_\epsilon$ as in section 3.4, we are in precisely the setting originally considered by Stark in [24]. We will see in sections 3.6 and 3.7 that it suffices to prove Conjecture 3.5.1 in these cases. Of course, it is obviously a significant technical advantage to be able to consider the conjecture over general base fields.

We now present a version of Stark's conjecture proposed by Deligne, where the representation of G is defined over an arbitrary field F which can be embedded in $\mathbf{C}$. Let χ be the character of an $F[G]$-module V of finite dimension over F. Instead of letting f arise from an isomorphism $\mathbf{Q}X \to \mathbf{Q}U$ as before, we now let f be any $F[G]$-module isomorphism $FX \to FU$.

For any $\alpha\colon F \hookrightarrow \mathbf{C}$, we construct the $\mathbf{C}[G]$-module $V^\alpha = \mathbf{C}\otimes_{F,\alpha} V$, whose character is $\chi^\alpha = \alpha\circ\chi$. To the character χ^α of $G = \mathrm{Gal}(K/k)$ there corresponds the Artin L-function $L_S(s, \chi^\alpha)$. We further define $f^\alpha\colon \mathbf{C}X \to \mathbf{C}U$ to be the extension of scalars of f by means of $\alpha\colon F \hookrightarrow \mathbf{C}$. As before, we have an induced endomorphism $(\lambda\circ f^\alpha)_{V^\alpha}$ of the $\mathbf{C}[G]$-module $\mathrm{Hom}_G((V^\alpha)^*, \mathbf{C}X)$. Its (nonzero) determinant is the Stark regulator $R(\chi^\alpha, f^\alpha)$, which is independent of the realization V of χ over F.

Conjecture 3.5.2 (Deligne). *There exists an element $A(\chi, f) \in F$ such that for all $\alpha\colon F \hookrightarrow \mathbf{C}$, we have*

$$R(\chi^\alpha, f^\alpha) = A(\chi, f)^\alpha \cdot c(\chi^\alpha).$$

Note that if $F = \mathbf{C}$, and $f(\mathbf{Q}X) \subset \mathbf{Q}U$, then $f(\mathbf{Q}X) = \mathbf{Q}U$ for dimension reasons and $f^\alpha = f$ for all $\alpha\colon \mathbf{C} \hookrightarrow \mathbf{C}$. Thus Stark's Conjecture 3.5.1 is a special case of the Conjecture 3.5.2. We will see in section 3.6 that the two conjectures are actually equivalent. Obviously, it suffices to prove Conjecture 3.5.2 for F finitely generated over $\mathbf{Q}$ in order to prove it in general. For our purposes, the significance of Conjecture 3.5.2 is that it will allow us to show that the choice of isomorphism f does not affect the truth of the conjecture (for fixed K/k and χ). It would be awkward to directly show that Conjecture 3.5.1 is independent of the choice of f.

3.6 Changing the isomorphism f

We now study the dependence of Stark's conjecture on the isomorphism f and the set of primes S. In this section, we show that if Conjecture 3.5.2 is true for one choice of f, then it is true for all f. As a corollary we will find that Conjectures 3.5.1 and 3.5.2 are equivalent. We begin by studying some formal properties satisfied by the regulator $R(\chi, f)$.

Definition 3.6.1. Let χ be the character of a $\mathbf{C}[G]$-module V. If θ is any $\mathbf{C}[G]$-endomorphism of a $\mathbf{C}[G]$-module M with finite $\mathbf{C}$-dimension, we define $\delta(\chi, \theta)$ to be the determinant of the automorphism θ_V of $\mathrm{Hom}_G(V^*, M)$ induced by θ.

Clearly $\delta(\chi, \theta)$ is independent of the realization V of χ. As an example, the Stark regulator is $R(\chi, f) = \delta(\chi, \lambda \circ f)$, where $\lambda \circ f$ is the automorphism $\mathbf{C}X \to \mathbf{C}X$ induced by a $\mathbf{C}[G]$-module isomorphism $f\colon \mathbf{C}X \to \mathbf{C}U$ defined over $\mathbf{Q}$.

Proposition 3.6.2. *The determinant δ satisfies the following properties:*

(1) $\delta(\chi + \chi', \theta) = \delta(\chi, \theta)\delta(\chi', \theta)$.

(2) $\delta(\mathrm{Ind}_H^G \chi, \theta) = \delta(\chi, \theta)$.

Here χ is the character of a subgroup $H \subset G$, and on the right side θ is considered as a $\mathbf{C}[H]$-endomorphism of M.

(3) $\delta(\mathrm{Infl}\, \chi, \theta) = \delta(\chi, \theta|_{M^H})$.

Here χ is a character of G/H for a normal subgroup $H \subset G$, and on the right side θ is considered as a $\mathbf{C}[G/H]$-endomorphism of M^H.

(4) $\delta(\chi, \theta \circ \theta') = \delta(\chi, \theta)\delta(\chi, \theta')$.

(5) $\delta(\chi, \theta)^\alpha = \delta(\chi^\alpha, \theta^\alpha)$ *for* $\alpha \in \mathrm{Aut}\, \mathbf{C}$.

Here θ^α is the $\mathbf{C}[G]$-endomorphism of $\mathbf{C} \otimes_{\mathbf{C},\alpha} M$ obtained by extension of scalars.

Proof. (1) This is clear since in the appropriate bases, the matrix for $\theta_{V \oplus V'}$ is a block matrix whose blocks are the matrices for θ_V and $\theta_{V'}$.

(2) Recall the fact that for any $\mathbf{C}[G]$-module M and any $\mathbf{C}[H]$-module W, there is a natural isomorphism $\mathrm{Hom}_G(\mathrm{Ind}_H^G W, M) \cong \mathrm{Hom}_H(W, M)$, where on the right hand side, M is considered as a $\mathbf{C}[H]$-module (see A.2.2). Since this isomorphism is functorial in M, the result follows.

(3) If V is a $\mathbf{C}[G/H]$-module and M is a $\mathbf{C}[G]$-module, then there is a natural isomorphism $\mathrm{Hom}_G(V^*, M) \cong \mathrm{Hom}_{G/H}(V^*, M^H)$ (see A.2.2). The result follows by functorality.

(4) This is clear since $(\theta \circ \theta')_V = \theta_V \circ \theta'_V$.

(5) Let $\alpha \in \operatorname{Aut} \mathbf{C}$. Then $V^\alpha = \mathbf{C} \otimes_{\mathbf{C},\alpha} V$ is a realization of χ^α. We wish to compare the determinant of θ_V on $M_1 = \operatorname{Hom}_G(V^*, M)$ and of $\theta^\alpha_{V^\alpha}$ on

$$M_2 = \operatorname{Hom}_G(\mathbf{C} \otimes_{\mathbf{C},\alpha} V^*, \mathbf{C} \otimes_{\mathbf{C},\alpha} M) \cong \mathbf{C} \otimes_{\mathbf{C},\alpha} \operatorname{Hom}_G(V^*, M).$$

Since the formation of determinant is compatible with extension of scalars, we are done. □

Proposition 3.6.3. *Suppose $F = \mathbf{C}$ and that Conjecture* 3.5.2 *is true for some $\mathbf{C}[G]$-module automorphism $f_0 \colon \mathbf{C}X \to \mathbf{C}U$. Then it is true for all $\mathbf{C}[G]$-module automorphisms $f \colon \mathbf{C}X \to \mathbf{C}U$.*

In particular, Conjecture 3.5.1 is equivalent to Conjecture 3.5.2 with $F = \mathbf{C}$.

Proof. Define the $\mathbf{C}[G]$-module automorphism $\theta = f_0^{-1} f \colon \mathbf{C}X \to \mathbf{C}X$. For any character χ of G over $\mathbf{C}$, we define

$$A(\chi, f) = A(\chi, f_0)\delta(\chi, \theta) \in \mathbf{C},$$

where $A(\chi, f_0)$ satisfies Conjecture 3.5.2. For any $\alpha \colon \mathbf{C} \hookrightarrow \mathbf{C}$ we may use Conjecture 3.5.2 for f_0 and Proposition 3.6.2 for θ to compute

$$\begin{aligned}
A(\chi, f)^\alpha &= A(\chi, f_0)^\alpha \delta(\chi, \theta)^\alpha \\
&= \frac{R(\chi^\alpha, f_0^\alpha)}{c(\chi^\alpha)} \delta(\chi^\alpha, \theta^\alpha) \\
&= \frac{\delta(\chi^\alpha, \lambda \circ f_0^\alpha)}{c(\chi^\alpha)} \delta(\chi^\alpha, (f_0^{-1} f)^\alpha) \\
&= \frac{\delta(\chi^\alpha, \lambda \circ f^\alpha)}{c(\chi^\alpha)} \\
&= \frac{R(\chi^\alpha, f^\alpha)}{c(\chi^\alpha)}
\end{aligned}$$

This proves Conjecture 3.5.2 for f. □

Proposition 3.6.4. *Conjecture* 3.5.1 *and Conjecture* 3.5.2 *are equivalent.*

Proof. We have seen that Conjecture 3.5.2 implies Conjecture 3.5.1 by taking the special case $F = \mathbf{C}$. Conversely, to show that Conjecture 3.5.2 follows from Conjecture 3.5.1, we may assume that F is finitely generated over $\mathbf{Q}$ and hence countable. Fix an injection $\alpha \colon F \hookrightarrow \mathbf{C}$. Suppose that $f \colon FX \to FU$ is an $F[G]$-module homomorphism and consider $f^\alpha \colon \mathbf{C}X \to \mathbf{C}U$. By Proposition 3.6.3 and the assumption that 3.5.1 holds, we may assume that Conjecture 3.5.2 holds for f^α (and the base field $\mathbf{C}$). Therefore, if χ is a representation of G over F, there exists $A(\chi^\alpha, f^\alpha) \in \mathbf{C}$ such that

$$R((\chi^\alpha)^\gamma, (f^\alpha)^\gamma) = A(\chi^\alpha, f^\alpha)^\gamma c((\chi^\alpha)^\gamma)$$

for all $\gamma \in \operatorname{Aut} \mathbf{C}$.

In particular, if γ is any automorphism fixing $\alpha(F) \subset \mathbf{C}$, then $A(\chi^\alpha, f^\alpha)$ is fixed by γ. Hence by Proposition A.11.3 there exists a unique $A(\chi, f) \in F$ such that $A(\chi, f)^\alpha = A(\chi^\alpha, f^\alpha) \in \mathbf{C}$. Furthermore, if $\beta \colon F \hookrightarrow \mathbf{C}$ is any injection then we may choose $\gamma \in \operatorname{Aut} \mathbf{C}$ such that $\gamma \circ \alpha = \beta$ by Proposition A.11.1. Then

$$A(\chi, f)^\beta = A(\chi^\alpha, f^\alpha)^\gamma = \frac{R((\chi^\alpha)^\gamma, (f^\alpha)^\gamma)}{c((\chi^\alpha)^\gamma)} = \frac{R(\chi^\beta, \chi^\beta)}{c(\chi^\beta)}$$

as desired. Thus, Conjecture 3.5.2 is satisfied. □

3.7 Independence of the set S

In the previous section, we saw that Stark's Conjecture 3.5.1 is independent of the choice of non-canonical isomorphism $f\colon \mathbf{Q}X \to \mathbf{Q}U$. In this section, we show that the conjecture is also independent of the choice of set of primes S. Essentially, the method of proof will follow the proof of Corollary 3.1.4, where we derived the generalized Dirichlet class number formula from the case $S = S_\infty$. More precisely, we will consider a given set S and relate the terms in the Stark conjecture for S and $S \cup \{\mathfrak{p}\}$, for a prime $\mathfrak{p} \notin S$. However, it will be easier to do this for characters of degree 1 than for general characters. To deduce the general result from the degree 1 case, we will apply Brauer's Theorem (cf. proof of Proposition 3.2.2; this "induction" technique is standard and will be used repeatedly in this thesis).

In order to apply Brauer's Theorem, we will need certain naturality properties of $A(\chi, f)$. We now derive these from the properties of the general determinants δ in Proposition 3.6.2 and from the naturality properties of L-functions.

Proposition 3.7.1. *Let the notation be as in Conjecture* 3.5.1*.*

- $A(\chi + \chi', f) = A(\chi, f)A(\chi', f)$
- $A(\operatorname{Ind}_H^G \chi, f) = A(\chi, f)$ *for a subgroup* $H \subset G$ *and a character* χ *of* $H = \operatorname{Gal}(K/K^H)$*.*
- $A(\operatorname{Infl} \chi, f) = A(\chi, f|_{\mathbf{C}X_L})$ *if* $k \subset L \subset K$*,* L/k *is Galois, and* χ *is a character of* $\operatorname{Gal}(L/k)$*.*
- $A(\overline{\chi}, f) = \overline{A(\chi, f)}$*.*

Proof. The first two assertions follow immediately from Proposition 3.6.2 and Proposition A.7.2. For the third assertion, let $k \subset L \subset K$ and let $H \subset G$ be the subgroup fixing L. We have $c(\operatorname{Infl} \chi) = c(\chi)$ from Proposition A.7.2. However, in dealing with the regulator we must be careful about simply writing

$$R(\operatorname{Infl} \chi, f) = \delta(\operatorname{Infl} \chi, \lambda \circ f) = \delta(\chi, \lambda \circ f|_{(\mathbf{C}X)^H}) = R(\chi, f|_{\mathbf{C}X_L}). \tag{13}$$

Indeed, in the last equality, we must check that the inclusion $X_L \hookrightarrow X_K = X$ induces the equality $\mathbf{C}X_L = (\mathbf{C}X)^H$. We must also check that this inclusion is compatible with λ; that is, the diagram

$$\begin{array}{ccc} U_L & \xrightarrow{\lambda_L} & \mathbf{R}X_L \\ \big\downarrow & & \big\downarrow \\ U_K & \xrightarrow{\lambda_K} & \mathbf{R}X_K \end{array}$$

commutes. As these two facts were noted in 3.2.1 and 3.3.3, we see that (13) holds and the proof of the third assertion is complete.

For the final statement of the proposition, we apply statement (5) of Proposition 3.6.2 with $\alpha \in \operatorname{Aut} \mathbf{C}$ chosen to be complex conjugation. Since $\lambda \circ f$ is defined over $\mathbf{R}$, it follows that $R(\overline{\chi}, f) = \overline{R(\chi, f)}$. Furthermore, from the Euler product representation of L_S and the continuity of complex conjugation, we find that $L_S(s, \overline{\chi}) = \overline{L_S(\overline{s}, \chi)}$. Thus $c(\overline{\chi}) = \overline{c(\chi)}$. This completes the proof. $\square$

Proposition 3.7.2. *The truth of the Conjectures* 3.5.1 *and* 3.5.2 *is independent of the set of primes* S*.*

Proof. We will work with Conjecture 3.5.1. Let $S' = S \cup \{\mathfrak{p}\}$ for a finite prime $\mathfrak{p} \notin S$. Let $U', X', f', c'(\chi), r'(\chi)$, and $A'(\chi, f')$ be the corresponding objects when S is replaced by S'. Since we have seen that the truth of the conjectures is independent of the isomorphisms f and f', we may assume by semi-simplicity considerations that $f'|_{\mathbf{C}X} = f$ and that f is defined over $\mathbf{Q}$. Letting

$$B(\chi) = \frac{A'(\chi, f')}{A(\chi, f)},$$

it suffices to show that $B(\chi^\alpha) = B(\chi)^\alpha$ for all automorphisms α of $\mathbf{C}$.

Using Brauer's Theorem and Proposition 3.7.1 we may reduce to the case where χ is 1-dimensional; by the third part of Proposition 3.7.1 we may replace K by $K^{\operatorname{Ker}\chi}$ and so assume that K/k is abelian. Let $\mathfrak{P} \in S'_K$ be a prime lying above $\mathfrak{p}$ and let $G_{\mathfrak{P}} \subset G$ be its decomposition group (which is independent of $\mathfrak{P}$ since G is abelian). We consider two cases.

Case 1: $\chi(G_{\mathfrak{P}}) \neq 1$. In this case, the formula in Proposition 3.2.2 shows that $r(\chi) = r'(\chi)$. Furthermore, if V is a realization of χ and φ is any G-invariant homomorphism from V^* to $\mathbf{C}X'$, then the composite map

$$V^* \longrightarrow \mathbf{C}X' \longrightarrow \mathbf{C}X'/\mathbf{C}X$$

of $G_{\mathfrak{P}}$-modules must be zero, since the right hand side has trivial $G_{\mathfrak{P}}$ action (recall $G_{\mathfrak{P}} = G_{\mathfrak{P}'}$ for all $\mathfrak{P}'$ over $\mathfrak{p}$) and the left hand side is 1-dimensional with non-trivial $G_{\mathfrak{P}}$ action. Thus, there is a natural identification $\operatorname{Hom}_G(V^*, \mathbf{C}X') = \operatorname{Hom}_G(V^*, \mathbf{C}X)$ and $R'(\chi, f') = R(\chi, f)$ since $f'|_{\mathbf{C}X} = f$.

Furthermore, we can compare the coefficients $c(\chi)$ and $c'(\chi)$. If $\chi(I_{\mathfrak{P}}) \neq 1$ then $L_S(s, \chi) = L_{S'}(s, \chi)$ and hence $c(\chi) = c'(\chi)$. Thus $B(\chi) = 1 = B(\chi^\alpha)$ and our desired equality is satisfied. If $\chi(I_{\mathfrak{P}}) = 1$, then $c(\chi') = (1 - \chi(\sigma_{\mathfrak{P}}))^{-1} c(\chi)$, and $B(\chi) = 1 - \chi(\sigma_{\mathfrak{P}})$. The result follows.

Case 2: $\chi(G_{\mathfrak{P}}) = 1$. By Proposition 3.7.1, we can replace K with $K^{G_{\mathfrak{P}}}$, so we may assume that $\mathfrak{p}$ splits completely in K. Then $\sigma_{\mathfrak{p}} = 1$, so $L_{S'}(s, \chi) = (1 - \mathrm{N}\mathfrak{p}^{-s}) L_S(s, \chi)$. Therefore $r'(\chi) = r(\chi) + 1$ and $c'(\chi) = (\log \mathrm{N}\mathfrak{p}) c(\chi)$.

Now let $\mathfrak{P}^h = \pi \mathcal{O}_K$ for some positive $h \in \mathbf{Z}$ and $\pi \in K^*$ (for example, we may take $h = h_K$). Clearly π is a unit away from $\mathfrak{P}$ and $v_{\mathfrak{P}}(\pi) = h$. So $\mathbf{Q}[G]\pi$ is a $\mathbf{Q}[G]$-submodule of $\mathbf{Q}U'$ whose intersection with $\mathbf{Q}U \subset \mathbf{Q}U'$ is $\{0\}$. By dimension counting, we find

$$\mathbf{Q}U' = \mathbf{Q}U \oplus \mathbf{Q}[G]\pi$$

as $\mathbf{Q}[G]$-modules, since $|S'_K| - 1 = (|S_K| - 1) + |G|$ and $\mathfrak{p}$ is totally split in K. By similar reasoning we find that

$$\mathbf{Q}X' = \mathbf{Q}X \oplus \mathbf{Q}[G]x,$$

where $x = \left(\mathfrak{P} - \frac{1}{|G|} \mathrm{N}G \cdot w_0\right)$, with w_0 an arbitrary archimedean place of K, and

$$\mathrm{N}G = \sum_{\sigma \in G} \sigma \in \mathbf{Q}[G].$$

The $\mathbf{Q}[G]$-modules $\mathbf{Q}[G]x$ and $\mathbf{Q}[G]\pi$ are isomorphic since each is the regular representation of G over $\mathbf{Q}$. We define the isomorphism $f' \colon \mathbf{Q}X' \to \mathbf{Q}U'$ by $f'|_{\mathbf{Q}X} = f$ and

$$f'(x^\sigma) = \pi^\sigma.$$

The matrices for λ' and f' with respect to suitable bases over $\mathbf{C}$, are

$$\begin{aligned} M(\lambda') &= \left(\begin{array}{c|c} M(\lambda) & * \\ \hline 0 & \log|\pi|_{\mathfrak{P}} \operatorname{Id}_{|G|} \end{array}\right), \\ M(f') &= \left(\begin{array}{c|c} M(f) & * \\ \hline 0 & \operatorname{Id}_{|G|} \end{array}\right), \end{aligned}$$

so

$$M(\lambda' \circ f') = \left(\begin{array}{c|c} M(\lambda \circ f) & * \\ \hline 0 & \log|\pi|_{\mathfrak{P}} \operatorname{Id}_{|G|} \end{array}\right).$$

Here, $M(\lambda)$ and $M(f)$ are the $(|S|-1) \times (|S|-1)$ matrices for λ and f respectively, and $\operatorname{Id}_{|G|}$ is the $|G| \times |G|$ identity matrix. We can now compute the regulator $R'(\chi, f')$. Define a basis for $\operatorname{Hom}_G(V^*, \mathbf{C}X')$ by starting with a basis for $\operatorname{Hom}_G(V^*, \mathbf{C}X)$ and adding a basis vector for the 1-dimensional space $\operatorname{Hom}_G(V^*, \mathbf{C}[G]x)$. With respect to this basis, the matrix for $(\lambda' \circ f')_V$ is

$$M((\lambda' \circ f')_V) = \left(\begin{array}{c|c} M((\lambda \circ f)_V) & * \\ \hline 0 & \log|\pi|_{\mathfrak{P}} \end{array}\right).$$

Thus $R'(\chi, f') = R(\chi, f) \cdot \log|\pi|_{\mathfrak{P}}$, so $B(\chi) = \frac{\log|\pi|_{\mathfrak{P}}}{\log \mathrm{N}\mathfrak{p}} = -h$, a rational number independent of χ. This completes the proof. □

Proposition 3.7.2 now allows us to see that it is enough to prove Stark's conjecture in the setting originally considered by Stark as in section 3.4. This is made precise by the following proposition.

Proposition 3.7.3. *If the Stark conjecture is true for every finite Galois extension $K/\mathbf{Q}$ then it is true in general. If the conjecture is true for all abelian extensions K/k then it is true for every finite Galois extension K/k.*

Proof. We work with Conjecture 3.5.1. For the first assertion, suppose that K/k is Galois and χ is a character of $\operatorname{Gal}(K/k)$. By Proposition 3.7.2, it suffices to consider the case when $S = S_\infty$, the set of infinite primes. Let K' be the Galois closure of K over $\mathbf{Q}$ and write $G = \operatorname{Gal}(K'/\mathbf{Q})$, $H = \operatorname{Gal}(K'/k)$. The assumption $S = S_\infty$ ensures that when we restrict the primes of S to $\mathbf{Q}$ and take the primes of K' lying over them, we will obtain the same set as if we had simply lifted the primes of S to K' directly.

If f is an isomorphism $\mathbf{C}X_{K'} \to \mathbf{C}U_{K'}$ defined over $\mathbf{Q}$, then we obtain from Proposition 3.7.1

$$A(\operatorname{Ind}_H^G \operatorname{Infl} \chi, f) = A(\operatorname{Infl} \chi, f) = A(\chi, f|_{\mathbf{C}X_K}).$$

Therefore, knowing Stark's conjecture for the Galois extension $K'/\mathbf{Q}$ will prove it for general K/k.

The second assertion follows from Brauer's Theorem, using Proposition 3.7.1 in a similar manner as in the proof of Proposition 3.7.2. □

Using Proposition 3.7.1 and the independence of f in the Stark conjecture, we can now prove the conjecture for the trivial character 1_G and arbitrary K/k.

Proposition 3.7.4. *Stark's conjecture is true for the trivial character 1_G. Moreover, if f arises from an injection $X \hookrightarrow U$ of $\mathbf{Z}[G]$-modules, then*

$$A(1_G, f) = \pm \frac{[U_k : f(X_k)]}{h_S} \in \mathbf{Q},$$

where $h_S = h_{k,S}$ is the class number of the ring $\mathcal{O}_S$ of S-integers of k.

Proof. First note that if $f\colon X \hookrightarrow U$ then $X^G \hookrightarrow X_k \hookrightarrow U^G = U_k$, so $[U_k : f(X_k)]$ makes sense. Furthermore, for $\chi = 1_G$, the third property of $A(\chi, f)$ in Proposition 3.7.1 implies that we may take $K = k$ without loss of generality. The desired formula now follows from Proposition 3.3.4. Since any isomorphism $f\colon \mathbf{Q}X \to \mathbf{Q}U$ can be scaled by an integer to arise from an injection $X \hookrightarrow U$ of G-modules, the proof of Proposition 3.6.3 shows that $A(1_G, f) \in \mathbf{Q}$ for arbitrary f. □

Since $A(\chi, f)$ behaves multiplicatively under direct sums of representations, Propositions 3.7.3 and 3.7.4 show that it suffices to prove Stark's conjecture when $k = \mathbf{Q}$ and χ is a character not containing the trivial character. In other words, the setting in section 3.4 is sufficient to imply the general case.

4 The cases $r(\chi) = 0$ and $r(\chi) = 1$

In this section we analyze the cases $r(\chi) = 0$ and $r(\chi) = 1$. In the case $r(\chi) = 0$, we prove Stark's conjecture by reducing it to a theorem of Siegel. In the case $r(\chi) = 1$, we give a reformulation of Stark's conjecture which provides an introduction to the "Stark units" in the abelian Stark conjectures. We conclude the section by stating the rank one abelian Stark conjecture and proving some basic facts about it.

4.1 The case $r(\chi) = 0$

We have seen that in many instances, Brauer's Theorem allows us to prove certain statements about general characters χ via reduction to the special case when χ is 1-dimensional. In this section, we will demonstrate how this technique can be used to reduce Stark's conjecture for $r(\chi) = 0$ to a theorem of Siegel on the rationality of partial zeta functions at zero (see A.6.2).

Proposition 3.7.2 implies that we may assume without loss of generality $S = S_\infty$, the set of infinite primes of k. Note that if $r_S(\chi) = 0$, then $r_{S_\infty}(\chi) = 0$. When $r(\chi) = 0$, the regulator is 1 and $L(0, \chi) \neq 0$. We must therefore prove that $L(0, \chi^\alpha) = L(0, \chi)^\alpha$ for all $\alpha \in \operatorname{Aut} \mathbf{C}$.

In reducing to the abelian case, we will need a slightly stronger version of Brauer's Theorem suggested by Serre [2, App.].

Lemma 4.1.1. *Let G be a finite group with center C and let χ be an irreducible character of G over $\mathbf{C}$. The restriction of χ to C is a multiple of a 1-dimensional character ψ of C, and we can write*

$$\chi = \sum_{H_i} n_i \operatorname{Ind}_{H_i}^G \chi_i,$$

where for each i, H_i is a subgroup of G containing C, χ_i is a 1-dimensional character of H_i whose restriction to C is ψ, and $n_i \in \mathbf{Z}$.

Proof. Write $\widehat{G}$ for the group of 1-dimensional representations of G. In general, $|\widehat{G}| = [G : [G, G]]$, where $[G, G]$ is the commutator subgroup of G. For any subgroup H of G, we have $[CH, CH] = [H, H]$ since C is the center of G. Thus,

$$|\widehat{CH}| = [CH : [H, H]] = [CH : H][H : [H, H]] = [CH : H]|\widehat{H}|.$$

Every element of $\widehat{CH}$ restricts to an element of $\widehat{H}$, and for each element of $\widehat{H}$ there are at most $[CH : H]$ elements of CH which restrict to it. Therefore, each $\theta \in \widehat{H}$ has *exactly* $[CH : H]$ elements of $\widehat{CH}$ restricting to it. Call these $\theta_i \in \widehat{CH}$ for $i = 1, \ldots, [CH : H]$. Frobenius reciprocity gives

$$\langle \operatorname{Ind}_H^{CH} \theta, \theta_i \rangle_{CH} = \langle \theta, \theta_i|_H \rangle_H = \langle \theta, \theta \rangle_H = 1.$$

Therefore, each θ_i appears as a summand of $\operatorname{Ind}_H^{CH} \theta$. As $\operatorname{Ind}_H^{CH} \theta$ is a $[CH : H]$-dimensional character, we find

$$\operatorname{Ind}_H^{CH} \theta = \sum_{i=1}^{[CH:H]} \theta_i$$

and therefore

$$\operatorname{Ind}_H^G \theta = \sum_{i=1}^{[CH:H]} \operatorname{Ind}_{CH}^G \theta_i. \tag{14}$$

Combining (14) with Brauer's Theorem, we can write

$$\chi = \sum n_i \operatorname{Ind}_{H_i}^G \chi_i \tag{15}$$

where $n_i \in \mathbf{Z}$, $H_i \supset C$, and $\chi_i(1) = 1$.

Now let V be a $\mathbf{C}[G]$-module realizing χ. Consider V as a representation of $C \subset G$ and let W be an irreducible component; W must be 1-dimensional since C is abelian. Consider the subspace

$$W' = \sum_{g \in G} gW \subset V.$$

Since W' is G-stable and V is irreducible, we must have $W' = V$. But as a representation of C, W' is a direct sum of copies of W, and hence $\chi|_C = m_\chi \psi_\chi$, where $m_\chi \in \mathbf{Z}$ and ψ_χ is the character of the $\mathbf{C}[C]$-module W.

Returning to our decomposition of χ as a linear combination of induced characters, we note that for any irreducible character ϕ of G, we have

$$\langle \phi, \operatorname{Ind}_{H_i}^G \chi_i \rangle_G = \langle \phi|_{H_i}, \chi_i \rangle_{H_i} \tag{16}$$

by Frobenius reciprocity. Since χ_i is irreducible, (16) implies that ϕ is a summand of $\operatorname{Ind}_{H_i}^G \chi_i$ only if χ_i is a summand of $\phi|_{H_i}$. This occurs only if $\phi|_C = m_\phi \psi_\phi$ contains a copy of $\chi_i|_C$, that is, only if $\chi_i|_C = \psi_\phi$. We therefore break up our decomposition for χ into two parts:

$$\chi = \left(\sum_{\chi_i|_C = \psi_\chi} n_i \operatorname{Ind}_{H_i}^G \chi_i \right) + \left(\sum_{\chi_i|_C \neq \psi_\chi} n_i \operatorname{Ind}_{H_i}^G \chi_i \right).$$

None of the irreducible representations of G appearing in one of the two terms of the sum appears in the other. Since χ is an irreducible character not appearing in the second term, it follows that this term is zero and that we have

$$\chi = \sum_{\chi_i|_C = \psi_\chi} n_i \operatorname{Ind}_{H_i}^G \chi_i.$$

This gives the desired result. $\square$

With this tool we can now prove:

Theorem 4.1.2. *Conjecture* 3.5.1 *is true if* $r(\chi) = 0$.

Proof. As usual, we may assume that $S = S_\infty$. If the character χ decomposes as a sum of irreducible characters θ_i, then since $r(\chi)$ behaves additively under direct sums we have $r(\theta_i) = 0$ for all i. Thus, it suffices to consider the case where χ is irreducible. Furthermore, by replacing K by $K^{\operatorname{Ker}\chi}$, we can assume that that χ is a faithful character; note that χ remains irreducible.

We may also assume $\chi \neq 1_G$, since we have proven the conjecture in this case (Proposition 3.7.4). By Proposition 3.2.2, we see that $V^{G_w} = \{0\}$ for each archimedean place w of K. In particular, k is totally real and K is totally complex. If $G_w = \{1, \tau_w\}$ for a complex place w of K, then τ_w acts as -1 on V, since $\tau_w^2 = 1$ and $V^{\tau_w} = \{0\}$. Since the representation V is faithful, all the τ_w must equal the same $\tau \in G$. Thus, K is a totally imaginary quadratic extension of the totally real subfield K^τ.

For any $\sigma \in G$, we have $\sigma\tau\sigma^{-1} = \sigma\tau_w\sigma^{-1} = \tau_{\sigma w} = \tau$ and therefore τ lies in the center C of G. We can apply Lemma 4.1.1 to conclude

$$\chi = \sum_i n_i \operatorname{Ind}_{H_i}^G \chi_i \tag{17}$$

where $n_i \in \mathbf{Z}$, $C \subset H_i$, $\chi_i \in \widehat{H_i}$, and $\chi_i|_C = \psi_\chi$. Thus,

$$\chi_i(\tau) = \psi_\chi(\tau) = \frac{\chi(\tau)}{\chi(1)} = -1,$$

and we obtain

$$L_{K/K^{H_i}}(0, \chi_i) \neq 0$$

since τ generates the decomposition group of the archimedean primes of K over the totally real field $K^{H_i} \subset K^\tau$. Since

$$L(0, \chi) = \prod_i L_{K/K^{H_i}}(0, \chi_i)^{n_i}$$

by (17), it suffices to prove the Conjecture 3.5.1 for the χ_i. In other words, we have reduced to the case where $k = K^{H_i}$ is totally real and $\chi = \chi_i$ is a 1-dimensional representation. Furthermore, by replacing K by $K^{\operatorname{Ker}\chi}$, we may assume that K/k is an abelian extension and χ is faithful.

In the abelian case, we can write L-functions as linear combinations of partial zeta functions (as in A.6.1):

$$L(s, \chi) = \sum_{\sigma \in G} \chi(\sigma)\zeta(s, \sigma).$$

The desired equation $L(0, \chi^\alpha) = L(0, \chi)^\alpha$ now follows from a theorem of Siegel which states that $\zeta(0, \sigma)$ is a rational number for all $\sigma \in G$ (see Theorem A.6.2). □

4.2 The case $r(\chi) = 1$

If $r(\chi) = 1$, then χ is the direct sum of irreducible characters θ_i with $r(\theta_i) = 0$ and one irreducible character θ with $r(\theta) = 1$. Since we have proven the conjecture for the θ_i, we may assume that χ irreducible. We no longer impose the condition $S = S_\infty$. From Proposition 3.2.2 we have

$$r(\chi) = \langle \chi, \chi_X \rangle_G = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\chi_X(\sigma).$$

It is clear that the values of $\chi_X = \chi_Y - 1$ are rational integers, and thus $r(\chi^\alpha) = r(\chi)^\alpha = 1$ for any $\alpha \in \operatorname{Aut}\mathbf{C}$.

Let F be a field of characteristic zero and let $\{\chi_i\}$ be the characters of the irreducible representations $\{V_i\}$ of G over F. Any $F[G]$-module V has a "canonical decomposition" $V = \oplus_i W_i$ where W_i is the sum of the subrepresentations of V isomorphic to V_i (see [18, I.2.6]).

Definition 4.2.1. Let χ be an irreducible character. Define

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \overline{\chi}(\sigma)\sigma \in \mathbf{C}[G].$$

The element e_χ is a central idempotent of $\mathbf{C}[G]$ and acts as projection onto the χ-component in the canonical decomposition of any $\mathbf{C}[G]$-module.

Note that

- if $r(\chi) > 1$, then $L'_S(0, \chi) = 0$;
- if $r(\chi) = 0$, then $e_\chi \mathbf{Q}X = 0$, since $\mathbf{Q}X$ contains no subrepresentation isomorphic to χ (as the inner product of χ and χ_X is $r(\chi) = 0$).

Therefore, if $\Gamma = \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$ and $a \in \mathbf{Q}(\chi)$, we define

$$\pi(a, \chi) = \sum_{\gamma \in \Gamma} a^\gamma L'_S(0, \chi^\gamma) e_{\overline{\chi}^\gamma} \in \mathbf{C}[G],$$

so $\pi(a, \chi)\mathbf{Q}X = 0$ unless $r(\chi) = 1$. Furthermore, since the e_χ are central, $\pi(a, \chi)$ is central and multiplication by $\pi(a, \chi)$ is a G-homomorphism $\mathbf{C}X \to \mathbf{C}X$.

The element $\pi(a, \chi)$ does more than isolate the χ for which $r(\chi) = 1$. Consider the image $\sum a_w \cdot w \in \mathbf{C}X$ of an element of X under $\pi(a, \chi)$. The coefficients a_w are $\mathbf{Q}(\chi)$-linear combinations of the values $L'_S(0, \chi^\gamma)$, for $\gamma \in \Gamma$. Therefore, if there is a unit $\epsilon \in U$ such that $\lambda(\epsilon) = \sum a_w \cdot w$, then the logarithms of the valuations of ϵ at the places of S_K are equal to these linear combinations of the $L'_S(0, \chi^\gamma)$. Therefore, we can get a better understanding of the Stark conjecture for $r(\chi) = 1$ by studying the intersection between $\pi(a, \chi)\mathbf{Q}X$ and $\lambda\mathbf{Q}U$. We now carry this through in detail.

Let V be a realization of χ over $\mathbf{C}$. With the notation of Lemma A.12.4, let m be the Schur index of χ over $\mathbf{Q}(\chi)$, so $m\chi$ is the character of an irreducible representation V' of G over $\mathbf{Q}(\chi)$. Furthermore, V' must appear as a subrepresentation in $\mathbf{Q}(\chi)X$ since

$$\langle m\chi, \chi_X \rangle_G = mr(\chi) > 0.$$

This implies that $\mathbf{C} \otimes_{\mathbf{Q}(\chi)} V' \cong mV$ appears as a subrepresentation of $\mathbf{C} \otimes_{\mathbf{Q}(\chi)} \mathbf{Q}(\chi)X = \mathbf{C}X$. Hence

$$m \leq \langle \chi, \chi_X \rangle_G = r(\chi) = 1,$$

so $m = 1$. Thus, χ is realizable over $\mathbf{Q}(\chi)$. Furthermore, Lemma A.12.4 shows that $\psi = \mathrm{Tr}_{\mathbf{Q}(\chi)/\mathbf{Q}}\, \chi$ is realizable as an irreducible representation of G over $\mathbf{Q}$. Let W be such a realization; i.e., a simple $\mathbf{Q}[G]$-module with character ψ. By an argument similar to the proof above that $m = 1$, one finds that the multiplicity of W in $\mathbf{Q}X$ is 1. We write X_W (resp. U_W) for the unique $\mathbf{Q}[G]$-submodule of $\mathbf{Q}X$ (resp. $\mathbf{Q}W$) isomorphic to W.

Proposition 4.2.2 (Tate). *Let χ be an irreducible character of G over $\mathbf{C}$ with $r(\chi) = 1$, and let $a \in \mathbf{Q}(\chi)^*$. The following statements are equivalent:*

(a) $\pi(a, \chi)\mathbf{Q}X \cap \lambda\mathbf{Q}U \neq \{0\}$ *in* $\mathbf{C}X$*;*

(b) $\pi(a, \chi)\mathbf{Q}X = \lambda U_W$ *in* $\mathbf{C}X$*;*

(c) *the Stark conjecture is true for* χ*.*

Proof. (a) $\Leftrightarrow$ (b): Consider the canonical decomposition of $\mathbf{Q}X$ into isotypical components:

$$\mathbf{Q}X = X_W \oplus \bigoplus_i W_i.$$

The canonical decomposition of $\mathbf{C}X$ is a refinement of the decomposition over $\mathbf{Q}$. That is,

$$\mathbf{C}X = \bigoplus_{\gamma \in \Gamma} V^\gamma \oplus \bigoplus_i \bigoplus_j W_{ij},$$

where

$$\bigoplus_j W_{ij} \cong \mathbf{C} \otimes_{\mathbf{Q}} W_i$$

for each i. Since none of the W_{ij}'s has a subrepresentation isomorphic to any V^γ, it follows that e_{χ^γ} annihilates each W_{ij} and hence that $\pi(a,\chi)$ annihilates each W_i. Thus $\pi(a,\chi)\mathbf{Q}X = \pi(a,\chi)X_W$ is either $\{0\}$ or a simple $\mathbf{Q}[G]$-module isomorphic to W. The same is true of its image under the isomorphism $\lambda^{-1}\colon \mathbf{C}X \to \mathbf{C}U$. Therefore, if $\lambda^{-1}(\pi(a,\chi)\mathbf{Q}X) \cap \mathbf{Q}U$ is not $\{0\}$, then it must be a $\mathbf{Q}[G]$-submodule of $\mathbf{Q}U$ isomorphic to W, since W is irreducible. Since the only such submodule is U_W, the equivalence of (a) and (b) is proven.

Before we establish the equivalence of the other parts of the proposition, a few preliminary considerations will be useful. Choose a decomposition of $\mathbf{Q}[G]$-modules: $\mathbf{Q}X = X_W \oplus X'$ and $\mathbf{Q}U = U_W \oplus U'$. Since $\mathbf{Q}X \cong \mathbf{Q}U$ and $X_W \cong U_W$, we also have $X' \cong U'$ by semi-simplicity of $\mathbf{Q}[G]$. Let

$$f'\colon X' \to U'$$

be any such $\mathbf{Q}[G]$-module isomorphism. We can now define a homomorphism $f(a,\chi)\colon \mathbf{C}X \to \mathbf{C}U$ via the decomposition $\mathbf{Q}X = X_W \oplus X'$. Writing $\mathbf{C}X_W$ for the $\mathbf{C}[G]$-module $\mathbf{C} \otimes_{\mathbf{Q}} X_W$ and defining $\mathbf{C}U_W$, $\mathbf{C}X'$, and $\mathbf{C}U'$ similarly, we have

$$\mathbf{C}X = \mathbf{C}X_W \oplus \mathbf{C}X', \qquad \mathbf{C}U = \mathbf{C}U_W \oplus \mathbf{C}U'.$$

We define

$$f(a,\chi) = \begin{cases} \lambda^{-1} \circ \pi(a,\chi) & \text{on } \mathbf{C}X_W \\ 1 \otimes f' & \text{on } \mathbf{C}X'. \end{cases}$$

Here

$$\pi(a,\chi)\colon \mathbf{C}X_W \to \mathbf{C}X_W$$

represents multiplication by the element $\pi(a,\chi)$ and

$$\lambda^{-1}\colon \mathbf{C}X_W \to \mathbf{C}U_W$$

is an isomorphism.

Since $\operatorname{Hom}_G((V^\gamma)^*, \mathbf{C}X)$ is a 1-dimensional space by Remark 3.2.3, and the image of any $\mathbf{C}[G]$-homomorphism

$$\varphi\colon (V^\gamma)^* \to \mathbf{C}X$$

lies in $\mathbf{C}X_W$, $\lambda \circ f(a,\chi)$ acts on $\operatorname{Hom}_G((V^\gamma)^*, \mathbf{C}X)$ as $\pi(a,\chi)$ acts on $(V^\gamma)^*$, namely by multiplication by the complex number $a^\gamma L'(0,\chi^\gamma)$. Therefore, we have

$$A(\chi^\gamma, f(a,\chi)) = a^\gamma \text{ for all } \gamma \in \Gamma. \tag{18}$$

(b) $\implies$ (c): If (b) is true, then $\lambda^{-1} \circ \pi(a,\chi)X_W = U_W$. Since $f(a,\chi)$ is defined over $\mathbf{Q}$, (18) gives the desired result.

(c) $\implies$ (a): Conversely, suppose that the Stark Conjecture 3.5.1 is true. By (18) and Proposition 3.6.4 we see that for $\alpha, \beta \in \operatorname{Aut}\mathbf{C}$,

$$A(\chi^\alpha, f(a,\chi)^\beta) = A(\chi^{\beta^{-1}\alpha}, f(a,\chi))^\beta = (a^{\beta^{-1}\alpha})^\beta = a^\alpha = A(\chi^\alpha, f(a,\chi)).$$

Therefore $(\lambda \circ f(a,\chi)^\beta)_{V^\alpha}$ and $(\lambda \circ f(a,\chi))_{V^\alpha}$ have the same determinant on the 1-dimensional space $\operatorname{Hom}_G((V^\alpha)^*, \mathbf{C}X)$, and are thus equal. This implies that the two automorphisms

$$\lambda \circ f(a,\chi), \quad \lambda \circ f(a,\chi)^\beta\colon \mathbf{C}X \to \mathbf{C}X \tag{19}$$

are equal on the subrepresentation of $\mathbf{C}X$ isomorphic to V^α. Since this is true for all α, the two automorphisms of (19) are equal on $\mathbf{C}X_W$. Therefore, since f is defined over $\mathbf{Q}$ on the complement X' of X_W in $\mathbf{Q}X$ by construction, and since $\lambda \circ f(a,\chi)$ and $\lambda \circ f(a,\chi)^\beta$ coincide on $\mathbf{C}X_W$ for all β, we see that $f(a,\chi)\colon \mathbf{C}X_W \to \mathbf{C}U_W$ is defined over $\mathbf{Q}$. Therefore

$$\pi(a,\chi)\mathbf{Q}X = \pi(a,\chi)X_W \subset \lambda\mathbf{Q}U.$$

But $\pi(a,\chi)$ acts as multiplication by $aL'_S(0,\overline{\chi}) \neq 0$ on any element of $\mathbf{Q}X$ lying in the subrepresentation of $\mathbf{C}X$ isomorphic to V. Hence $\pi(a,\chi)\mathbf{Q}X \neq 0$, and (a) follows. $\square$

We now use Proposition 4.2.2 to reformulate Stark's Conjecture in the case $r(\chi) = 1$, by introducing the notion of a Stark unit. Let Ψ be a finite set of irreducible representations χ of G over $\mathbf{C}$ such that:

- $1_G \notin \Psi$,
- if $\chi \in \Psi$, then $\chi^\alpha \in \Psi$ for $\alpha \in \operatorname{Aut}\mathbf{C}$.

Let $(a_\chi)_{\chi\in\Psi}$ be a set of complex numbers such that $a_\chi^\alpha = a_{\chi^\alpha}$ for $\alpha \in \operatorname{Aut}\mathbf{C}$. *Assume* Stark's Conjecture is true for all $\chi \in \Psi$. Proposition 4.2.2 and the observation that $\pi(a_\chi,\chi)\mathbf{Q}X = 0$ if $r(\chi) \neq 1$ then imply

$$\sum_{\chi\in\Psi} a_\chi L'_S(0,\chi)e_{\overline{\chi}}X \subset \lambda\mathbf{Q}U = \mathbf{Q}\lambda U. \tag{20}$$

Since $1_G \notin \Psi$, the trivial representation is annihilated by each $e_{\overline{\chi}}$ with $\chi \in \Psi$. Therefore, the inclusion (20) holds when X is replaced by Y. Given any place $w \in S_K$ lying above a place $v \in S$, we can then write

$$m\sum_{\chi\in\Psi} a_\chi L'_S(0,\chi)e_{\overline{\chi}}w = \lambda(\epsilon) \tag{21}$$

with nonzero $m \in \mathbf{Z}$ and a unit $\epsilon \in U$. This ϵ is called a *Stark unit*.

Once the integer m is fixed, the unit ϵ is uniquely determined by w up to a root of unity. For $\sigma \in G_w$, ϵ^σ satisfies (21), so we may write $\epsilon^\sigma = \zeta_\sigma\epsilon$ for some root of unity $\zeta_\sigma \in \mu(K)$. Furthermore, since

$$\epsilon^{\tau\sigma} = \zeta_\sigma^\tau\epsilon^\tau = \zeta_\sigma^\tau\zeta_\tau\epsilon,$$

the ζ_σ define an element of $H^1(G_w,\mu(K))$. This element is trivial if and only if $\zeta_\sigma = \zeta^{\sigma-1}$ for some $\zeta \in \mu(K)$, which is to say $\zeta\epsilon \in K^{G_w}$. Since $H^1(G_w,\mu(K))$ has exponent dividing $n = \gcd(|G_w|, e_K)$, we may replace m by mn and ϵ by $\zeta\epsilon^n$ in equation (21) for some $\zeta \in \mu(K)$ and so can assume that $\epsilon \in K^{G_w}$ (see Proposition A.9.2).

From the definition of e_χ in 4.2.1 and of λ in 3.3.1, equation (21) can be written

$$\sum_{w\in S_K} \log|\epsilon|_w w = \frac{m\chi(1)}{|G|}\sum_{\substack{\chi\in\Psi\\ \sigma\in G}} a_\chi L'_S(0,\chi)\chi(\sigma)\sigma w.$$

Equating coefficients, we find

$$\begin{cases} |\epsilon|_{w'} = 1 & \text{for } w' \nmid v\\ \log|\epsilon|_{\sigma w} = \log|\epsilon^{\sigma^{-1}}|_w = \frac{m\chi(1)}{|G|}\sum_{\substack{\chi\in\Psi\\ \tau\in G_w}} a_\chi L'_S(0,\chi)\chi(\sigma\tau) & \text{for } \sigma \in G. \end{cases} \tag{22}$$

This remarkable formula states that the values of $L'(0,\chi)$ are related in a linear relationship to the valuations of a unit ϵ in K.

For example, suppose that K/k is abelian and that the place v of k lying below w splits completely in K. Assume also that $|S| \geq 3$. Note that in such a situation we can drop the restriction $1_G \notin \Psi$. Indeed, in such a situation $L'(0, 1_G) = 0$, and the inclusion (20) from which we began our argument still holds. We let $\Psi = \widehat{G}$ and choose $a_\chi = 1$ for all χ. Equation (22) then becomes

$$\log |\epsilon^{\sigma^{-1}}|_w = \frac{m}{|G|} \sum_{\chi \in \widehat{G}} L'_S(0, \chi)\chi(\sigma),$$

or equivalently, for $\theta \in \widehat{G}$,

$$\begin{aligned}
\sum_{\sigma \in G} \theta(\sigma) \log |\epsilon^{\sigma}|_w &= \sum_{\sigma \in G} \overline{\theta}(\sigma) \log |\epsilon^{\sigma^{-1}}|_w \\
&= \frac{m}{|G|} \sum_{\substack{\chi \in \widehat{G} \\ \sigma \in G}} L'_S(0, \theta)\chi(\sigma)\overline{\theta}(\sigma) \\
&= mL'_S(0, \theta). \qquad (23)
\end{aligned}$$

As we will see in the next section, the "abelian Stark conjecture" essentially says that with a few extra conditions, we may take $m = e_K = |\mu(K)|$, and that for a Stark unit ϵ, $K(\epsilon^{1/e_K})$ is an *abelian* extension of k.

4.3 The rank one abelian Stark conjecture

In the case where K/k is an *abelian* extension (and with a few other assumptions), Stark has given a conjecture which makes the case $r(\chi) = 1$ more explicit. The abelian Stark conjecture refines the general non-abelian Stark conjecture in the case $r(\chi) = 1$ by specifying the exact value $m = e_K$ in equation (23) of section 4.2. The abelian conjecture further predicts that $K(\epsilon^{1/e_K})/k$ will be an abelian extension when ϵ is appropriately chosen. For $r(\chi) > 1$, Rubin has given a generalization of Stark's "rank one" abelian Conjecture [13].

Notation 4.3.1. In this section, K/k will denote an abelian extension of number fields with Galois group G. Let e and e_k denote the size of the groups $\mu(K)$ and $\mu(k)$ of roots of unity in K and k, respectively.

Let S be a set of primes of k satisfying the conditions below:

- S contains the archimedean places of k and the non-archimedean places which ramify in K.
- S contains at least one place which splits completely in K.
- $|S| \geq 2$.

Fix a place v of S which splits completely in K, and fix an extension w of v to K. If $|S| \geq 3$, we define

$$U^v = \{u \in U_{K,S_K} : |u|_{w'} = 1 \text{ for all } w' \nmid v\}.$$

If $S = \{v, v'\}$ and w' is an extension of v' to K, then we define

$$U^v = \{u \in U_{K,S_K} : |u|_{\sigma w'} = |u|_{w'} \text{ for all } \sigma \in G\}.$$

Finally, define

$$U^{\text{ab}}_{K/k} = \{\epsilon \in U_{K,S_K} : K(\epsilon^{1/e})/k \text{ is an abelian extension}\}.$$

Conjecture 4.3.2 (Stark). *With notation as above, there exists an S-unit $\epsilon \in U^{\mathrm{ab}}_{K/k} \cap U^v$ such that*

$$\log|\epsilon^\sigma|_w = -e\zeta'_S(0,\sigma) \text{ for all } \sigma \in G, \tag{24}$$

or equivalently,

$$L'_S(0,\chi) = -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w \text{ for all } \chi \in \widehat{G}. \tag{25}$$

Remark 4.3.3. Clearly the choice of place w lying above v does not effect the truth of the conjecture. The S-unit ϵ in the conjecture is called a *Stark unit*. The conditions on ϵ specify its absolute value at every place of K, so ϵ is uniquely determined up to multiplication by a root of unity in K. This is clear if $|S| \geq 3$, and follows from the product formula if $|S| = 2$. Furthermore, if G is *cyclic* and S contains only one place which splits completely in K, then any ϵ satisfying Conjecture 4.3.2 generates K over k. Indeed, Proposition 3.2.4 implies that for any faithful character $\chi\colon G \hookrightarrow \mathbf{C}^*$ (note $\chi \neq 1_G$ since $K \neq k$) we have $r(\chi) = 1$ and hence $L'_S(0,\chi) \neq 0$. If any $\tau \in G$ satisfies $\epsilon^\tau = \epsilon$, then by replacing σ by $\sigma\tau$ in (25) we obtain

$$\begin{aligned} L'_S(0,\chi) &= -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w \\ &= -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma\tau)\log|\epsilon^{\sigma\tau}|_w \\ &= -\frac{\chi(\tau)}{e}\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w \\ &= \chi(\tau)L'_S(0,\chi). \end{aligned}$$

Thus $\chi(\tau) = 1$, and hence $\tau = 1$. Therefore $K = k(\epsilon)$ as desired. Suppose we also have that k is totally real, and the place v which splits completely is real. Fix an embedding $k \subset K \subset K_w = \mathbf{R}$ corresponding to a real place w above v, and suppose ϵ is a positive Stark unit. Then (24) yields

$$\epsilon = \exp(-2\zeta'_S(0,1_G))$$

and hence

$$K = k(\exp(-2\zeta'_S(0,1_G))).$$

This is very reminiscent of Hilbert's 12th problem. Unfortunately, however, the function $\zeta_S(s,1_G)$ depends on the extension K/k.

We write $\mathrm{St}(K/k,S,v)$ or simply $\mathrm{St}(K/k,S)$ for Stark's Conjecture 4.3.2.

Proposition 4.3.4. *The conjecture* $\mathrm{St}(K/k,S)$ *is true if S contains at least two places which split completely.*

Proof. Let v and v' be two places of S which split completely in K. If $|S| \geq 3$, then $\epsilon = 1$ is a Stark unit since $L'_S(0,\chi) = 0$ for all $\chi \in \widehat{G}$ by Proposition 3.2.4.

It remains to consider the more interesting case $S = \{v, v'\}$. The rank of the group of S-units is then 1 by Proposition 3.3.2. Let $\eta \in U_{k,S}$ be a fundamental S-unit with $|\eta|_v > 1$. The Dirichlet class number formula (Theorem 3.1.4) implies

$$L'_{K/k,S}(0,1_G) = \zeta'_{k,S}(0) = -\frac{h_{k,S}\log|\eta|_v}{e_k}.$$

Since $\mu(k)$ is a subgroup of $\mu(K)$, e is a multiple of e_k. Proposition A.5.3 implies that $h_S = h_{k,S}$ is a multiple of $[K : k]$. Hence

$$m = \frac{e \cdot h_S}{e_k \cdot [K : k]}$$

is an integer and we can define $\epsilon = \eta^m$. It is obvious that $\epsilon \in U^v$ (as defined in 4.3.1). Furthermore, $K(\epsilon^{1/e})$ lies inside of the compositum $K(\eta^{1/e_k})$ of the abelian extension K/k and the Kummer extension $k(\eta^{1/e_k})/k$, so $K(\epsilon^{1/e})/k$ is abelian. Thus, $\epsilon \in U^{\mathrm{ab}}_{K/k} \cap U^v$.

Finally, since $\epsilon \in k$,

$$L'_S(0, 1_G) = -\frac{h_S \log |\eta|_v}{e_k} = -\frac{[K : k]}{e} \log |\epsilon|_v = -\frac{1}{e} \sum_{\sigma \in G} 1_G(\sigma) \log |\epsilon^\sigma|_w$$

and for $\chi \neq 1_G$,

$$-\frac{1}{e} \sum_{\sigma \in G} \chi_G(\sigma) \log |\epsilon^\sigma|_w = -\frac{\log |\epsilon|_v}{e} \cdot \sum_{\sigma \in G} \chi_G(\sigma) = 0 = L'_S(0, \chi).$$

Here the last equality follows from Proposition 3.2.4 and our hypothesis that v and v' are totally split in K. Thus, ϵ is the desired Stark unit. □

Since Stark's conjecture is true when S contains two places which split completely, the truth of the conjecture is independent of choice of v. This justifies the notation $\mathrm{St}(K/k, S)$ for the conjecture. Tate's reformulation of $\mathrm{St}(K/k, S, v)$ in section 6 will make the "independence of v" more explicit by not mentioning v at all.

Corollary 4.3.5. $\mathrm{St}(k/k, S)$ *is true.*

Corollary 4.3.6. *If S contains two complex places then* $\mathrm{St}(K/k, S)$ *is true. If S contains a finite place v which splits completely and k is not totally real, then* $\mathrm{St}(K/k, S)$ *is true.*

Note that if S satisfies the three conditions in 4.3.1, then so does any $S' \supset S$. Also, if S satisfies these conditions for K/k, then it does for F/k as well if $k \subset F \subset K$.

Proposition 4.3.7. $\mathrm{St}(K/k, S)$ *implies* $\mathrm{St}(K/k, S')$ *for* $S \subset S'$.

Proof. Without loss of generality, $S' \neq S$. Fix $v \in S$ which splits completely in K, and choose any $\mathfrak{p} \in S' - S$. By the conditions on S, $\mathfrak{p}$ is a finite prime of k unramified in K. Let ϵ be a Stark unit for $\mathrm{St}(K/k, S, v)$, and let

$$\epsilon' = \epsilon^{1-\sigma_{\mathfrak{p}}^{-1}}.$$

It is clear that ϵ' is an S'-unit since ϵ is an S-unit. Furthermore, if $|S| \geq 3$ then it is also clear that $\epsilon' \in U^v_{S'}$ since $\epsilon \in U^v_S$. If $S = \{v, v'\}$, we see that $\epsilon' \in U^v_{S'}$ because for any w' lying above v',

$$\left|\epsilon'\right|_{w'} = \frac{\left|\epsilon\right|_{w'}}{\left|\epsilon\right|_{\sigma_{\mathfrak{p}}^{-1} w'}} = 1$$

since $\epsilon \in U^v_S$. Also, $K((\epsilon')^{1/e_K})/k$ is an abelian extension since $K((\epsilon')^{1/e_K}) \subset K(\epsilon^{1/e_K})$. It remains to check that ϵ' has the correct valuations at the places w lying above v. From the Euler product representation for the L-functions, we have

$$L_{S'}(s, \chi) = (1 - \chi(\sigma_{\mathfrak{p}})\, \mathrm{N}\mathfrak{p}^{-s}) L_S(s, \chi).$$

for all $\chi \in \widehat{G}$. Since $L_S(0,\chi) = 0$,

$$\begin{aligned}
L'_{S'}(0,\chi) &= (1-\chi(\sigma_{\mathfrak{p}}))L'_S(0,\chi) \\
&= -\frac{1-\chi(\sigma_{\mathfrak{p}})}{e}\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w \\
&= -\frac{1}{e}\left[\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w - \sum_{\sigma\in G}\chi(\sigma\sigma_{\mathfrak{p}})\log|\epsilon^\sigma|_w\right] \\
&= -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma)\log\left|(\epsilon^{1-\sigma_{\mathfrak{p}}^{-1}})^\sigma\right|_w \\
&= -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma)\log|(\epsilon')^\sigma|_w
\end{aligned}$$

as desired. □

Once we give the arithmetic preliminaries for Tate's reformulation of Stark's Conjecture 4.3.2 in section 6, we will be able to fill in the gap in the proof of:

Proposition 4.3.8. *If $k \subset F \subset K$ then* $\mathrm{St}(K/k,S)$ *implies* $\mathrm{St}(F/k,S)$.

Proof. Fix a place $v \in S$ which splits completely and a place w of K lying over v. Let ϵ be a Stark unit for $(K/k,S,w)$. In section 6.4, we will prove that there exists an $\epsilon_F \in U^{\mathrm{ab}}_{F/k}$ such that

$${\epsilon_F}^{e/e_F} = \zeta\,\mathrm{N}_{K/F}\epsilon$$

for some root of unity $\zeta \in F$. This ϵ_F is a Stark unit for $(F/k,S,w|_F)$. If $|S| \geq 3$, then $\epsilon \in U^v_K$ means that $|\epsilon|_{w'} = 1$ for all places w' of K not dividing v. Hence, for any place w'_F of F not dividing v, we compute

$$|\epsilon_F|^{e/e_F}_{w'_F} = |\,\mathrm{N}_{K/F}\epsilon|_{w'} = 1$$

so $\epsilon_F \in U^v_F$. A similar argument holds if $|S| = 2$.

Define $G = \mathrm{Gal}(K/k)$ and $H = \mathrm{Gal}(K/F)$ so $G/H = \mathrm{Gal}(F/k)$. Let χ be a character of G/H, and let $\mathrm{Infl}\,\chi$ be the corresponding character of G obtained by projection. For every $\sigma \in G/H$ choose a lifting $\widehat{\sigma} \in G$. By the naturality of L-series (Proposition A.7.2) we obtain

$$\begin{aligned}
L'_{F/k,S}(0,\chi) &= L'_{K/k,S}(0,\mathrm{Infl}\,\chi) \\
&= -\frac{1}{e}\sum_{\gamma\in G}\mathrm{Infl}\,\chi(\gamma)\log|\epsilon^\gamma|_w \\
&= -\frac{1}{e}\sum_{\sigma\in G/H}\chi(\sigma)\sum_{\tau\in H}\log|\epsilon^{\widehat{\sigma}\tau}|_w \\
&= -\frac{1}{e}\sum_{\sigma\in G/H}\chi(\sigma)\log|(\mathrm{N}_{K/F}\epsilon)^{\widehat{\sigma}}|_w \\
&= -\frac{1}{e_F}\sum_{\sigma\in G/H}\chi(\sigma)\log|(\epsilon_F)^\sigma|_{w'}
\end{aligned}$$

as desired. □

Remark 4.3.9. In the case where $e_F = e$, we obtain $\epsilon_F = \zeta\,\mathrm{N}_{K/F}\epsilon$. As Stark units are only defined up to a root of unity, we may take $\epsilon_F = \mathrm{N}_{K/F}\epsilon$.

For certain cases where the abelian extensions of k can be explicitly constructed by class field theory, one can actually prove Stark's conjecture. The proof of the following theorem can be found in [26] and [27].

Theorem 4.3.10 (Stark). *$\mathrm{St}(K/k, S)$ is true if $k = \mathbf{Q}$ or if k is an imaginary quadratic field.*

Proposition 4.3.11. *$\mathrm{St}(K/k, S)$ is true if $|S| = 2$.*

Proof. The case when k has only one archimedean place is handled by Theorem 4.3.10, so we need only consider $S = \{v, v'\}$ where both places of S are archimedean. Since v splits completely in K and K/k is unramified at all finite places, v' also splits completely in K by Lemma A.3.7. Hence, the result follows from Proposition 4.3.4. □

Before analyzing the abelian Stark conjecture in greater detail, we will entertain the reader in the next section with a numerical confirmation in a specific case. We will return to the abstract theory, including Tate's reformulation of Stark's abelian conjecture, in section 6.

5 A numerical confirmation

Recall from Theorem 4.3.10 that Stark proved his Conjecture 4.3.2 in the cases when $k = \mathbf{Q}$ or k is an imaginary quadratic field. It is possible to computationally "verify" the conjecture in specific cases where it is not mathematically proven; Stark did this originally in [22] and [26]. Such a verification involves the numerical computation of the values $\zeta_S'(0,\gamma)$ and the construction of a Stark unit ϵ based on these values. While the algebraic properties of $\mathrm{St}(K/k,S)$ can be proven, a computational verification does not *prove* the conjecture in the specific case because the values of $\zeta_S'(0,\gamma)$ are only numerical approximations and not exact values. Nevertheless, the values of $\zeta_S'(0,\gamma)$ given in [6] and in this section are proven accurate to a large number of decimal places, so the results give very strong corroboration for the conjecture in these cases.

Stark, Fogel, Shintani, and Hayes have each done numerical confirmations in many cases where k is a real quadratic field, and Dummit, Sands, and Tangedal have done numerical confirmations in many cases where k a totally real cubic field [6]. In [6] it is remarked that no numerical confirmations of $\mathrm{St}(K/k,S,v)$ have been obtained for cases where k has more than one archimedean place and the place v of k which splits completely in K is complex. In this section we provide such a confirmation, following the techniques of [6].

We will take k to be a cubic extension of $\mathbf{Q}$ with one real place and one complex place. Since the conjecture is known to be true when S contains more than one place which splits completely, we will take v to be the complex place of S, and the real place of k will ramify in K. Furthermore, the conjecture is known to be true when $|S| = 2$ and also $\mathrm{St}(K/k,S) \implies \mathrm{St}(K/k,S')$ if $S' \supset S$; thus in our example we will have a finite prime of k ramifying in K. Finally, since Sands has studied the case where $|G|$ has exponent two (see section 7), we will take $|G|$ to be divisible by 3. These considerations lead to the following example.

5.1 The example

Let $k = \mathbf{Q}(\beta)$, where β is a root of $f(x) = x^3 - x^2 + 5x + 1$. This is the cubic field of smallest discriminant (in absolute value) with class number 3, one real place, and one complex place (tables of number fields are available via anonymous ftp at `megrez.math.u-bordeaux.fr/pub/numberfields`). We denote these archimedean places $\infty_{\mathbf{R}}$ and $\infty_{\mathbf{C}}$, respectively. The ring of integers $\mathcal{O}_k$ is $\mathbf{Z}[\beta]$ and the discriminant of k over $\mathbf{Q}$ is -588. The primes which ramify are

$$\begin{aligned}
(3) &= \mathfrak{p}_1\mathfrak{p}_2^2 \text{ where } \mathfrak{p}_1 = (3,\beta+1) \text{ and } \mathfrak{p}_2 = (3,\beta-1),\\
(2) &= \mathfrak{p}_3^3 \text{ where } \mathfrak{p}_3 = (2,\beta-1), \text{ and}\\
(7) &= \mathfrak{p}_4^3 \text{ where } \mathfrak{p}_4 = (7,\beta+2).
\end{aligned}$$

With this notation, the different of k over $\mathbf{Q}$ is $\mathfrak{D}_{k/\mathbf{Q}} = \mathfrak{p}_2\mathfrak{p}_3^2\mathfrak{p}_4^2$. The Galois closure of k over $\mathbf{Q}$ is $k(\sqrt{-3})$. The Hilbert class field is $H = k(\theta)$ where $\theta = \zeta_7 + \zeta_7^{-1}$ satisfies the equation $\theta^3 + \theta^2 - 2\theta - 1 = 0$. We take K to be $H(\sqrt{-3})$. Then K/k is abelian, since it is the compositum of $k(\sqrt{-3})$ and H, and G is cyclic of order 6. Also, K has $e = 6$ roots of unity. All of these facts can be checked on the Pari-GP calculator (or by hand).

Proposition 5.1.1. *K is the ray class field over k with modulus $\mathfrak{m} = \mathfrak{p}_1\infty_{\mathbf{R}}$.*

Proof. First note that $K = H(\sqrt{-3})$ is everywhere unramified over $k(\sqrt{-3})$ since H/k is unramified, so the only primes of k which ramify in K are those which ramify in $k(\sqrt{-3})$. These are the real place $\infty_{\mathbf{R}}$ and possibly the primes lying over 3 (since $\mathbf{Q}(\sqrt{-3})/\mathbf{Q}$ is ramified only at the real place and 3). Consider the factorization of the ideal $(3) = P_1^r P_2^r \cdots P_g^r$ in $k(\sqrt{-3})$. Since $(3) = \mathfrak{p}_1\mathfrak{p}_2^2$ in k, the ramification index r must be divisible by 2 and $g \geq 2$. Since $rfg = [k(\sqrt{-3}) : \mathbf{Q}] = 6$, where f is the residue field degree, the only possibility is $r = 2, f = 1, g = 3$. This implies that $\mathfrak{p}_1$ ramifies in $k(\sqrt{-3})$ and that $\mathfrak{p}_2$ splits. Thus the primes of k which ramify in K are precisely those dividing $\mathfrak{m} = \mathfrak{p}_1\infty_{\mathbf{R}}$.

The conductor of the extension $\mathbf{Q}(\sqrt{-3})/\mathbf{Q}$ is $(3)\infty$. By Proposition A.3.8, the conductor of $k(\sqrt{-3})/k$ divides $\mathfrak{p}_1\mathfrak{p}_2^2\infty_{\mathbf{R}}$. Since $\mathfrak{p}_2$ is unramified in $k(\sqrt{-3})$ while $\mathfrak{p}_1$ and $\infty_{\mathbf{R}}$ ramify, the conductor of $k(\sqrt{-3})/k$ must be $\mathfrak{m} = \mathfrak{p}_1\infty_{\mathbf{R}}$. Moreover, the conductor of H/k is 1, so Proposition A.3.9 implies that K/k has conductor $\mathfrak{m}$.

Since $[K : k] = 6$, the conclusion will now follow if we show that $[I_{k,\mathfrak{m}} : P_{k,\mathfrak{m}}] \leq 6$. Now

$$[I_{\mathfrak{m}} : P_{\mathfrak{m}}] = [I_{\mathfrak{m}} : P_1 \cap I_{\mathfrak{m}}] \cdot [P_1 \cap I_{\mathfrak{m}} : P_{\mathfrak{m}}] \leq [I_{\mathfrak{m}}P_1 : P_1] \cdot [P_{\mathfrak{p}_1} : P_{\mathfrak{m}}].$$

In the last step, we used that fact that $P_1 \cap I_{\mathfrak{m}} \subset P_{\mathfrak{p}_1}$; to see this, note that since $\mathrm{N}\mathfrak{p}_1 = 3$, if $(\alpha) \in I_{\mathfrak{m}}$ then either $\alpha \equiv 1$ or $-\alpha \equiv 1$ modulo $\mathfrak{p}_1$ and so $(\alpha) = (-\alpha) \in P_{\mathfrak{p}_1}$. Furthermore, we have $[P_{\mathfrak{p}_1} : P_{\mathfrak{m}}] \leq 2$ since if $\alpha \equiv 1 \pmod{\mathfrak{p}_1}$ and $\infty_{\mathbf{R}}(\alpha) < 0$ then any element of $P_{\mathfrak{p}_1}$ lies either in $P_{\mathfrak{m}}$ or the coset of (α) mod $P_{\mathfrak{m}}$. Thus

$$[I_{\mathfrak{m}} : P_{\mathfrak{m}}] \leq 2 \cdot [I_{\mathfrak{m}}P_1 : P_1] \leq 2 \cdot [I_1 : P_1] = 6$$

as desired. □

Remark 5.1.2. Note that $K/\mathbf{Q}$ is a Galois extension since it is the compositum of the Galois extensions $k(\sqrt{-3})$ and $\mathbf{Q}(\zeta_7 + \zeta_7^{-1})$ over $\mathbf{Q}$. Hence we can factor the ideal (3) in K as $(3) = P_1^r P_2^r \cdots P_g^r$. Based on our factorization of (3) in $k(\sqrt{-3})$ from above, we see that $r = 2$ since K is the compositum of $k(\sqrt{-3})$ and the unramified extension H/k. We also see from the factorization in $k(\sqrt{-3})$ that $g \geq 3$. Since $rfg = 18$ we have $g = 3$ or $g = 9$. But $g = 9$ would imply $f = 1$, which would in turn imply that $\mathfrak{p}_1$ splits completely in H. But $\mathfrak{p}_1$ is not principal, so this is not the case. Hence $(3) = P_1^2P_2^2P_3^2$. In particular $\mathfrak{p}_1\mathcal{O}_K = \mathfrak{P}^2$ for a prime $\mathfrak{P}$ of $\mathcal{O}_K$ and $\mathfrak{p}_2$ factors as the product of two distinct primes. The inertia group $I_{\mathfrak{p}_1}$ therefore has size two.

By Proposition 4.3.7, we are led to take S to be the minimal possible set for the extension K/k, namely $S = \{\mathfrak{p}_1, \infty_{\mathbf{R}}, \infty_{\mathbf{C}}\}$. We define $v = \infty_{\mathbf{C}}$. The conjecture $\mathrm{St}(K/k, S, v)$ is not known for this case. We will computationally verify it in sections 5.2 and 5.3.

5.2 Calculating $\zeta'_S(0, \gamma)$

The construction of a Stark unit can be broken down into two distinct steps. The first step is the numerical calculation of the values $\zeta'_S(0, \gamma)$ for $\gamma \in G$. The second step is the construction of a unit ϵ with the desired valuations. We describe the first step in this section. Our methods follow those of [6].

Let τ be the unique element of order two in G, so τ generates the decomposition group of $\infty_{\mathbf{R}}$. Proposition 3.2.4 implies that $L_S(s, \chi)$ has a zero of order 2 at $s = 0$ if $\chi(\tau) = 1$ and of order 1 if $\chi(\tau) = -1$. In particular, $\zeta_S(0, \chi) = 0$ for all χ.

5.2.1 The functional equation

We first consider the functional equation for the L-functions of K/k. With notation as in Theorem A.8.1, we have $n = 3$, $\chi(1) = 1$ for all χ, and $(a_1, a_2) = (0, 1)$ if $\chi(\tau) = -1$. Since $\mathfrak{p}_1$ is tamely ramified in K with inertia group of order 2 generated by τ, by Proposition A.4.3 we see that $\mathfrak{f}(\chi) = \mathfrak{p}_1$ when $\chi(\tau) = -1$. Hence, in this case we obtain

$$\Lambda(s,\chi) = \frac{2}{\sqrt{\pi}} \cdot a^s \cdot \Gamma(s)\Gamma\left(\frac{1+s}{2}\right) L(s,\chi), \text{ where } a = \sqrt{\frac{588\cdot 3}{4\pi^3}} = \frac{21}{\pi^{3/2}}.$$

5.2.2 The auxiliary function Λ_S

Definition 5.2.1. For any $\gamma \in G$ we define

$$\Lambda_S(s,\gamma) = a^s\Gamma(s)\Gamma\left(\frac{1+s}{2}\right)\left(\zeta_S(s,\gamma) - \zeta_S(s,\tau\gamma)\right),$$

where $a = \frac{21}{\pi^{3/2}}$ as above.

We can find an alternative expression for Λ_S in terms of the completed L-functions Λ:

$$\begin{aligned}
\Lambda_S(s,\gamma) &= a^s\Gamma(s)\Gamma\left(\frac{1+s}{2}\right)\left(\zeta_S(s,\gamma) - \zeta_S(s,\tau\gamma)\right) \\
&= a^s\Gamma(s)\Gamma\left(\frac{1+s}{2}\right)\left(\frac{1}{3}\sum_{\chi:\chi(\tau)=-1} \overline{\chi}(\gamma) L_S(s,\chi)\right) \\
&= \frac{1}{3}\sum_{\chi:\chi(\tau)=-1} \frac{\sqrt{\pi}}{2}\overline{\chi}(\gamma)\Lambda(s,\chi).
\end{aligned} \tag{26}$$

In the last equality, we used the fact that $L_S(s,\chi) = L(s,\chi)$ for $\chi(\tau) = -1$, since the inertia group of $\mathfrak{p}_1$ is $\{1,\tau\}$ (see 5.1.2).

Taking the limit as $s \to 0$ in the definition of Λ_S shows that

$$\Lambda_S(0,\gamma) = \sqrt{\pi}\left(\zeta_S'(0,\gamma) - \zeta_S'(0,\tau\gamma)\right)$$

since $\Gamma(s)$ has a simple pole of residue 1 at $s = 0$ and $\Gamma(1/2) = \sqrt{\pi}$. Furthermore,

$$\zeta_S(s,\gamma) + \zeta_S(s,\tau\gamma) = \frac{1}{3}\sum_{\chi:\chi(\tau)=1} \overline{\chi}(\gamma) L_S(s,\chi)$$

has a zero of order at least 2 at $s = 0$ (as noted earlier), so $\zeta_S'(0,\gamma) + \zeta_S'(0,\tau\gamma) = 0$. Thus

$$\Lambda_S(0,\gamma) = 2\sqrt{\pi}\zeta_S'(0,\gamma).$$

We have now reduced the calculation of $\zeta_S'(0,\gamma)$ to that of $\Lambda_S(0,\gamma)$, and the advantage of this is that Λ_S satisfies a functional equation.

5.2.3 The functional equation for Λ_S

To obtain the functional equation for Λ_S, we must first evaluate the root numbers $W(\chi)$ appearing in the functional equations of the L-functions. For an ideal $\mathfrak{U} \in I_{k,\mathfrak{m}}$ we write $\chi(\mathfrak{U})$ for $\chi(\sigma_{\mathfrak{U}})$, where $\sigma_{\mathfrak{U}}$ denotes the image of $\mathfrak{U}$ under the Artin map.

Proposition 5.2.2. *If $\chi(\tau) = -1$, then $W(\chi) = \chi(\mathfrak{D}_{k/\mathbf{Q}}\mathfrak{p}_2)$.*

Proof. Recall from Tate's Thesis [1] that the global root number is the product of local root numbers:

$$W(\chi) = \prod_v W(\chi_v).$$

Definitions of the local and global root numbers are given in [11]. For an archimedean place v, $W(\chi_v) = i^{-n(\chi_v)}$, where $n(\chi_v) = \frac{1}{2}(\chi_v(1) - \chi_v(\sigma_v))$ and σ_v generates a decomposition group over v. Thus, $n(\chi_{\infty_\mathbf{C}}) = 0$ and, since $\chi(\tau) = -1$, we have $n(\chi_{\infty_\mathbf{R}}) = 1$.

Now consider finite v lying over a prime p of $\mathbf{Q}$; there is an explicit expression for $W(\chi_v)$ in terms of a Gauss sum, which we now describe. Define U_i to be the subgroup of the group of units of $\mathcal{O}_v$ which are congruent to 1 modulo $\mathfrak{p}_v^i$, where $\mathfrak{p}_v$ is the unique prime ideal of $\mathcal{O}_v$ (and $U_0 = \mathcal{O}_v^*$). The local conductor $\mathfrak{f}(\chi_v)$ is equal to $\mathfrak{p}_v^{f_v}$, where $f_v = f(\chi, v)$ is defined in A.4.1. We then have

$$W(\chi_v) = \frac{\tau(\overline{\chi}_v)}{\sqrt{\mathrm{N}\mathfrak{f}(\chi_v)}}$$

where

$$\tau(\overline{\chi}_v) = \sum_x \overline{\chi}_v\left(\frac{x}{c}\right) \exp\left(2\pi i \operatorname{Tr}_{k_v/\mathbf{Q}_p}\left(\frac{x}{c}\right)\right)$$

is the "local Gauss sum." Here the sum runs over representatives x of U_0/U_{f_v}, c is a generator of the ideal $\mathfrak{f}(\chi_v)\mathfrak{D}_{k_v/\mathbf{Q}_p}$, and $\chi_v \colon k_v^* \to \mathbf{C}^*$ is the composite of the map

$$k_v^* \longrightarrow G_v \subset G$$

from local class field theory with $\chi \colon G \to \mathbf{C}^*$. In particular, $f_v = 0$ when χ_v is unramified and $\operatorname{Tr}_{k_v/\mathbf{Q}_p}(\mathfrak{D}_{k_v/\mathbf{Q}_p}^{-1}) \subset \mathcal{O}_v$ by definition of the different, so for unramified χ_v this sum reduces to one term: $\tau(\overline{\chi}_v) = \overline{\chi}_v(\mathfrak{D}_{k_v/\mathbf{Q}_p}^{-1})$.

In our case, the only finite place where K/k is ramified is $\mathfrak{p}_1$, and $\mathfrak{p}_1$ does not divide the different $\mathfrak{D}_{k/\mathbf{Q}}$. Hence the contribution to the global root number $W(\chi)$ by the finite places $v \neq \mathfrak{p}_1$ is

$$\prod_{v \neq \mathfrak{p}_1} W(\chi_v) = \prod_{v \neq \mathfrak{p}_1} \overline{\chi}_v(\mathfrak{D}_{k_v/\mathbf{Q}_p}^{-1}) = \chi(\mathfrak{D}_{k/\mathbf{Q}}).$$

It remains to compute $W(\chi_{\mathfrak{p}_1})$. We see from Proposition A.4.3 that $f_{\mathfrak{p}_1} = 1$. Also,

$$U_0/U_1 \cong (\mathcal{O}_k/\mathfrak{p}_1)^* \cong \mathbf{F}_3^*,$$

so we take $\pm 1 \in U_0$ as representatives of U_0/U_1. The ideal $\mathfrak{f}(\chi_{\mathfrak{p}_1})\mathfrak{D}_{k_{\mathfrak{p}_1}/\mathbf{Q}_3} = \mathfrak{p}_1\mathcal{O}_{\mathfrak{p}_1}$ is generated by $c = 3$, since $\mathfrak{p}_1$ appears once in the factorization of (3) in k, so

$$\tau(\overline{\chi}_{\mathfrak{p}_1}) = \overline{\chi}_{\mathfrak{p}_1}\left(-\frac{1}{3}\right) e^{-2\pi i/3} + \overline{\chi}_{\mathfrak{p}_1}\left(\frac{1}{3}\right) e^{2\pi i/3} = \chi_{\mathfrak{p}_1}(3)(e^{2\pi i/3} + \chi_{\mathfrak{p}_1}(-1)e^{-2\pi i/3}).$$

The map $k_{\mathfrak{p}_1}^* \to G_{\mathfrak{p}_1}$ from local class field theory is used to define $\chi_\mathfrak{p}$, but this map is "mysterious" since $\mathfrak{p}_1$ is ramified in K.

We will compute values of $\chi_{\mathfrak{p}_1}$ indirectly. Since $-1 < 0$ is a local unit at each place v of k, $\chi_v(-1) = 1$ for $v \neq \mathfrak{p}_1, \infty_\mathbf{R}$ (as such v are unramified in K). Furthermore, by the compatibility of local and global class field theory, χ acts trivially on principal ideles, giving

$$1 = \chi(-1) = \chi_{\mathfrak{p}_1}(-1)\chi_{\infty_\mathbf{R}}(-1).$$

Now $\chi_{\infty_{\mathbf{R}}}(-1) = -1$ since -1 is not a norm from $K_{w'} \cong \mathbf{C}$ for any w' over $\infty_{\mathbf{R}}$, so $\chi_{\mathfrak{p}_1}(-1) = -1$ and therefore

$$\tau(\overline{\chi}_{\mathfrak{p}_1}) = \chi_{\mathfrak{p}_1}(3) \cdot i\sqrt{3}.$$

We calculate $\chi_{\mathfrak{p}_1}(3)$ similarly, observing that 3 is a local unit away from $\mathfrak{p}_1$ and $\mathfrak{p}_2$, so

$$1 = \chi(3) = \chi_{\infty_{\mathbf{R}}}(3)\chi_{\mathfrak{p}_1}(3)\chi_{\mathfrak{p}_2}(3) = \chi_{\mathfrak{p}_1}(3)\chi_{\mathfrak{p}_2}(3)$$

since $3 > 0$. Now $\mathfrak{p}_2$ has decomposition group of size 3 (see Remark 5.1.2), so $\chi_{\mathfrak{p}_2}^3 = 1$. Combining this with the fact that χ is unramified at $\mathfrak{p}_2$ and $v_{\mathfrak{p}_2}(3) = 2$, we compute

$$\chi_{\mathfrak{p}_1}(3) = \chi_{\mathfrak{p}_2}(3)^{-1} = \chi(\mathfrak{p}_2)^{-2} = \chi(\mathfrak{p}_2).$$

Thus $\tau(\overline{\chi}_{\mathfrak{p}_1}) = \chi(\mathfrak{p}_2) \cdot i\sqrt{3}$, so by definition $W(\chi_{\mathfrak{p}_1}) = \tau(\overline{\chi}_{\mathfrak{p}_1})/\sqrt{3} = i\chi(\mathfrak{p}_2)$.

Putting together our local computations,

$$W(\chi) = W(\chi_{\infty_{\mathbf{C}}})W(\chi_{\infty_{\mathbf{R}}})W(\chi_{\mathfrak{p}_1}) \prod_{v \neq \mathfrak{p}_1} W(\chi_v) = 1 \cdot i^{-1} \cdot i\chi(\mathfrak{p}_2) \cdot \chi(\mathfrak{D}_{k/\mathbf{Q}}) = \chi(\mathfrak{D}_{k/\mathbf{Q}}\mathfrak{p}_2).$$

□

Proposition 5.2.3. *For $\gamma \in G$, $\Lambda_S(1-s,\gamma) = \Lambda_S(s,\widehat{\gamma})$ where $\widehat{\gamma} = \gamma^{-1}\sigma_{\mathfrak{D}_{k/\mathbf{Q}}\mathfrak{p}_2}$.*

Proof. By equation (26) of section 5.2.2 and Proposition 5.2.2,

$$\begin{aligned}
\Lambda_S(1-s,\gamma) &= \frac{1}{3}\sum_{\chi:\chi(\tau)=-1} \frac{\sqrt{\pi}}{2}\overline{\chi}(\gamma)\Lambda(1-s,\chi) \\
&= \frac{1}{3}\sum_{\chi:\chi(\tau)=-1} \frac{\sqrt{\pi}}{2}\overline{\chi}(\gamma)W(\chi)\Lambda(s,\overline{\chi}) \\
&= \frac{1}{3}\sum_{\chi:\chi(\tau)=-1} \frac{\sqrt{\pi}}{2}\chi(\gamma)W(\overline{\chi})\Lambda(s,\chi) \\
&= \frac{1}{3}\sum_{\chi:\chi(\tau)=-1} \frac{\sqrt{\pi}}{2}\overline{\chi}(\gamma^{-1})\overline{\chi}(\sigma_{\mathfrak{D}_{k/\mathbf{Q}}\mathfrak{p}_2})\Lambda(s,\chi) \\
&= \Lambda_S(s,\widehat{\gamma}).
\end{aligned}$$

□

5.2.4 The main proposition for the calculation

Let $a_n(\gamma)$ be the number of integral ideals $\mathfrak{U}$ of k such that $\mathfrak{U}$ is relatively prime to S, $\sigma_{\mathfrak{U}} = \gamma$, and $\mathrm{N}\mathfrak{U} = n$. If $A_n(\gamma) = a_n(\gamma) - a_n(\gamma\tau)$, then the Dirichlet series for $\zeta_S(s,\gamma) - \zeta_S(s,\tau\gamma)$ is given by $\sum_{n=1}^{\infty} A_n(\gamma)/n^s$. Since $\sum_{n=1}^{N} A_n(\gamma) = O(N^{2/3})$, this Dirichlet series converges for $\mathrm{Re}(s) > 2/3$ (see Theorems VI.3.3 and VIII.1.2 of [10]). The following proposition is essentially a special case of the calculation in Proposition 2.3 of [7].

Proposition 5.2.4. *For $a = \frac{21}{\pi^{3/2}}$ as above, we have*

$$\begin{aligned}
2\sqrt{\pi}\zeta_S'(0,\gamma) = \Lambda_S(0,\gamma) =& \sum_{n=1}^{\infty}\left[A_n(\gamma)\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(\frac{a}{n}\right)^z \Gamma(z)\Gamma\left(\frac{1+z}{2}\right)\frac{dz}{z}\right] \\
&+\sum_{n=1}^{\infty}\left[A_n(\widehat{\gamma})\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\left(\frac{a}{n}\right)^z \Gamma(z)\Gamma\left(\frac{1+z}{2}\right)\frac{dz}{z-1}\right].
\end{aligned}$$

Proof. Note that the line of integration in the first integral above can be shifted from $\mathrm{Re}(s) = 1/2$ to $\mathrm{Re}(s) = 3/2$ without altering the value, since the integrand has no poles with positive real part. By using the definition of Λ_S and the Dirichlet series for $\zeta_S(z,\gamma) - \zeta_S(z,\tau\gamma)$, and interchanging the order of summation and integration, the result will follow if we can show

$$\Lambda_S(0,\gamma) = \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\Lambda_S(z,\gamma)}{z}dz + \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\Lambda_S(z,\widehat{\gamma})}{z-1}dz.$$

The functional equation for Λ_S from Proposition 5.2.3 gives

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\Lambda_S(z,\widehat{\gamma})}{z-1}dz = \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\Lambda_S(1-z,\gamma)}{z-1}dz = -\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Lambda_S(z,\gamma)}{z}dz$$

where the last equality uses the change of variables $1 - z \mapsto z$. Thus, it suffices to show

$$\Lambda_S(0,\gamma) = \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\Lambda_S(z,\gamma)}{z}dz - \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Lambda_S(z,\gamma)}{z}dz.$$

This holds because shifting the line of integration for $\Lambda_S(z,\gamma)/z$ from $\mathrm{Re}(z) = 3/2$ to $\mathrm{Re}(z) = -1/2$ only picks up the simple pole at $z = 0$ with residue $\Lambda_S(0,\gamma)$. This completes the proof. □

5.2.5 How to calculate the relevant integrals

Proposition 5.2.4 reduces the calculation of $\zeta_S'(0,\gamma)$ to the evaluation of integrals of the form

$$F(t) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} t^z\Gamma(z)\Gamma\left(\frac{1+z}{2}\right)\frac{dz}{z}$$

and

$$G(t) = \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} t^z\Gamma(z)\Gamma\left(\frac{1+z}{2}\right)\frac{dz}{z-1}$$

for $t = \frac{a}{n}$, $n = 1, 2, 3, \dots$. We evaluate these integrals by shifting the lines of integration to the left and summing up the resulting residues.

Proposition 5.2.5. *For any integer $J > 0$, we have*

$$F(t) = \sqrt{\pi}\left(-\frac{3\gamma}{2} - \log 2 + \log t\right) + \sum_{j=1}^{J} f_j + \frac{1}{2\pi i}\int_{-J-\frac{1}{2}-i\infty}^{-J-\frac{1}{2}+i\infty} t^z\Gamma(z)\Gamma\left(\frac{1+z}{2}\right)\frac{dz}{z},$$

where

$$f_j = \begin{cases} \frac{\sqrt{\pi}(-1)^{j/2+1}(j/2-1)!2^{j-1}}{t^j j!^2} & j \text{ even} \\ \frac{(-1)^{(j-1)/2}}{(\frac{j-1}{2})!t^j j!j}\cdot\left(2\log t + \frac{2}{j} + 2H_j + H_{(j-1)/2} - 3\gamma\right) & j \text{ odd,} \end{cases}$$

with $H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j}$ ($H_0 = 0$) and $\gamma = -\Gamma'(1) = 0.577\ldots$ equal to Euler's constant.

A similar result holds for $G(t)$. The proofs of these formulae are given in Appendix B.

5.2.6 Number of integrals and residues to calculate

In this section we determine how many terms of the sum in Proposition 5.2.4 to evaluate and how many residues to calculate for each integral to get a provably accurate numerical approximation for $\Lambda_S(0,\gamma)$. By Proposition 2.3 of [7], we have as $t \to 0$ the estimate

$$F(t) \sim \sqrt{\frac{4\pi}{3}} \left(\frac{t^2}{2}\right)^{1/3} \exp(-3(2t)^{-2/3}).$$

Using a more precise version of this estimate and the very crude bound $|A_n|, |B_n| \leq n^{2/3}$ (which follows since no primes of $\mathbf{Q}$ less than $3^{3/2}$ split completely in k), we obtain the following upper bound on the error by estimating $\Lambda_S(0,\gamma)$ with the sum of the first N terms of Proposition 5.2.4:

$$\sqrt{\frac{1}{3\pi}}(2a^4N)^{1/3} \exp\left(-3\left(\frac{N}{2a}\right)^{2/3}\right).$$

To obtain an error of less than 10^{-25}, it suffices to take $N \geq 705$.

We now calculate the number of residues needed in Proposition 5.2.5 to obtain an accurate estimate for the integrals $F(t)$ and $G(t)$. This is easily computed using a suitable version of Stirling's formula. One finds for J large

$$\left|\frac{1}{2\pi i}\int_{-J-\frac{1}{2}-i\infty}^{-J-\frac{1}{2}+i\infty} t^z \Gamma(z)\Gamma\left(\frac{1+z}{2}\right)\frac{dz}{z}\right| \leq 2\cdot\left(\frac{8}{J^2 t}\right)^J.$$

To achieve an error in the calculation of $F(t)$ of less than 10^{-31}, it suffices to take $J \geq 69$. Note that since we will take $N < 1000$, an error of less than 10^{-31} in the calculation of $F(t)$ will contribute an error of less than 10^{-25} to the calculation of $\Lambda_S(0,\gamma)$.

5.2.7 The results for $\zeta'_S(0,\gamma)$

All calculations were done on the Pari-GP calculator. In our case, $\sigma_{\mathfrak{p}_3}$ generates G, and in terms of this generator one can compute on Pari that $\sigma_{\mathfrak{D}_{k/\mathbf{Q}}\mathfrak{p}_2} = \sigma_{\mathfrak{p}_3}^2$. Therefore $\widehat{\gamma} = \gamma^{-1}\sigma_{\mathfrak{p}_3}^2$ for $\gamma \in G$. The values of $A_n(\gamma)$ were computed using Pari's `ideallist` and `bnrisprincipal` commands. An internal accuracy of 77 digits was used with $N = 800$ and $J = 200$. This insures that the values of $\zeta'_S(0,\gamma)$ below are correct to within the 25 decimal places shown (truncated):

$$\begin{aligned}
\zeta'_S(0,\gamma^0) &= -\zeta'_S(0,\gamma^3) = 0.4384785858524408926911022\ldots \\
\zeta'_S(0,\gamma^1) &= -\zeta'_S(0,\gamma^4) = 0.3999812299583346413364528\ldots \\
\zeta'_S(0,\gamma^2) &= -\zeta'_S(0,\gamma^5) = 0.7885047438188618467230391\ldots
\end{aligned}$$

where $\gamma = \sigma_{\mathfrak{p}_3}$.

5.3 Finding the Stark unit ϵ

In Dummit/Sands/Tangedal's examples and in Stark's examples ([6] and [22]), the alleged Stark unit ϵ was obtained by considering its irreducible polynomial over the Hilbert class field H of k. We first describe this method (with notation adapted to our situation).

Let $\gamma = \sigma_{\mathfrak{p}_3}$, and let $\tau = \gamma^3$ be the unique element of G with order 2. Consider a complex place w' of K which lies above the real place $\infty_{\mathbf{R}}$ of k. A Stark unit must satisfy $|\epsilon|_{w'} = 1$. Furthermore, complex conjugation is an order 2 automorphism of K fixing k, so it must correspond to τ; since in the embedding $K \hookrightarrow K_w$, ϵ is a complex number of absolute value 1, we must have $\epsilon^{-1} = \epsilon^{\tau}$.

This implies that the minimal polynomial of ϵ over H is of the form $x^2 - Ax + 1$, where $A = \epsilon + \epsilon^{-1} = \operatorname{Tr}_{K/H} \epsilon$. Now consider the minimal polynomial of A over k, say $x^3 - s_1 x^2 + s_2 - s_3$, where the s_i's are algebraic integers in k. In the cases previously considered, when K has a real place, the information about the desired valuations of ϵ gave accurate estimates for the values of s_i in a real place of k. In [6], a standard recognition algorithm was then used to construct the minimal polynomials for the algebraic integers s_i; constructing the Stark unit ϵ was then straightforward.

However, in our case, K has no real places. Thus the information about the valuations of ϵ only give *bounds* on the complex valuations of the s_i. In our example, these bounds give too many possibilities to check feasibly on a computer. To solve this problem, we reduce the number of cases to check by studying the irreducible polynomial of ϵ over $k(\sqrt{-3})$ instead of over H.

5.3.1 The polynomial of ϵ over $k(\sqrt{-3})$

Recall that $\epsilon^{\tau} = \epsilon^{\gamma^3} = \epsilon^{-1}$. The minimal polynomial of ϵ over $k(\sqrt{-3})$ is then $x^3 - Bx^2 + Cx - D = 0$, where

$$
\begin{aligned}
B &= \operatorname{Tr}_{K/k(\sqrt{-3})} \epsilon = \epsilon + \epsilon^{\gamma^2} + \epsilon^{\gamma^4}, \\
D &= \operatorname{N}_{K/k(\sqrt{-3})} \epsilon = \epsilon^{1+\gamma^2+\gamma^4}, \text{ and} \\
C &= \epsilon^{1+\gamma^2} + \epsilon^{1+\gamma^4} + \epsilon^{\gamma^2+\gamma^4} = \epsilon^{1+\gamma^2+\gamma^4}(\epsilon^{-1} + \epsilon^{-\gamma^2} + \epsilon^{-\gamma^4}) \\
&= D \cdot B^{\tau} = D \cdot (\operatorname{Tr}_{k(\sqrt{-3})/k} B - B).
\end{aligned}
$$

Furthermore, B satisfies the equation $x^2 - t_1 x + t_2 = 0$ over k, where

$$
\begin{aligned}
t_1 &= \operatorname{Tr}_{k(\sqrt{-3})/k} B = \operatorname{Tr}_{K/k} \epsilon = \sum_{i=0}^{5} \epsilon^{\gamma^i}, \text{ and} \\
t_2 &= \operatorname{N}_{k(\sqrt{-3})/k} B = (\epsilon + \epsilon^{\gamma^2} + \epsilon^{\gamma^4})(\epsilon^{\gamma} + \epsilon^{\gamma^3} + \epsilon^{\gamma^5}).
\end{aligned}
$$

Also, $\operatorname{N}_{k(\sqrt{-3})/k} D = \operatorname{N}_{K/k} \epsilon = 1$ (since $\epsilon^{1+\tau} = 1$), so D satisfies the equation $x^2 - t_3 x + 1 = 0$, where

$$
t_3 = \operatorname{Tr}_{k(\sqrt{-3})/k} D = \epsilon^{1+\gamma^2+\gamma^4} + \epsilon^{\gamma+\gamma^3+\gamma^5}.
$$

Here $t_1, t_2, t_3 \in \mathcal{O}_k = \mathbf{Z}[\beta]$.

5.3.2 Bounds on the t_i

For a complex place w of K, recall that $|x|_w = w(x)\overline{w(x)}$. Hence we introduce the notation $\|x\|_w = \sqrt{|x|_w}$ for the magnitude of the complex number $w(x)$. Since K has $e = 6$ roots of unity, a Stark unit ϵ should satisfy

$$
\|\epsilon\|_{\gamma^i w} = \|\epsilon^{\gamma^{-i}}\|_w = \sqrt{\exp(-6\zeta_S'(0, \gamma^{-i}))} = \exp(-3\zeta_S'(0, \gamma^{-i})).
$$

where w is some fixed place above $v = \infty_{\mathbf{C}}$. Furthermore, we must have $\|\epsilon\|_{w'} = 1$ for any place w' of K above $\infty_{\mathbf{R}}$. Using section 5.2.7, this gives the following bounds on the t_i:

$$\|t_1\|_{\infty_{\mathbf{C}}} \le \sum_{i=0}^{5} \|\epsilon^{\gamma^i}\|_w < 18.37, \qquad |t_1|_{\infty_{\mathbf{R}}} \le 6,$$

$$\|t_2\|_{\infty_{\mathbf{C}}} \le \left(\|\epsilon\|_w + \|\epsilon^{\gamma^2}\|_w + \|\epsilon^{\gamma^4}\|_w\right)\left(\|\epsilon^{\gamma}\|_w + \|\epsilon^{\gamma^3}\|_w + \|\epsilon^{\gamma^5}\|_w\right) < 54.05, \qquad |t_2|_{\infty_{\mathbf{R}}} \le 9,$$

$$\|t_3\|_{\infty_{\mathbf{C}}} \le \|\epsilon\|_w \cdot \|\epsilon^{\gamma^2}\|_w \cdot \|\epsilon^{\gamma^4}\|_w + \|\epsilon^{\gamma}\|_w \cdot \|\epsilon^{\gamma^3}\|_w \cdot \|\epsilon^{\gamma^5}\|_w < 12.04, \qquad |t_3|_{\infty_{\mathbf{R}}} \le 2.$$

Here all the bounds have been "rounded up" along every step of the computation in order to ensure accuracy.

5.4 Creating a finite list of possibilities

Each t_i can be written as $t_i = a_i + b_i\beta + c_i\beta^2$ with $a_i, b_i, c_i \in \mathbf{Z}$. Furthermore, for each $x \in k$ we can consider the vector

$$\overrightarrow{x} \in k \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathbf{R} \times \mathbf{C} \cong \mathbf{R} \times (\mathbf{R}\cdot 1 \times \mathbf{R}\cdot i)$$

corresponding to x by

$$x \mapsto \overrightarrow{x} = (\infty_{\mathbf{R}}(x), \mathrm{Re}(\infty_{\mathbf{C}}(x)), \mathrm{Im}(\infty_{\mathbf{C}}(x))),$$

where $\infty_{\mathbf{R}}(x) \in \mathbf{R}$ and $\infty_{\mathbf{C}}(x) \in \mathbf{C}$ are the images of x in the real and complex embedding of k (of the two conjugate possibilities, choose a fixed complex embedding of k; here we choose the one determined by $\mathrm{Im}\,\beta > 0$). The image of $\mathcal{O}_k = \mathbf{Z}[\beta]$ under this $\mathbf{Q}$-linear map is a lattice in $\mathbf{R}^3$ generated by $\overrightarrow{1}$, $\overrightarrow{\beta}$, and $\overrightarrow{\beta^2}$. Our bounds on t_i give bounds on the possibilities for a_i, b_i, and c_i. With $|\cdot|_{\mathbf{R}^3}$ denoting the standard magnitude of a vector in $\mathbf{R}^3$ we find for $\overrightarrow{t_i} = a_i\overrightarrow{1} + b_i\overrightarrow{\beta} + c_i\overrightarrow{\beta^2}$:

$$|\overrightarrow{t_1}|_{\mathbf{R}^3} = \sqrt{|t_1|^2_{\infty_{\mathbf{R}}} + \|t_1\|^2_{\infty_{\mathbf{C}}}} < 19.32.$$

We similarly find $|\overrightarrow{t_2}|_{\mathbf{R}^3} < 54.79$ and $|\overrightarrow{t_3}|_{\mathbf{R}^3} < 12.21$.

If $\times$ denotes the standard cross product in $\mathbf{R}^3$ then the Cauchy-Schwartz inequality yields

$$|a_i| \cdot |\overrightarrow{1} \cdot (\overrightarrow{\beta} \times \overrightarrow{\beta^2})| = |\overrightarrow{t_i} \cdot (\overrightarrow{\beta} \times \overrightarrow{\beta^2})| \le |\overrightarrow{t_i}|_{\mathbf{R}^3} |\overrightarrow{\beta} \times \overrightarrow{\beta^2}|_{\mathbf{R}^3}.$$

Hence the bounds on $|\overrightarrow{t_i}|_{\mathbf{R}^3}$ give bounds on the possible size of a_i, b_i, and c_i. These bounds are given below:

i	a_i	b_i	c_i
1	18.46	9.37	5.13
2	52.36	26.58	14.55
3	11.67	5.92	3.24

Thus we have reduced our calculation to the checking of finitely many cases. One need not check all of these cases, however. If $\delta = -0.191\ldots$ is the real root of $f(x)$, then we also have from the bounds on t_i in the place $\infty_{\mathbf{R}}$ the inequality

$$-d_i - b_i\delta - c_i\delta^2 \le a_i \le -b_i\delta - c_i\delta^2 + d_i,$$

where $(d_1, d_2, d_3) = (6, 9, 2)$. This reduces the number of a_i to check.

The list of all these possible triples (t_i) was reduced by first checking that the resulting B and D values were elements of $k(\sqrt{-3})$ (using the `nfroots` function in Pari). Next, the valuations of the corresponding ϵ in the archimedean places of K were checked with the values of ζ'_S obtained earlier. This reduces to exactly one possible Stark unit ϵ (up to a root of unity and the choice of the place w lying above $v = \infty_{\mathbf{C}}$), which we now describe.

5.5 The results

Following the method above, we can reduce to a unique possible Stark unit ϵ, up to a choice of root of unity. Furthermore, this ϵ has the desired valuations at the complex places of K up to at least 25 decimal places. The minimal polynomials for the possible ϵ are given below:

$$\begin{aligned}
f_1(x) &= x^6 - (\beta^2 - 4\beta + 1)x^5 + (-7\beta^2 + 10\beta + 5)x^4 - \\
&\quad (22\beta^2 + 17\beta + 6)x^3 + (-7\beta^2 + 10\beta + 5)x^2 - (\beta^2 - 4\beta + 1)x + 1, \\
f_2(x) &= x^6 - (3\beta^2 - 2\beta + 1)x^5 + (-12\beta^2 - 14\beta - 2)x^4 - \\
&\quad (6\beta^2 + 49\beta + 8)x^3 + (-12\beta^2 - 14\beta - 2)x^2 - (3\beta^2 - 2\beta + 1)x + 1, \\
f_3(x) &= x^6 - (2\beta^2 + 2\beta)x^5 + (\beta^2 - 14\beta - 3)x^4 - \\
&\quad (-22\beta^2 + 35\beta + 8)x^3 + (\beta^2 - 14\beta - 3)x^2 - (2\beta^2 + 2\beta)x + 1, \\
f_4(x) &= -f_1(-x), f_5(x) = -f_2(-x), \text{ and } f_6(x) = -f_3(-x).
\end{aligned}$$

For each fixed $f_i(x)$ and fixed place w above v, there is unique root ϵ of $f_i(x)$ such that

$$\log |\epsilon^\gamma|_w \approx -6\zeta'_S(0, \gamma),$$

where $\approx$ represents an error proven to be less than 10^{-25}. Furthermore, for each root of unity ζ, there is a unique j such that $\zeta\epsilon$ is a root of $f_j(x)$.

The `nfroots` command of Pari-GP verified that the polynomials f_i indeed have roots in K, and that these roots do not lie in any subfield of K containing k. Furthermore, any root ϵ is clearly a unit of $\mathcal{O}_K$ since $\mathrm{N}_{K/k}\,\epsilon = 1$. We also see that $\epsilon^\tau = \epsilon^{-1}$ since $x^6 \cdot f_i(1/x) = f_i(x)$; it follows that $|\epsilon|_{w'} = 1$ for every place w' of K lying above the real prime $\infty_{\mathbf{R}}$ of k. To complete our numerical confirmation, it remains to prove that $K(\epsilon^{1/6})/k$ is abelian. The following general lemma will be useful for this purpose.

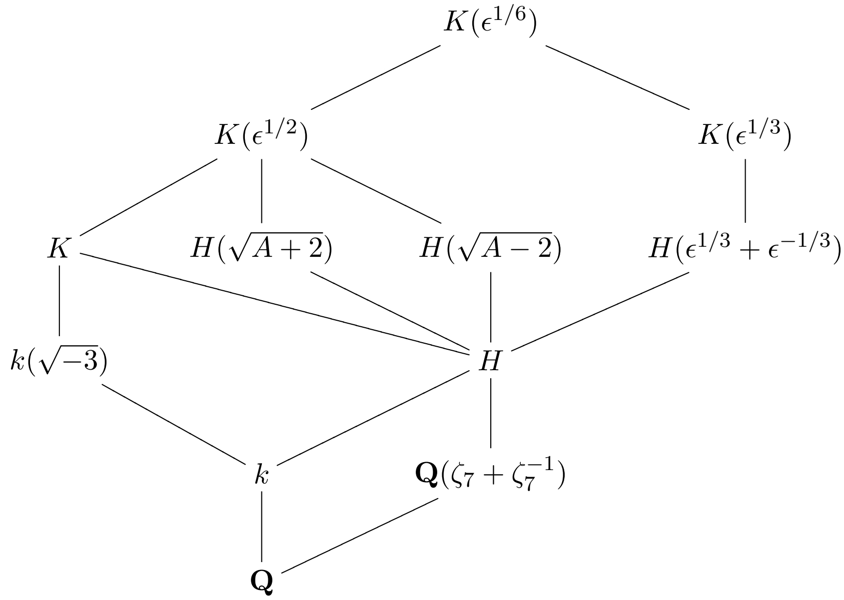
Lemma 5.5.1. *Let $F \supset L \supset H \supset k$ be a chain of finite extensions of fields such that $F/k, L/H$, and H/k are Galois. Suppose further that $[F : L]$ is the largest power of some prime p dividing $[F : H]$. Then L/k is Galois.*

Proof. We need to show that $\sigma(L) = L$ for every $\sigma \in \mathrm{Gal}(F/k)$. Since H/k is Galois, we have $\sigma(H) = H$ and hence $\sigma(L) \supset H$. Thus, $\sigma(L)$ is a subfield of F containing H and having index in F equal to that of L. We conclude that L and $\sigma(L)$ are the fixed fields of p-Sylow subgroups of $\mathrm{Gal}(F/H)$, say $L = F^{G_1}$ and $\sigma(L) = F^{G_2}$. Since these two subgroups are conjugate, say $G_2 = \tau G_1 \tau^{-1}$ with $\tau \in \mathrm{Gal}(F/H)$, we obtain $\sigma(L) = F^{G_2} = F^{\tau G_1 \tau^{-1}} = \tau(F^{G_1}) = \tau(L)$. But L/H is Galois, so $\tau(L) = L$, thus completing the proof. □

Proposition 5.5.2. *Let ϵ be a fixed root of one of the $f_i(x)$. Then $K(\epsilon^{1/6})/k$ is an abelian extension.*

Proof. When we choose an nth root $\epsilon^{1/n}$ of ϵ, we will write $\epsilon^{-1/n}$ for $1/(\epsilon^{1/n})$.

It suffices to show that $K(\epsilon^{1/2})/k$ and $K(\epsilon^{1/3})/k$ are abelian extensions. To demonstrate the first of these, we follow the methods of [6]. Since $(\epsilon^{1/2} \pm \epsilon^{-1/2})^2 = A \pm 2$, we have $K(\epsilon^{1/2}) = k(\sqrt{A+2}, \sqrt{A-2})$. Hence it suffices to show that $k(\sqrt{A+2})/k$ and $k(\sqrt{A-2})/k$ are abelian.



Clearly $A = \mathrm{Tr}_{K/H}\,\epsilon \in H$ does not lie in k or else $K = k(\epsilon)$ would have degree at most 2 over k (since $\epsilon^2 - A\epsilon + 1 = 0$). Thus $H = k(A)$ and $k(\sqrt{A+2}) = H(\sqrt{A+2}) \supset H$. Hence $H(\sqrt{A+2})/k$ must be abelian if it is Galois, since H/k is Galois and S_3 has no normal subgroups of size 2. Now $H(\sqrt{A+2})/k$ will be Galois if and only if $\sigma(A+2)/(A+2)$ is a square in H for a generator σ of $\mathrm{Gal}(H/k)$. This is easily checked (and similarly for $A-2$) in Pari-GP.

To show that $K(\epsilon^{1/3})/k$ is abelian, we first check that it is Galois. By Kummer theory, this is equivalent to $\sigma(\epsilon)/\epsilon^i$ being a cube in K with $i = 1$ or $i = 2$ for each $\sigma \in \mathrm{Gal}(K/k)$ (but i can depend on σ). Again, one can verify this on Pari-GP.

The element $\epsilon^{1/3} + \epsilon^{-1/3}$ satisfies the polynomial $x^3 - 3x - A$ over H. It can be checked on Pari that this polynomial has no roots in H. Hence $K(\epsilon^{1/3})$ is the compositum of K and $H(\epsilon^{1/3} + \epsilon^{-1/3})$ by degree considerations. Thus we have reduced to showing that $H(\epsilon^{1/3} + \epsilon^{-1/3})/k$ is abelian; this is equivalent to the extension being Galois since there are no non-abelian groups of order 9. Lemma 5.5.1 with $L = H(\epsilon^{1/3} + \epsilon^{-1/3})$, $F = K(\epsilon^{1/3})$, and $p = 2$ will give the desired result if we can check that the cubic extension $H(\epsilon^{1/3} + \epsilon^{-1/3})/H$ is Galois. This is equivalent to the discriminant of the irreducible cubic $x^3 - 3x - A \in H[x]$ being a square in H, which one can verify on Pari-GP. $\square$

The following proposition summarizes the results of this section.

Proposition 5.5.3. *Fix a place w of K lying above $\infty_{\mathbf{C}}$. If there is a Stark unit ϵ for $\mathrm{St}(K/k, S)$, then after scaling by a root of unity, it must a root of $f_1(x)$. Furthermore, for each w, there is a root ϵ of $f_1(x)$ satisfying all of the properties of a Stark unit except for possibly*

$$\log|\epsilon^\sigma|_w = -6\zeta_S'(0,\sigma).$$

Finally, this last property is satisfied by ϵ up to an error proven to be less than 10^{-25}.

We conclude this section by posing the following question, the answer to which would have made the calculations in this example much less computationally intensive.

Question. Consider the Stark conjecture $\mathrm{St}(K/k, S, v)$ when v is a complex prime. Is there a formula for the actual value in $K_w = \mathbf{C}$ of a (conjectured) Stark unit ϵ ? In other words, can one propose a putative formula for the *argument* of a Stark unit ϵ in addition to the Stark formula for its absolute value?

Not only would answering this question make numerical confirmations easier (by pinpointing the actual *values* of the s_i and t_i in the archimedean places, instead of just giving *bounds* on them), but it would also enable one to make a reference to Hilbert's 12th problem in the complex v case, as one can do in the real v case.

6 Tate's reformulation of the rank one abelian Stark conjecture

In this section, we present Tate's reformulation of Conjecture 4.3.2. We demonstrate the equivalence between Stark's formulation and Tate's formulation, and we give alternate proofs of the propositions in section 4.3 in Tate's language.

6.1 Definitions for Tate's reformulation

In this section and the next, S will be any finite set of primes of k containing the archimedean places and $G = \mathrm{Gal}(K/k)$ is assumed to be abelian. We do not impose any of the other conditions of the abelian Stark conjecture given in 4.3.1 until we state Tate's reformulation of the conjecture in section 6.3.

Remark 6.1.1. Any $\chi \in \widehat{G}$ extends by linearity to a linear functional (in fact a $\mathbf{C}$-algebra homomorphism) $\mathbf{C}[G] \to \mathbf{C}$. The $\chi \in \widehat{G}$ form a basis for the space of linear functionals $\mathbf{C}[G] \to \mathbf{C}$. It follows that if $x, y \in \mathbf{C}[G]$ and $\chi(x) = \chi(y)$ for all $\chi \in \widehat{G}$, then $x = y$.

Definition 6.1.2. Let θ be the meromorphic function on $\mathbf{C}$ with values in $\mathbf{C}[G]$ given by

$$\theta(s) = \theta_{K/k,S}(s) = \sum_{\chi \in \widehat{G}} L_S(s,\chi) e_{\overline{\chi}},$$

where e_χ is defined as in 4.2.1. For every finite prime $\mathfrak{p}$ of k, define

$$F_{\mathfrak{p}} = \frac{1}{|I_{\mathfrak{p}}|} \sum_{\tau \in \sigma_{\mathfrak{p}}} \tau^{-1} \in \mathbf{Q}[G],$$

where $\sigma_{\mathfrak{p}}$ is the coset of $I_{\mathfrak{p}}$ in $G_{\mathfrak{p}}$ corresponding to the Frobenius automorphism.

Proposition 6.1.3. *The function $\theta(s)$ satisfies the following properties.*

- *For $s \in \mathbf{C}$ and any character χ of G we have*

$$\chi(\theta^{(n)}(s)) = L_S^{(n)}(s, \overline{\chi}),$$

  where $\theta^{(n)}$ and $L_S^{(n)}$ denote nth derivatives.

- *For* $\mathrm{Re}(s) > 1$*, we have*

$$\theta(s) = \prod_{\mathfrak{p} \notin S} \left(1 - F_{\mathfrak{p}} \,\mathrm{N}\mathfrak{p}^{-s}\right)^{-1}, \tag{27}$$

  with the product absolutely and uniformly convergent in right half-planes $\mathrm{Re}(s) \geq a > 1$*.*

- *If S contains the ramified primes of K/k, then for* $\mathrm{Re}(s) > 1$*, we have*

$$\theta(s) = \sum_{\sigma \in G} \zeta_S(s,\sigma)\sigma^{-1}.$$

Proof. The definition of the e_ψ along with the orthogonality relations for irreducible characters over $\mathbf{C}$ gives $\chi(e_\psi) = 0$ for $\chi \neq \psi$ and $\chi(e_\chi) = 1$. This implies the first assertion.

We similarly find that for $\chi \in \widehat{G}$,

$$\chi(F_{\mathfrak{p}}) = \begin{cases} \chi(\sigma_{\mathfrak{p}})^{-1} = \overline{\chi}(\sigma_{\mathfrak{p}}) & \text{if } \chi(I_{\mathfrak{p}}) = 1 \\ 0 & \text{otherwise} \end{cases},$$

and hence for $\mathrm{Re}(s) > 1$, we have

$$\chi\left(\prod_{\mathfrak{p}\notin S}\left(1 - F_{\mathfrak{p}}\,\mathrm{N}\mathfrak{p}^{-s}\right)^{-1}\right) = \prod_{\substack{\mathfrak{p}\notin S\\ \chi(I_{\mathfrak{p}})=1}}\left(1 - \overline{\chi}(\sigma_{\mathfrak{p}})\,\mathrm{N}\mathfrak{p}^{-s}\right)^{-1} = L_S(s,\overline{\chi}) = \chi(\theta(s)).$$

The second assertion now follows from Remark 6.1.1. The final assertion is similarly obtained by expanding the Euler product in (27), since $F_{\mathfrak{p}} = \sigma_{\mathfrak{p}}$ for $\mathfrak{p}$ unramified (and considering $\mathrm{Re}(s) > 1$ without loss of generality). Note that one could have deduced the final statement directly from the definition of θ and the orthogonality relations without using Euler products (for $\mathrm{Re}(s) > 1$). □

Corollary 6.1.4. *For $s \in \mathbf{C}$ and $\mathfrak{p} \notin S$, we have*

$$\begin{aligned}\theta_{S\cup\{\mathfrak{p}\}}(s) &= \theta_S(s)\left(1 - F_{\mathfrak{p}}\,\mathrm{N}\mathfrak{p}^{-s}\right),\\ \theta'_{S\cup\{\mathfrak{p}\}}(0) &= \theta'_S(0)\,(1 - F_{\mathfrak{p}}) + \log \mathrm{N}\mathfrak{p}\cdot F_{\mathfrak{p}}\cdot\theta_S(0).\end{aligned}$$

Definition 6.1.5. Let H be a subgroup of G and let $K' = K^H$ be the fixed field of H. There is a natural algebra homomorphism $\pi\colon \mathbf{C}[G] \to \mathbf{C}[G/H]$ induced by the projection $G \to G/H$. Also, any $x \in \mathbf{C}[G]$ gives by multiplication an endomorphism of the free $\mathbf{C}[H]$-module $\mathbf{C}[G]$. The determinant of this endomorphism is called the norm $\mathrm{N}(x) \in \mathbf{C}[H]$.

In order to prove that θ satisfies certain naturality properties we need the following facts from the representation theory of finite abelian groups.

Proposition 6.1.6. *Let $\chi \in \widehat{H}$ and let χ^* be the induced representation on G. Let $\{\chi_1, \ldots, \chi_n\}$ be the set of characters in $\widehat{G}$ which restrict to χ on H. Then we have:*

- *$n = [G : H]$ and $\chi^* = \sum_{i=1}^{[G:H]} \chi_i$.*
- *There is an equality of maps $\mathbf{C}[G] \to \mathbf{C}$ given by*

$$\chi \circ \mathrm{N} = \prod_{i=1}^{[G:H]} \chi_i.$$

Proof. For $\psi \in \widehat{G}$, Frobenius reciprocity gives

$$\langle \chi^*, \psi\rangle_G = \langle \chi, \psi|_H\rangle_H = \begin{cases} 1 & \text{if } \psi = \chi_i \text{ for some } i \\ 0 & \text{otherwise}\end{cases}.$$

Hence $\chi^* = \sum_{i=1}^{n} \chi_i$ and $n = [G : H]$ follows by considering dimension.

For the second assertion, let $x \in \mathbf{C}[G]$ and let $\{g_i\}$ be a set of coset representatives for H in G. Then $\mathrm{N}(x) = \det(a_{ij})$ where $xg_i = \sum_j a_{ij}g_j$ is the unique such expression with $a_{ij} \in \mathbf{C}[H]$. Furthermore, the fact that χ is an algebra homomorphism $\mathbf{C}[H] \to \mathbf{C}$ implies that

$$\chi \circ \mathrm{N}(x) = \det M,$$

where $M = (\chi(a_{ij}))$. Hence it suffices to prove that the n-dimensional square matrix M has $\{\chi_1(x), \ldots, \chi_n(x)\}$ as its set of eigenvalues (with multiplicities). Applying χ_l to the equation defining the a_{ij} gives

$$\chi_l(x)\chi_l(g_i) = \sum_{j=1}^{n} \chi(a_{ij})\chi_l(g_j)$$

so $v_l = (\chi_l(g_1), \ldots, \chi_l(g_n))$ is an eigenvector of M with eigenvalue $\chi_l(x)$. Any linear dependence $\sum c_i v_i = 0$ gives a dependence among the linear functionals χ_i, as follows. For any $g \in G$, write $g = g_j h$ with $h \in H$. Then

$$\sum c_i \chi_i(g) = \chi(h) \sum c_i \chi_i(g_j) = 0.$$

Since the functionals χ_i are linearly independent, the v_i are linearly independent and the second assertion is proven. □

Proposition 6.1.7. *We have*

$$\begin{aligned} \theta_{K'/k,S}(s) &= \pi(\theta_{K/k,S}(s)) \text{ and} \\ \theta_{K/K',S_{K'}}(s) &= \mathrm{N}(\theta_{K/k,S}(s)). \end{aligned}$$

Proof. These equations result from the naturality properties of L-functions, cf. A.7.2, as we now explain. For $\chi \in \widehat{G/H}$ we have, using Proposition 6.1.3 twice,

$$\chi(\theta_{K'/k,S}(s)) = L_{K'/k,S}(s, \overline{\chi}) = L_{K/k,S}(s, \overline{\chi} \circ \pi) = \chi \circ \pi \left(\theta_{K/k,S}(s)\right).$$

The first assertion now follows from Remark 6.1.1.

For the second equality, let $\chi \in \widehat{H}$ and let the notation be as in Proposition 6.1.6. Then (by Proposition 6.1.3)

$$\chi\left(\theta_{K/K',S_{K'}}(s)\right) = L_{K/K',S_{K'}}(s, \overline{\chi}) = L_{K/k,S}(s, \overline{\chi}^*).$$

Applying Propositions 6.1.3 and 6.1.6 we find

$$L_{K/k,S}(s, \overline{\chi}^*) = \prod_{i=1}^{[G:H]} L_{K/k,S}(s, \overline{\chi}_i) = \prod_{i=1}^{[G:H]} \chi_i(\theta_{K/k,S}(s)) = \chi \circ \mathrm{N}(\theta_{K/k,S}(s)).$$

The result now follows from Remark 6.1.1. □

6.2 Arithmetic preliminaries for Tate's reformulation

Proposition 6.2.1. *Let T be a finite set of places of k containing the archimedean primes, the primes which ramify in K, and the primes dividing $e = |\mu(K)|$. Then the annihilator A of the $\mathbf{Z}[G]$-module $\mu(K)$ is generated over $\mathbf{Z}$ by the elements $\sigma_\mathfrak{p} - \mathrm{N}\mathfrak{p}$, as $\mathfrak{p}$ ranges over the primes of k not in T. We furthermore have*

$$e = \gcd_{\mathfrak{p} \notin T, \sigma_\mathfrak{p} = 1} (1 - \mathrm{N}\mathfrak{p}).$$

Proof. If $\eta \in \mu(K)$, then for $\mathfrak{p} \notin T$ and $\mathfrak{P}$ a prime of K lying over $\mathfrak{p}$, we have $\eta^{\sigma_\mathfrak{p} - \mathrm{N}\mathfrak{p}} \equiv 1 \pmod{\mathfrak{P}}$ by the definition of the Frobenius automorphism. Now $\eta^{\sigma_\mathfrak{p} - \mathrm{N}\mathfrak{p}}$ is a solution of the equation $x^e - 1 = 0$ in $\mathcal{O}_K$, hence in $\mathcal{O}_K/\mathfrak{P}$. Since $\mathfrak{P}$ does not divide e, Hensel's Lemma implies that this lifts uniquely to a solution in $K_\mathfrak{P}$. Thus we actually have $\eta^{\sigma_\mathfrak{p} - \mathrm{N}\mathfrak{p}} = 1$, so $\sigma_\mathfrak{p} - \mathrm{N}\mathfrak{p} \in A$. The Cebotarev Density Theorem implies that any $\sigma \in G$ is of the form $\sigma = \sigma_\mathfrak{p}$ with $\mathfrak{p} \notin T$. Hence, any $a \in A$ can be written as

$$a = \sum_{\mathfrak{p} \notin T} a_\mathfrak{p}(\sigma_\mathfrak{p} - \mathrm{N}\mathfrak{p}) + a'$$

where $a_\mathfrak{p}, a' \in \mathbf{Z}$ and all but finitely many of the $a_\mathfrak{p}$ are zero. Then $a' \in A$, so a' must be divisible by e. Therefore, the first assertion of the proposition follows from the second, since it implies that e can be written as a finite $\mathbf{Z}$-linear combination of the $1 - \mathrm{N}\mathfrak{p}$ with $\mathfrak{p} \notin T$ and $\sigma_\mathfrak{p} = 1$.

We have seen above that if $\mathfrak{p} \notin T$ and $\sigma_{\mathfrak{p}} = 1$, then $1 - \mathrm{N}\mathfrak{p} \in A$, so $1 - \mathrm{N}\mathfrak{p}$ is divisible by e. It remains to demonstrate that these $(1 - \mathrm{N}\mathfrak{p})$'s have no greater common divisor e'. Let e' be such a common divisor and let ζ be a primitive e'-th root of unity in an extension of K. Consider the abelian extension $K(\zeta)/k$. To avoid confusion about which extension is involved, we use the notation $(\mathfrak{p}, K/k)$ for the Frobenius automorphism associated to $\mathfrak{p}$ for the extension K/k.

Let σ be an arbitrary element of $\mathrm{Gal}(K(\zeta)/K) \subset \mathrm{Gal}(K(\zeta)/k)$. By Cebotarev, $\sigma = (\mathfrak{p}, K(\zeta)/k)$ for some $\mathfrak{p} \notin T$. By the functorial properties of the Artin map (Proposition A.3.1), the restriction of σ to K is $(\mathfrak{p}, K/k)$. However, σ acts trivially on K and hence $(\mathfrak{p}, K/k) = 1$; thus by definition we have $e' | (1 - \mathrm{N}\mathfrak{p})$ and so $\sigma = (\mathfrak{p}, K(\zeta)/k)$ acts trivially on ζ. Therefore σ is the identity. As σ was arbitrary in $\mathrm{Gal}(K(\zeta)/K)$, we conclude that $\zeta \in K$ and $e'|e$. This concludes the proof. □

Definition 6.2.2. Let k^{ab} denote an abelian closure of k containing K. For any L with $k \subset L \subset k^{\mathrm{ab}}$ we define the canonical map $\sim : U_{L,S} \to \mathbf{Q}U_{L,S}$ given by $\widetilde{x} = 1 \otimes x$.

Proposition 6.2.3. *For each $\sigma \in G$, choose $n_\sigma \in \mathbf{Z}$ such that $\zeta^\sigma = \zeta^{n_\sigma}$ for each $\zeta \in \mu(K)$. Then for $u \in \mathbf{Q}U_{K,S}$, the following three statements are equivalent:*

(a) *There exists $\epsilon \in U^{\mathrm{ab}}_{K/k}$ such that $\widetilde{\epsilon} = eu$.*

(b) *There exists L with $k \subset L \subset k^{\mathrm{ab}}$ and $u \in \widetilde{L}$; equivalently, $u \in \widetilde{k^{\mathrm{ab}}}$.*

(c) *There exists a finite set of places T of k containing the archimedean places and those ramifying in K, such that for $\mathfrak{p} \notin T$ there exists $\epsilon_{\mathfrak{p}} \in U_K$ with $\epsilon_{\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}\mathcal{O}_K}$ and*

$$\widetilde{\epsilon_{\mathfrak{p}}} = u^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}}.$$

(d) *There exist $\epsilon \in U_{K,S}$ and $\alpha_\sigma \in U_{K,S}$ for each $\sigma \in G$ such that*

$$eu = \widetilde{\epsilon}, \quad \alpha_\sigma^{\sigma' - n_{\sigma'}} = \alpha_{\sigma'}^{\sigma - n_\sigma}, \quad \epsilon^{\sigma - n_\sigma} = \alpha_\sigma^e$$

for $\sigma, \sigma' \in G$.

Proof. We once again drop the subscripts S for ease of notation.

(a) $\implies$ (b): Take $L = K(\epsilon^{1/e})$, which satisfies $L \subset k^{\mathrm{ab}}$ by the assumption $\epsilon \in U^{\mathrm{ab}}_{K/k}$. Then $u = \frac{1}{e}\widetilde{\epsilon} = \widetilde{\epsilon^{1/e}}$ as desired.

(b) $\implies$ (c): We are given L/k with $\eta \in L \subset k^{\mathrm{ab}}$ and $u = \widetilde{\eta}$. Since $\widetilde{\eta^{\tau-1}} = u^{\tau-1} = 1$, we obtain $\eta^{\tau-1} \in \mu(L)$. Let T be a finite set of places of k containing S, the primes which ramify in L, and those dividing e_L (so T satisfies the conditions of Proposition 6.2.1 for L/k). For $\mathfrak{p} \notin T$, we define $\epsilon_{\mathfrak{p}} = \eta^{\sigma_{\mathfrak{p}} - N\mathfrak{p}}$, so $\widetilde{\epsilon_{\mathfrak{p}}} = u^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}}$.

To demonstrate that $\epsilon_{\mathfrak{p}} \in K$, we consider $\tau \in \mathrm{Gal}(L/K)$. Then

$$\epsilon_{\mathfrak{p}}^{\tau-1} = \left(\eta^{\tau-1}\right)^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}} = 1$$

by Proposition 6.2.1. Thus $\epsilon_{\mathfrak{p}} \in U_K$. Furthermore, $\eta \in U_L$ is obviously a T_L-unit, so

$$\epsilon_{\mathfrak{p}} = \eta^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}\mathcal{O}_L}$$

by the definition of the Frobenius automorphism. Since $\epsilon_{\mathfrak{p}} \in K$, we have $\epsilon_{\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}\mathcal{O}_K}$, as desired.

(c) $\Longrightarrow$ (d): Enlarge T if necessary to include the primes dividing e, so the conditions of Proposition 6.2.1 are satisfied. We can then find integers $b_{\mathfrak{p}}$ and $b_{\mathfrak{p},\sigma}$ for $\mathfrak{p} \notin T$ and $\sigma \in \mathrm{Gal}(K/k)$, with all but finitely many of the $b_{\mathfrak{p}}$ and $b_{\mathfrak{p},\sigma}$ equal to zero, such that

$$\begin{aligned} e &= \sum_{\mathfrak{p}\notin T} b_{\mathfrak{p}}(\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}), \\ \sigma - n_\sigma &= \sum_{\mathfrak{p}\notin T} b_{\mathfrak{p},\sigma}(\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}) \text{ for } \sigma \in \mathrm{Gal}(K/k). \end{aligned}$$

We define

$$\epsilon = \prod_{\mathfrak{p}\notin T} \epsilon_{\mathfrak{p}}^{b_{\mathfrak{p}}}, \quad \alpha_\sigma = \prod_{\mathfrak{p}\notin T} \epsilon_{\mathfrak{p}}^{b_{\mathfrak{p},\sigma}}.$$

Then

$$\widetilde{\epsilon} = \sum_{\mathfrak{p}\notin T} \widetilde{\epsilon_{\mathfrak{p}}}^{b_{\mathfrak{p}}} = \sum_{\mathfrak{p}\notin T} u^{b_{\mathfrak{p}}(\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p})} = eu.$$

For $\mathfrak{p}, \mathfrak{q} \notin T$, note that

$$\widetilde{\epsilon_{\mathfrak{p}}^{\sigma_{\mathfrak{q}} - \mathrm{N}\mathfrak{q}}} = u^{(\sigma_{\mathfrak{q}} - \mathrm{N}\mathfrak{q})(\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p})} = \widetilde{\epsilon_{\mathfrak{q}}^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}}}.$$

Hence $\epsilon_{\mathfrak{p}}^{\sigma_{\mathfrak{q}} - \mathrm{N}\mathfrak{q}}$ and $\epsilon_{\mathfrak{q}}^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}}$ are the same up to a factor which is a root of unity $\zeta_{\mathfrak{p},\mathfrak{q}} \in \mu(K)$. Since this root of unity is congruent to 1 modulo $\mathfrak{p}$ and $\mathfrak{q}$ (e.g. mod $\mathfrak{p}$, we are given $\epsilon_{\mathfrak{p}} \equiv 1$, while $\epsilon_{\mathfrak{q}}^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}} \equiv 1$ by the definition of the Frobenius automorphism), the Hensel's Lemma argument of the previous proposition shows that $\zeta_{\mathfrak{p},\mathfrak{q}} = 1$. That is,

$$\epsilon_{\mathfrak{p}}^{\sigma_{\mathfrak{q}} - \mathrm{N}\mathfrak{q}} = \epsilon_{\mathfrak{q}}^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}}.$$

Hence,

$$\alpha_\sigma^e = \prod_{\mathfrak{p}\notin T} \epsilon_{\mathfrak{p}}^{eb_{\mathfrak{p},\sigma}} = \prod_{\mathfrak{p},\mathfrak{q}\notin T} \epsilon_{\mathfrak{p}}^{b_{\mathfrak{p},\sigma} b_{\mathfrak{q}}(\sigma_{\mathfrak{q}} - \mathrm{N}\mathfrak{q})} = \prod_{\mathfrak{p},\mathfrak{q}\notin T} \epsilon_{\mathfrak{q}}^{b_{\mathfrak{p},\sigma} b_{\mathfrak{q}}(\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p})} = \prod_{\mathfrak{p}\notin T} \epsilon^{b_{\mathfrak{p},\sigma}(\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p})} = \epsilon^{\sigma - n_\sigma}.$$

Furthermore, for $\sigma, \sigma' \in \mathrm{Gal}(K/k)$, we have

$$\alpha_\sigma^{\sigma' - n_{\sigma'}} = \prod_{\mathfrak{p},\mathfrak{q}\notin T} \epsilon_{\mathfrak{p}}^{b_{\mathfrak{p},\sigma} b_{\mathfrak{q},\sigma'}(\sigma_{\mathfrak{q}} - \mathrm{N}\mathfrak{q})} = \prod_{\mathfrak{p},\mathfrak{q}\notin T} \epsilon_{\mathfrak{q}}^{b_{\mathfrak{p},\sigma} b_{\mathfrak{q},\sigma'}(\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p})} = \alpha_{\sigma'}^{\sigma - n_\sigma}.$$

Thus property (d) is satisfied.

(d) $\Longrightarrow$ (a): We will show that $K(\epsilon^{1/e})$ is an abelian extension of k. Let η be an e-th root of ϵ in an algebraic closure $\overline{k}$ of k and fix an embedding $K \hookrightarrow \overline{k}$ over k. Let τ be an automorphism of $\overline{k}$ over k, so we get $\tau|_K \in \mathrm{Gal}(K/k)$; we write n_τ for $n_{\tau|_K}$ and α_τ for $\alpha_{\tau|_K}$. Now

$$(\eta^\tau)^e = \epsilon^\tau = \epsilon^{n_\tau} \alpha_\tau^e = (\eta^{n_\tau} \alpha_\tau)^e.$$

Hence $\eta^\tau = \zeta \cdot \eta^{n_\tau} \alpha_\tau$ for some eth root of unity ζ, which necessarily lies in K by definition of $e = |\mu(K)|$. This implies $\eta^\tau \in K(\eta)$ for all τ, so $K(\eta)/k$ is a Galois extension. Now recall that the definition of the n_τ yields $\zeta^{\tau - n_\tau} = 1$ for all τ, and hence

$$(\eta^{\tau - n_\tau})^{\tau' - n_{\tau'}} = (\alpha_\tau \zeta^{m_\tau})^{\tau' - n_{\tau'}} = \alpha_\tau^{\tau' - n_{\tau'}} = \alpha_{\tau'}^{\tau - n_\tau} = (\eta^{\tau' - n_{\tau'}})^{\tau - n_\tau}$$

for $\tau, \tau' \in \mathrm{Gal}(\overline{k}/k)$. We conclude from this that $\eta^{\tau\tau'} = \eta^{\tau'\tau}$, finishing the proof (since K/k is abelian). $\square$

Corollary 6.2.4. *In $\mathbf{Q}U_{k^{\mathrm{ab}}}$ we have $\mathbf{Q}U_K \cap \widetilde{U_{k^{\mathrm{ab}}}} = \frac{1}{e}\widetilde{U^{\mathrm{ab}}_{K/k}}$.*

Proof. The inclusion $\frac{1}{e}\widetilde{U^{\mathrm{ab}}_{K/k}} \subset \mathbf{Q}U_K \cap \widetilde{U_{k^{\mathrm{ab}}}}$ is clear: if $\epsilon \in U^{\mathrm{ab}}_{K/k}$, then $\frac{1}{e}\widetilde{\epsilon} \in \mathbf{Q}U_K$ and $\frac{1}{e}\widetilde{\epsilon} = \widetilde{\epsilon^{1/e}} \in \widetilde{U_{k^{\mathrm{ab}}}}$. The reverse inclusion follows from the implication (b) $\implies$ (a) of the proposition. If $u \in \mathbf{Q}U_K$ has image in $\mathbf{Q}U_{k^{\mathrm{ab}}}$ satisfying $u = \widetilde{\eta}$ for $\eta \in U_{k^{\mathrm{ab}}}$, then statement (b) of the proposition holds with $L = k(\eta)$. Thus statement (a) holds, and $eu \in \widetilde{U^{\mathrm{ab}}_{K/k}}$. □

Corollary 6.2.5. *If L is an extension of k contained in K and $u \in \frac{1}{e}\widetilde{U^{\mathrm{ab}}_{K/k}}$ in $\mathbf{Q}U_K$, then $\mathrm{N}_{K/L}u \in \frac{1}{e_L}\widetilde{U^{\mathrm{ab}}_{L/k}}$ in $\mathbf{Q}U_L$.*

Proof. The given u satisfies condition (a) of the proposition, so by condition (c) there exists a finite set of places T of k containing the archimedean places and those ramifying in K, such that for $\mathfrak{p} \notin T$ there exists $\epsilon_{\mathfrak{p}} \in U_K$ with $\epsilon_{\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}\mathcal{O}_K}$ and $\widetilde{\epsilon_{\mathfrak{p}}} = u^{\sigma_p - \mathrm{N}\mathfrak{p}}$. Let $\epsilon'_{\mathfrak{p}} = \mathrm{N}_{K/L}\epsilon_{\mathfrak{p}} \in U_L$. Since for any $\sigma \in \mathrm{Gal}(K/L)$ we have $\epsilon_{\mathfrak{p}}^{\sigma} \equiv 1 \pmod{\mathfrak{p}\mathcal{O}_K}$, the product of such congruences over $\sigma \in \mathrm{Gal}(K/L)$ gives $\epsilon'_{\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}\mathcal{O}_K}$. Hence we have $\epsilon'_{\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}\mathcal{O}_L}$. Furthermore

$$\widetilde{\epsilon'_{\mathfrak{p}}} = \widetilde{\prod_{\sigma} \epsilon_{\mathfrak{p}}^{\sigma}} = \widetilde{\epsilon_{\mathfrak{p}}}^{\sum \sigma} = u^{(\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p})\sum \sigma} = (\mathrm{N}_{K/L}u)^{\sigma_{\mathfrak{p}} - \mathrm{N}\mathfrak{p}},$$

where the product and sums run over all $\sigma \in \mathrm{Gal}(K/L)$. Thus the $\epsilon'_{\mathfrak{p}}$ satisfy condition (c) for $\mathrm{N}_{K/L}u \in \mathbf{Q}U_L$, and we conclude from condition (a) that $\mathrm{N}_{K/L}u \in \frac{1}{e_L}\widetilde{U^{\mathrm{ab}}_{L/k}}$. □

6.3 Tate's reformulation of Stark's Conjecture

Conjecture 6.3.1. *Let S satisfy the conditions of* 4.3.1. *Recall the definitions of X and λ from* 3.2.1 *and* 3.3.1*, respectively. Then we have*

$$\theta'(0)X \subset \frac{1}{e}\lambda\left(U^{\mathrm{ab}}_{K/k}\right),$$

or equivalently,

$$\lambda^{-1}(\theta'(0)X) \subset \widetilde{U_{k^{\mathrm{ab}}}}.$$

Remark 6.3.2. The equivalence of the conclusions in the conjecture above follows from Corollary 6.2.4.

Proposition 6.3.3. *Conjectures* 4.3.2 *and* 6.3.1 *are equivalent.*

Proof. Suppose Conjecture 6.3.1 is true. By the third condition on S in 4.3.1, namely $|S| \geq 2$, there exists a $w' \in S_K$ which does not lie above v. Then there exists an $\epsilon \in U^{\mathrm{ab}}_{K/k}$ such that

$$\theta'(0)(w' - w) = \frac{1}{e}\lambda(\epsilon).$$

Since S contains the primes of k ramifying in K, we can apply Proposition 6.1.3 to obtain

$$\sum_{\chi \in \widehat{G}} L'_S(0,\chi)e_{\overline{\chi}} = \theta'(0) = \sum_{\sigma \in G} \zeta'(0,\sigma)\sigma^{-1},$$

and hence
$$\frac{1}{e}\lambda(\epsilon) = \sum_{\chi\in\widehat{G}} L'_S(0,\chi)e_{\overline{\chi}}w' - \sum_{\sigma\in G}\zeta'(0,\sigma)\sigma^{-1}w.$$

Let $\chi \in \widehat{G}$ and $\chi \neq 1_G$. If the multiplicity of χ in $\mathbf{C}Y$ is at least 2, then Proposition 3.2.2 implies that $L'_S(0,\chi) = 0$. Otherwise, the χ component of $\mathbf{C}Y$ is contained in $\bigoplus_{\sigma\in G}\mathbf{C}\sigma w$, which is the regular representation of G since $G_w = 1$. Hence $e_{\overline{\chi}}w' = 0$ in this case. Therefore,

$$\sum_{\chi\in\widehat{G}} L'_S(0,\chi)e_{\overline{\chi}}w' = L'_S(0,1_G)e_{1_G}w' = \zeta'_{k,S}(0)\frac{1}{|G|}\sum_{\sigma\in G}\sigma w'.$$

The equation
$$\frac{1}{e}\lambda(\epsilon) = \zeta'_{k,S}(0)\frac{1}{|G|}\sum_{\sigma\in G}\sigma w' - \sum_{\sigma\in G}\zeta'_S(0,\sigma)\sigma^{-1}w$$

implies that $\epsilon \in U^v$. Indeed, if $|S| \geq 3$, then $\zeta'_{k,S}(0) = 0$ by Proposition 3.2.4, and if $S = \{v', v\}$ then evidently the coefficients of $\sigma w'$ in $\lambda(\epsilon)$, for $\sigma \in G$, are all equal to each other. Furthermore, we see that for any $\sigma \in G$,

$$\frac{1}{e}\log|\epsilon^\sigma|_w = \frac{1}{e}\log|\epsilon|_{\sigma^{-1}w} = -\zeta'_S(0,\sigma).$$

Conjecture 4.3.2 is therefore satisfied.

Conversely, suppose that an ϵ exists satisfying Conjecture 4.3.2. Then reversing the steps from above we find that for any w' not lying above v,

$$\frac{1}{e}\lambda(\epsilon) = \theta'(0)(w' - w)$$

and similarly $\frac{1}{e}\lambda(\epsilon^\sigma) = \theta'(0)(w' - w^\sigma)$ for $\sigma \in G$. Yet the set of all $w' - w^\sigma$ generates X over $\mathbf{Z}$, so that
$$\theta'(0)X \subset \frac{1}{e}\lambda\left(U^{\mathrm{ab}}_{K/k}\right),$$

giving Conjecture 6.3.1. □

We noted earlier that Stark's conjecture is independent of the choices of v and w. Tate's formulation of the conjecture shows this directly by making no explicit references to any such choices.

6.4 Tate's version of the proofs from section 4.3

Tate's Conjecture 6.3.1 allows for quick proofs of the results in section 4.3. From Tate's point of view, the proof of Proposition 6.2.3 takes care at once of many intricate arguments which one needs to carry out when studying Stark's formulation of 4.3.2.

Notation 6.4.1. For any place v of k, we write $\mathrm{N}G_v$ for the sum of the elements of the decomposition group G_v in $\mathbf{Z}[G]$:
$$\mathrm{N}G_v = \sum_{\sigma\in G_v}\sigma \in \mathbf{Z}[G].$$

Proposition 6.4.2. *If $|S| \geq n+1$, then for distinct places $v_1, \ldots, v_n \in S$, we have*

$$\left(\prod_{i=1}^{n} \mathrm{N}G_{v_i}\right) \cdot \theta^{(n-1)}(0) = 0.$$

Proof. Let $\chi \in \widehat{G}$ and $\chi \neq 1_G$. Now $\chi(\mathrm{N}G_{v_i}) = 0$ unless $\chi(G_{v_i}) = 1$, and if $\chi(G_{v_i}) = 1$ for all $i = 1, \ldots, n$, then $L^{(n-1)}(0, \overline{\chi}) = 0$ by Proposition 3.2.4. Hence in all cases

$$\left(\prod_{i=1}^{n} \chi(\mathrm{N}G_{v_i})\right) L^{(n-1)}(0, \overline{\chi}) = 0.$$

Furthermore, Proposition 3.2.4 shows that $L^{(n-1)}(0, 1_G) = 0$ since $|S| \geq n+1$, so the above equation holds for all $\chi \in \widehat{G}$. The proposition now follows from Proposition 6.1.3 and Remark 6.1.1. □

Recall that if $|S| \geq 3$ and S contains two places v and v' which split completely, then $\epsilon = 1$ is a Stark unit for $\mathrm{St}(K/k, S)$. In Tate's language, this is represented by the formula

$$\theta'(0) = \mathrm{N}G_v \, \mathrm{N}G_{v'} \theta'(0) = 0$$

from Proposition 6.4.2. We can also give another proof of Proposition 4.3.7.

Proposition 6.4.3 (Tate's proof of 4.3.7). $\mathrm{St}(K/k, S)$ *implies* $\mathrm{St}(K/k, S')$ *for* $S \subset S'$.

Proof. Without loss of generality, $S' \neq S$. Fix $v \in S$ which splits completely in K, and choose any $\mathfrak{p} \in S' - S$. By the conditions on S, $\mathfrak{p}$ is a finite prime of k unramified in K. Proposition 6.4.2 shows that $0 = \mathrm{N}G_v \theta_S(0) = \theta_S(0)$, so

$$\theta'_{S \cup \{\mathfrak{p}\}}(0) = \theta'_S(0)(1 - F_{\mathfrak{p}}) \in \theta'_S(0) \cdot \mathbf{Z}[G]$$

by Corollary 6.1.4 (note that $F_{\mathfrak{p}} = \sigma_{\mathfrak{p}}^{-1} \in G$). Thus $\theta'_{S \cup \{\mathfrak{p}\}}(0) X \subset \theta'_S(0) \cdot \mathbf{Z}[G] X = \theta'_S(0) X$. Hence by induction on $|S' - S|$ we have $\theta'_{S'}(0) X \subset \theta'_S(0) X$ and and the proposition follows. □

We can also complete the proof of Proposition 4.3.8, now that we have built up the proper arithmetic machinery.

Proposition 6.4.4 (Tate's proof of 4.3.8). *If* $k \subset F \subset K$ *then* $\mathrm{St}(K/k, S)$ *implies* $\mathrm{St}(F/k, S)$.

Proof. Recall the embedding $X_{F,S} \hookrightarrow X_{K,S}$ given in 3.2.1. We have

$$\theta_{F/k,S}(0) X_{F,S} = (\pi \theta_{K/k,S}(0)) X_{F,S} \subset \theta_{K/k,S}(0) X_{K,S}$$

and the result follows. □

We now fill in the missing details of the proof of Proposition 6.4.4 given in 4.3.8. Fix a place $v \in S$ which splits completely and a place w of K lying over v. Let ϵ be a Stark unit for $(K/k, S, w)$. By By Corollary 6.2.5, we see that since $u = \frac{1}{e}\widetilde{\epsilon} \in \frac{1}{e}\widetilde{U^{\mathrm{ab}}_{K/k}}$, we have $\mathrm{N}_{K/F} u \in \frac{1}{e_F}\widetilde{U^{\mathrm{ab}}_{F/k}}$. Hence there exists an $\epsilon_F \in U^{\mathrm{ab}}_{F/k}$ such that

$$\widetilde{(\epsilon_F)^{e/e_F}} = e \, \mathrm{N}_{K/F} u = \widetilde{\mathrm{N}_{K/F} \epsilon}.$$

Therefore, $\epsilon_F^{e/e_F} = \zeta \, \mathrm{N}_{K/F} \epsilon$ for some root of unity $\zeta \in F$. The proof of 4.3.8 now shows that ϵ_F is a Stark unit for $(F/k, S, w_F)$.

7 Galois groups of exponent two

In this section we consider the work of Sands ([14] and [15]) in the case where $G = \mathrm{Gal}(K/k)$ has exponent two. Sands was able to prove Stark's conjecture in this case with the extra assumption that either $|S| > m + 1$ or K/k is tame. The basic idea of the exponent two case is that a Stark unit for K/k can be built up from Stark units of K_i/k, as K_i ranges over the relative quadratic extensions of k contained in K. Furthermore, one can easily compute the (unique) non-trivial L-function in the relative quadratic case because it is equal to a ratio of zeta-functions:

$$L_{K_i/k}(s,\chi) = \frac{\zeta_{K_i}(s)}{\zeta_k(s)}.$$

We begin with some definitions and lemmatta.

7.1 Basic results

Lemma 7.1.1. *Let K/k be a Galois extension with Galois group of prime exponent q. If $\mathfrak{P}$ is a finite prime of K which is ramified over k and $\mathfrak{P} \nmid q$, then the inertia group $I_{\mathfrak{P}}$ is cyclic of order q.*

Proof. The ramification index of $\mathfrak{P}$ over k divides $[K : k]$, which is a power of q. The size of the finite field $\mathcal{O}_K/\mathfrak{P}$ is a power of the positive prime rational integer p contained in $\mathfrak{P}$. Hence the fact that $q \notin \mathfrak{P}$ implies that p divides the ramification index of $\mathfrak{P}$ over k, and so $\mathfrak{P}$ is tamely ramified over k. By the structure of totally tame ramified extensions, $I_{\mathfrak{P}}$ is cyclic (Proposition A.4.3). Since $I_{\mathfrak{P}}$ is a subgroup of $\mathrm{Gal}(K/k)$, which has exponent q, $I_{\mathfrak{P}}$ has size q. □

Notation 7.1.2. Let F/k be a quadratic extension with $\mathrm{Gal}(F/k) = \{1, \tau\}$, and let d_F be the number of finite primes of k which ramify in F. Write I_F for the group of fractional ideals of F, and write P_F for the group of principal ideals in I_F. Let $I_{F/k}$ be the subgroup of I_F given by the fractional ideals of I_k extended to F. Define $P_{F/k}$ similarly. The set of "ambiguous ideals" is $A_F = \{\mathfrak{U} \in I_F : \mathfrak{U}^\tau = \mathfrak{U}\}$. The inclusion $A_F \subset I_F$ induces a map

$$\phi_F\colon A_F/P_{F/k} \to I_F/P_F = \mathrm{Cl}(F).$$

Remark 7.1.3. Note that $I_{F/k} \subset A_F$, and in fact A_F is generated by $I_{F/k}$ and the prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_{d_F}\}$ of F which are ramified over k. Furthermore, each of the $\mathfrak{p}_i$'s has order two in the quotient $A_F/I_{F/k}$, and they are independent (over $\mathbf{Z}$). Thus $[A_F : I_{F/k}] = 2^{d_F}$, with $A_F/I_{F/k}$ a group of exponent 2. Since $\mathrm{Cl}(k) = I_k/P_k \to I_{F/k}/P_{F/k}$ is a surjection, we see that $A_F/P_{F/k}$ is also finite, so ϕ_F is a map between finite groups. It is easy to see that $I_k/P_k \to I_{F/k}/P_{F/k}$ is injective, and hence an isomorphism, so $[A_F : P_{F/k}] = 2^{d_F} h_k$.

Proposition 7.1.4. *Suppose L is an everywhere unramified extension of F with L/k Galois of degree 2^{h+1}, such that $\mathrm{Gal}(L/k)$ is abelian with exponent 2. Let H be the kernel of the Artin map $I_F \to \mathrm{Gal}(L/F)$, so*

$$[HA_F : P_F A_F] \le [H : P_F] \le [I_F : P_F] \le \infty.$$

Then

- *$H \supset I_{F/k}$.*
- *$\frac{[I_F : P_F A_F]}{2^{h-d_F}}$ is an integer divisible by $[HA_F : P_F A_F]$.*
- *If $A_F \cap H \ne I_{F/k}$ then $\frac{[I_F : P_F A_F]}{2^{h-d_F+1}}$ is an integer.*

Proof. For the first assertion, let $\mathfrak{U} \in I_k$. Then $(\mathfrak{U}\mathcal{O}_F, L/F) = (\mathfrak{U}, L/k)^2$ by the consistency of the Artin map (Proposition A.3.1). Since $\mathrm{Gal}(L/k)$ has exponent 2, this is the identity and thus $I_{F/k} \subset H$.

For the second assertion, note that $[I_F : H] = [L : F] = 2^h$. Furthermore, $[HA_F : H] = [A_F : A_F \cap H]$ divides $[A_F : I_{F/k}]$ by the first assertion. We have seen that this last index is 2^{d_F}. Hence

$$\frac{[I_F : P_F A_F]}{2^{h-d_F}} = \left(\frac{[I_F : H]}{[HA_F : H]} \cdot [HA_F : P_F A_F] \right) \cdot \frac{1}{2^{h-d_F}} = \frac{2^{d_F}}{[HA_F : H]} \cdot [HA_F : P_F A_F]$$

and the result follows.

The final assertion of the proposition is a refinement of the previous one, for if $A_F \cap H \neq I_{F/k}$, then we have $[HA_F : H]$ is a proper divisor of 2^{d_F} and the result follows by the above calculation. □

Lemma 7.1.5. *Let F be such that $\zeta^{1+\tau} = 1$ for $\zeta \in \mu(F)$. If $\epsilon \in F$ and $\epsilon^{\tau+1} = 1$, then $F(\epsilon^{1/e_F})/k$ is abelian.*

Proof. With the notation of Proposition 6.2.3 (taking S a sufficiently large set so $\epsilon \in U_{F,S_F}$), we can take $n_1 = 1, n_\tau = -1, \alpha_1 = \alpha_\tau = 1$ so that the desired properties are satisfied. Thus by the *proof* of (d) $\implies$ (a), we see that $F(\epsilon^{1/e_F})/k$ is abelian. □

Throughout the remainder of this section, we consider $\mathrm{St}(K/k, S, v)$ and assume that the place v of S (which splits completely in K) is infinite; the finite case will be considered as part of the Brumer-Stark conjecture in section 8.2. As usual, w is an extension of v to K. We assume further that:

- $|S| \geq 3$.
- S contains at least one other infinite prime.
- S contains no primes which split completely other than v. In particular, the infinite primes of $S - \{v\}$ are real, and they ramify into complex places in K.

We have seen that $\mathrm{St}(K/k, S)$ is true if any of these assumptions do not hold (from 4.3.11, 4.3.10, and 4.3.4 respectively). We define $S(K/k)$ to be the minimal possible set S for the extension K/k, namely the set of $r \geq 2$ archimedean places of k and $d = d_K$ finite primes of k which ramify in K.

7.2 Relative quadratic extensions

For the first part of this section, we assume that $[K : k] = 2$. Let $G = \{1, \tau\}$, and take $S = S(K/k)$.

Since $|S| \geq 3$, Proposition 3.2.4 implies $L'_S(0, 1_G) = 0$, so we analyze $L'_S(0, \chi)$ where χ is the non-trivial character of G. From the product formula, we obtain

$$L_S(s, \chi) = L(s, \chi) = \frac{\zeta_K(s)}{\zeta_k(s)},$$

where the first equality holds since the only finite primes of S are those which ramify, and χ acts non-trivially on the corresponding inertia groups (which are $G = \{1, \tau\}$). The $r - 1$ real primes of k become complex primes of K, and the prime v splits into two primes of K. Thus K has $r + 1$ (inequivalent) archimedean places, so the unit groups of $\mathcal{O}_k$ and $\mathcal{O}_K$ have ranks $r - 1$ and r respectively (Proposition 3.3.2). Thus, and we obtain from 3.1.2 that

$$L'_S(0, \chi) = \lim_{s \to 0} \frac{L_S(s, \chi)}{s} = \lim_{s \to 0} \frac{\zeta_K(s)/s^r}{\zeta_k(s)/s^{r-1}} = \frac{e_k \cdot h_K \cdot R_K}{e_K \cdot h_k \cdot R_k}.$$

Our goal is to understand the ratios h_K/h_k and R_K/R_k in terms of the unit groups U_k and U_K of $\mathcal{O}_k$ and $\mathcal{O}_K$, respectively.

Lemma 7.2.1. *The finitely generated group $U_K/(U_k\mu(K))$ has rank one. If Q is the order of its torsion subgroup, and if $\delta \in U_K$ with $|\delta^{1-\tau}|_w \geq 1$ represents a generator of the maximal torsion-free quotient of $U_K/(U_k\mu(K))$, then*

$$\frac{R_K}{R_k} = \frac{2^{r-2}}{Q}\log|\delta^{1-\tau}|_w.$$

Proof. We have seen above that K has $r+1$ infinite places, while k has r such places, so U_K and U_k have ranks r and $r-1$ respectively. Since $\mu(k)$ is finite, we see that $U_K/(U_k\mu(K))$ has rank one.

Let the archimedean places of k other than v be $v_1,\ldots,v_{r-1}$, and let the places of K lying above these be $w_1,\ldots,w_{r-1}$. Let $w_r = w$. If $\epsilon_1,\ldots,\epsilon_r$ are independent units of U_K, we define the regulator $R(\{\epsilon_i\})$ as the determinant of an $r\times r$ matrix

$$R(\epsilon_1,\ldots,\epsilon_r) = |\det(\log|\epsilon_i|_{w_j})|.$$

If the ϵ_i are a set of *fundamental* units, then $R(\{\epsilon_i\})$ is simply the regulator $R_K \neq 0$. In general,

$$R(\epsilon_1,\ldots,\epsilon_r) = R_K \cdot [U_K : \mu(K)\langle\epsilon_1,\ldots,\epsilon_r\rangle] \neq 0,$$

where $\langle\epsilon_1,\ldots,\epsilon_r\rangle \subset U_k$ denotes the subgroup generated by $\epsilon_1,\ldots,\epsilon_r$. In particular, if we choose $\epsilon_1,\ldots,\epsilon_{r-1}$ to be fundamental units for U_k, then

$$R(\epsilon_1,\ldots,\epsilon_{r-1},\delta) = R_K\cdot[U_K : \mu(K)U_k\langle\delta\rangle] = R_K\cdot[U_K/(\mu(K)U_k) : \langle\delta\rangle] = Q\cdot R_K.$$

Now $\delta^{1-\tau} = \delta^{-(1+\tau)}\delta^2$, and $\delta^{-(1+\tau)} = \mathrm{N}\delta^{-1} \in U_k$. Hence

$$R(\epsilon_1,\ldots,\epsilon_{r-1},\delta^{1-\tau}) = R(\epsilon_1,\ldots,\epsilon_{r-1},\delta^2) = 2Q\cdot R_K.$$

However, we can also compute this regulator directly. For $j \leq r-1$, the decomposition group of w_j is $\{1,\tau\}$, so $|\delta^{1-\tau}|_{w_j} = 1$ for these places. Furthermore, for these places we have $|\epsilon_i|_{w_j} = |\epsilon_i|_{v_j}^2$ by our conventions on the normalization of archimedean absolute values (section 2). We conclude that

$$\begin{aligned} R(\epsilon_1,\ldots,\epsilon_{r-1},\delta^{1-\tau}) &= \log|\delta^{1-\tau}|_w \left|\det(\log|\epsilon_i|_{w_j})\right| \\ &= 2^{r-1}\log|\delta^{1-\tau}|_w \left|\det(\log|\epsilon_i|_{v_j})\right| \\ &= 2^{r-1}\log|\delta^{1-\tau}|_w R_k. \end{aligned}$$

Thus we have shown that $2Q\cdot R_K = 2^{r-1}\log|\delta^{1-\tau}|_w R_k$, proving the lemma. □

Definition 7.2.2. Let $U^- = \{u \in U_K : \mathrm{N}u = u^{1+\tau} = 1\}$. Note that $U_K^{1-\tau}$ has finite index in U^-.

Lemma 7.2.3. *With the map of finite groups $\phi_K\colon A_K/P_{K/k} \to I_K/P_K$ defined as in* 7.1.2,

$$|\operatorname{Coker}\phi_K| = [I_K : P_KA_K] = \frac{2^{1-d}}{Q}\cdot\frac{h_K}{h_k}[U^- : \mu(K)U_K^{1-\tau}].$$

Proof. Since $|\operatorname{Coker}\phi_K| = \frac{h_K}{[A_K:P_{K/k}]}|\operatorname{Ker}\phi_K|$, and $[A_K : P_{K/k}] = 2^d h_k$, it suffices to show

$$|\operatorname{Ker}\phi| = \frac{2}{Q}[U^- : \mu(K)U_K^{1-\tau}].$$

Note that $\operatorname{Ker}\phi_K = (P_K \cap A_K)/P_{K/k}$. Any ideal $(a) \in (P_K \cap A_K)/P_{K/k}$ corresponds to a well defined coset $a^{1-\tau} \in (K^*)^{1-\tau}/U_K^{1-\tau}$. Since $(a) \in A_K$ we also have $a^{1-\tau} \in U_K$, and $(K^*)^{1-\tau} \cap U_K \subset U^-$. To summarize, we have a well defined map

$$(P_K \cap A_K)/P_{K/k} \to U^-/U_K^{1-\tau}$$

given by $(a) \mapsto a^{1-\tau}$. Hilbert's Theorem 90 states that $(K^*)^{1-\tau} \cap U_K = U^-$, so the above map is surjective. Injectivity is not difficult to see either: if $a^{1-\tau} = b^{1-\tau}$ with $b \in U_K$, then a/b is fixed by τ and hence lies in k. Thus $(a) = (a/b) \in P_{K/k}$.

Hence, we have $|\operatorname{Ker}\phi_K| = [U^- : U_K^{1-\tau}]$. Note that if $\zeta \in \mu(K)$, then τ acts as complex conjugation in any complex place w_j of K, so $\zeta^{1+\tau} = |\zeta|_{w_j} = 1$. Thus $\mu(K) \subset U^-$, so we have

$$[U^- : U_K^{1-\tau}] = [U^- : \mu(K)U_K^{1-\tau}][\mu(K)U_K^{1-\tau} : U_K^{1-\tau}].$$

To complete the proof, we have to show that

$$[\mu(K)U_K^{1-\tau} : U_K^{1-\tau}] = \frac{2}{Q}.$$

Since $\zeta^2 = \zeta^{1-\tau}$ for $\zeta \in \mu(K)$, so $\mu(K)^2 \subset U_K^{1-\tau}$, we compute

$$[\mu(K)U_K^{1-\tau} : U_K^{1-\tau}] = [\mu(K) : \mu(K) \cap U_K^{1-\tau}] = \frac{[\mu(K) : \mu(K)^2]}{[\mu(K) \cap U_K^{1-\tau} : \mu(K)^2]}.$$

The group $\mu(K)$ is cyclic and of even order, so $[\mu(K) : \mu(K)^2] = 2$. It therefore suffices to prove that $(\mu(K) \cap U_K^{1-\tau})$ is isomorphic to $\mu(K)^2 \cong (U_K/\mu(K)U_k)_{\text{tors}}$, the torsion subgroup of $U_K/(\mu(K)U_k)$. Consider $u \in U_K$ representing a class in $(U_K/(\mu(K)U_k))_{\text{tors}}$. Since τ acts as inversion on roots of unity, $u^{1-\tau}$ represents a well-defined element of $U_K^{1-\tau}/\mu(K)^2$. Furthermore, if $u^t \in \mu(K)U_k$, then $u^{te_K} \in U_k$, so $(u^{1-\tau})^{te_K} = (u^{te_K})^{1-\tau} = 1$. Thus $u^{1-\tau} \in \mu(K)$. Hence, we have a well-defined map

$$(U_K/(\mu(K)U_k))_{\text{tors}} \to (\mu(K) \cap U_K^{1-\tau})/\mu(K)^2$$

given by $u \mapsto u^{1-\tau}$. It remains to prove that this map is surjective and injective. For the former, suppose that $u \in U_K$ and $u^{1-\tau} \in \mu(K)$; we need to show that $u^t \in \mu(K)U_k$ for some $t \in \mathbf{Z}$. But $(u^{e_K})^{1-\tau} = (u^{1-\tau})^{e_K} = 1$, so $u^{e_K} \in k \cap U_K = U_k$ as desired. To see injectivity, suppose $u \in U_K$ and $u^{1-\tau} = \zeta^2$ for $\zeta \in \mu(K)$. Then $(u/\zeta)^{1-\tau} = 1$, so $u \in \mu(K)U_k$. This completes the proof. □

We now consider a general $S \supset S(K/k)$. Let $M = M_S = M_{K,S} = [I_K : P_K A_K] \cdot 2^{|S|-3}$.

Theorem 7.2.4. *With the assumptions of this section,* $\operatorname{St}(K/k, S)$ *is true when* $[K : k] = 2$. *A Stark unit can to be taken to be* $\epsilon = \eta^M$ *for some* $\eta \in U^v \cap U_{K/k}^{\text{ab}}$.

Proof. First note that $U^v = U^-$, since the decomposition group in $\operatorname{Gal}(K/k)$ of any archimedean prime of k other than v is $\{1, \tau\}$. Recall that U^- is the kernel of the norm map $\mathrm{N}\colon U_K \to U_k$. The image $\mathrm{N}U_K$ lies between $U_k^2 = \mathrm{N}U_k$ and U_k, which each have rank $r - 1$, so $\mathrm{N}U_K$ has rank $r - 1$. Since U_K has rank r, we conclude that U^- has rank 1. Let η be a generator of the free abelian group $U^-/\mu(K)$ satisfying $|\eta|_w < 1$. With δ as in Lemma 7.2.1, $\delta^{1-\tau}$ generates the torsion-free rank 1 group $\mu(K)U_K^{1-\tau}/\mu(K)$, so

$$[U^- : \mu(K)U_K^{1-\tau}] \cdot \log|\eta|_w = -\log|\delta^{1-\tau}|_w. \tag{28}$$

Combining Lemma 7.2.1, Lemma 7.2.3, and (28), we obtain

$$\begin{aligned}
L'_{S(K/k)}(0,\chi) &= \frac{e_k \cdot h_K \cdot R_K}{e_K \cdot h_k \cdot R_k} \\
&= \frac{2}{e_K} \cdot \frac{h_K}{h_k} \cdot \frac{2^{r-2}}{Q} \log|\delta^{1-\tau}|_w \\
&= -\frac{2}{e_K} \cdot \frac{h_K}{h_k} \cdot \frac{2^{r-2}}{Q} [U^- : \mu(K)U_K^{1-\tau}] \cdot \log|\eta|_w \\
&= -\frac{2}{e_K} [I_K : P_K A_K] \frac{Q}{2^{1-d}} \frac{2^{r-2}}{Q} \log|\eta|_w \\
&= -\frac{2^{d+r-2}}{e_K} [I_K : P_K A_K] \log|\eta|_w \\
&= -\frac{2^{d+r-3}}{e_K} [I_K : P_K A_K] (\log|\eta|_w - \log|\eta|_{\tau w}).
\end{aligned}$$

Recalling the definition $M_{S(K/k)} = 2^{d+r-3}[I_K : P_K A_K]$, the previous equation can be written

$$L'_{S(K/k)}(0,\chi) = -\frac{1}{e_K} \sum_{\sigma \in G} \chi(\sigma) \log\left| \left(\eta^{M_{S(K/k)}}\right)^\sigma \right|_w.$$

Furthermore, we have

$$L'_{S(K/k)}(0,1_G) = 0 = -\frac{1}{e_K} \sum_{\sigma \in G} 1_G(\sigma) \log\left| \left(\eta^{M_{S(K/k)}}\right)^\sigma \right|_w$$

since $\eta^{1+\tau} = 1$.

We noted at the beginning of the proof that $U^- = U^v$, so $\eta \in U^v$. By Lemma 7.1.5, $U^- \subset U^{\mathrm{ab}}_{K/k}$, so $\eta \in U^v \cap U^{\mathrm{ab}}_{K/k}$. Thus, $\epsilon_{S(K/k)} = \eta^M$ is our desired Stark unit for $S(K/k)$. For general $S \supset S(K/k)$, choose $\mathfrak{p} \in S - S(K/k)$. If $\mathfrak{p}$ splits completely, then we have seen that $\epsilon = 1$ is a Stark unit. If $\mathfrak{p}$ does not split completely, then $\sigma_{\mathfrak{p}} = \tau$ since $\mathfrak{p}$ remains inert in K. Then the proof of Proposition 4.3.7 shows that (with the first product over $\mathfrak{p} \in S - S(K/k)$)

$$\epsilon_{S(K/k)}^{\prod_{\mathfrak{p}}(1-\tau)} = \left(\eta^{M_{S(K/k)}}\right)^{(1-\tau)^{|S-S(K/k)|}} = \eta^{M_{S(K/k)} \cdot 2^{|S-S(K/k)|}} = \eta^{M_S}$$

is a Stark unit for S. $\square$

7.3 General K/k of exponent 2

We now consider general K/k with Galois group G of exponent 2 and order 2^m. Pick an element $\tau \in G$ corresponding to complex conjugation in some complex place of K. It is an easy exercise to check that G has $2^m - 1$ subgroups of index 2, since these correspond to elements of order 2 in $\widehat{G}$ (which is non-canonically isomorphic to G). From this it follows that each element of G other than the identity lies in $2^{m-1} - 1$ of these subgroups. Thus there exist 2^{m-1} quadratic extensions K_i/k contained in K which are not fixed by τ. The compositum of the K_i must be K, for otherwise this compositum would have degree over k at most 2^{m-1}, so it could contain at most $2^{m-1} - 1$ different quadratic extensions of k, an absurdity.

Lemma 7.3.1. *Let K/k as above be tame (i.e. unramified at primes dividing 2), and let F be one of the K_i's. Assume that v is the only archimedean prime of k which splits completely in F. We let d_F be the number of finite primes of k which ramify in F and s be the number of primes of k which ramify in K but not F (all of these must be finite). Then*

- *$[I_F : P_F A_F]$ is an integer multiple of 2^{m-d_F-s-1}, and is an integer multiple of 2^{m-d_F-s} when $r = 2$.*
- *When $S = S(K/k)$ and $M = M_{S,F}$ as above, we have $\frac{M}{2^{m+r-4}} \in \mathbf{Z}$, and $\frac{M}{2^{m-1}} \in \mathbf{Z}$ when $r = 2$.*

Proof. Consider the subgroup of G generated by the inertia groups $I_\mathfrak{p}$ of the s primes of k which ramify in K but not F. The subfield $L \subset K$ fixed by this subgroup is the maximal unramified extension of F contained in K since no prime of k ramifying in F can ramify further in K, by Lemma 7.1.1 (with $q = 2$). This subgroup has size dividing 2^s, since each of the s inertia groups has size 2. Thus $[L : k] \geq 2^{m-s}$. Proposition 7.1.4 now implies that $[I_F : P_F A_F]$ is an integer multiple of 2^{m-d_F-s-1}.

When $r = 2$, a stronger result holds. Let $F = k(\sqrt{\alpha})$ with $\alpha \in \mathcal{O}_k$. Lemma A.3.7 implies that a finite place $\mathfrak{p}$ of k must ramify in F, since v is the only archimedean place of k which splits completely in F. By the tameness hypothesis, $\mathfrak{p}$ has odd residue characteristic, so by Kummer theory, an odd power of $\mathfrak{p}$ appears in the factorization of (α). Indeed, if this is not the case, then after completing at $\mathfrak{p}$ we see that $k_\mathfrak{p}(\sqrt{\alpha}) = k_\mathfrak{p}(\sqrt{u})$ for a unit $u \in \mathcal{O}_\mathfrak{p}$. Such an extension is unramified, contradicting our choice of $\mathfrak{p}$. Since an odd power of $\mathfrak{p}$ appears in (α), an odd power of $\mathfrak{P}$ appears in $(\sqrt{\alpha})$, where $\mathfrak{P}^2 = \mathfrak{p}\mathcal{O}_F$. Thus $(\sqrt{\alpha}) \notin I_{F/k}$ while evidently $(\sqrt{\alpha}) \in A_F$ since $(\sqrt{\alpha}) = (-\sqrt{\alpha})$. By Artin's Reciprocity Law, the principal ideal $(\sqrt{\alpha})$ lies in the kernel H of the Artin map $I_F \to \mathrm{Gal}(L/F)$. Thus the last part of Proposition 7.1.4 implies that $[I_F : P_F A_F]$ is an integer multiple of 2^{m-d_F-s} when $r = 2$.

The second assertion follows directly from the first since $|S| = r + d_F + s$ and $r - 3 \geq -1$. ☐

Theorem 7.3.2 (Sands). *Let $S = S(K/k)$ for K/k as above. If (1) $|S| > m + 1$ or (2) K/k is tame, then $\mathrm{St}(K/k, S)$ is true. In case (1) let $N = 2^{|S|-m-2}$, and in case (2) let $N = 2^{r-3}$ when $r \geq 3$ and let $N = 1$ when $r = 2$. Then there exists $\eta \in U^v \cap U^{\mathrm{ab}}_{K/k}$ such that $\epsilon = \eta^N$ is a Stark unit for S.*

Proof. Fix τ and the K_i as above, and let w_i be the restriction of w to K_i. Define $e_i = \mu(K_i)$. Note that $S \supset S(K_i/k)$, so we may let η_i and $M_i = M_{S,K_i}$ be as in Theorem 7.2.4 for the K_i which are ramified at the archimedean places of k other than v.

For the K_i which are unramified at some archimedean place other than v, let $\eta_i = 1$ and $M_i = 2^{m-1}N$. In any case, we have $M_i/(2^{m-1}N) \in \mathbf{Z}$ (using Lemma 7.3.1 in the second case) so $\eta_i^{M_i/(2^{m-1}N)} \in K_i$ makes sense and we can define

$$\eta = \prod_{i=1}^{2^{m-1}} \eta_i^{\frac{M_i}{2^{m-1}N}\frac{e}{e_i}} \in K.$$

Since the K_i generate K, we see that $K(\eta^{1/e})$ is contained in the compositum of the abelian extensions $K_i(\eta_i^{1/e_i})$ of k, so $K(\eta^{1/e})$ is abelian over k as well. Also, $\eta \in U^v$ since each $\eta_i \in U^v$. It remains to show that the conjugates of $\epsilon = \eta^N$ have the appropriate valuations at w.

Let χ be a character of $G = \mathrm{Gal}(K/k)$. If $\chi(\tau) = 1$ then $L'_S(0, \chi) = 0$ since τ generates the decomposition group of some real place in S. Also, since $\mathrm{Gal}(K_i/k) = \{1, \tau|_{K_i}\}$ and $\eta_i \in U^v_{K_i/k}$, we have $\eta_i^{1+\tau} = 1$. Hence if the σ_j are representatives for $G/\langle\tau\rangle$,

$$\sum_{\sigma\in G} \chi(\sigma)\log|\epsilon^\sigma|_w = \sum_{\sigma_j} \chi(\sigma_j)\log|\eta^{N(\sigma_j+\sigma_j\tau)}|_w = 0 = L'_S(0,\chi).$$

Now suppose that $\chi(\tau) \neq 1$. Since G has exponent two, $\chi^2 = 1_G$ and $\chi\colon G \to \{\pm 1\}$ is surjective with an index two kernel. The fixed field of this kernel is one of the K_i, say $K_{i(\chi)}$, since τ is not in the kernel. If $G_i = \mathrm{Gal}(K/K_i)$ for $i = 1, \ldots, 2^{m-1}$, then

$$\sum_{\sigma\in G_i} \chi(\sigma) = \begin{cases} 2^{m-1} & \text{if } \chi \text{ is trivial on } G_i \\ 0 & \text{otherwise} \end{cases}.$$

Since $|G_i| = |\ker\chi|$, we see that the first condition holds if and only if $G_i = \ker\chi$; that is, when $i = i(\chi)$. Hence

$$\begin{aligned} -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma)\log\left|\eta^{N\sigma}\right|_w &= -\frac{1}{e}\sum_{\sigma\in G}\sum_{i=1}^{2^{m-1}}\chi(\sigma)\log\left|\eta_i^{M_i\sigma}\right|_w^{\frac{1}{2^{m-1}}\frac{e}{e_i}} \\ &= -\sum_{i=1}^{2^{m-1}}\frac{1}{e_i\cdot 2^{m-1}}\sum_{\sigma\in G}\chi(\sigma)\log\left|\eta_i^{M_i\sigma}\right|_{w_i} \\ &= -\sum_{i=1}^{2^{m-1}}\frac{1}{e_i\cdot 2^{m-1}}\sum_{\sigma\in G_i}\chi(\sigma)\log\left|\eta_i^{M_i\sigma(1-\tau)}\right|_{w_i}, \end{aligned}$$

where the last equality holds because τ represents the non-trivial element of G/G_i and $\chi(\tau) = -1$.

Since $\sigma \in G_i$ fixes η_i, this last expression becomes

$$\begin{aligned} -\sum_{i=1}^{2^{m-1}}\frac{1}{e_i}\left(\frac{1}{2^{m-1}}\sum_{\sigma\in G_i}\chi(\sigma)\right)\log\left|\eta_i^{M_i(1-\tau)}\right|_{w_i} &= -\frac{1}{e_{i(\chi)}}\log\left|\eta_{i(\chi)}^{M_{i(\chi)}(1-\tau)}\right|_{w_{i(\chi)}} \\ &= -\frac{1}{e_{i(\chi)}}\sum_{\sigma\in\mathrm{Gal}(K_{i(\chi)}/k)}\chi(\sigma)\left|\eta_{i(\chi)}^{M_{i(\chi)}}\right|_{w_{i(\chi)}}, \end{aligned}$$

because $\mathrm{Gal}(K_{i(\chi)}/k) = \{1, \tau|_{K_{i(\chi)}}\}$. As $\eta_i^{M_i}$ is a Stark unit for K_i/k, this last expression equals

$$L'_S(0,\chi,K_{i(\chi)}/k) = L'_S(0,\chi,K/k)$$

by Proposition A.7.2. This completes the proof. $\square$

8 The Brumer-Stark conjecture

In this section, we consider the case of Conjecture 4.3.2 where the place v of S which splits completely in K is a *finite* prime $\mathfrak{p}$. We begin by giving a reformulation of Stark's conjecture in this case, which leads to a generalization that Tate dubbed the "Brumer-Stark conjecture".

8.1 Statement of the Brumer-Stark conjecture

Notation 8.1.1. Let T be a finite set of primes of k containing the archimedean places and the primes which ramify in K. Let T_K be the set of primes of K lying above those in T. We define

$$K_T = \begin{cases} \{x \in K : |x|_w = 1 \text{ for all } w \in T_K\} & \text{if } |T| \geq 2 \\ \{x \in K : |x|_{\sigma w} = |x|_w \text{ for all } \sigma \in G\} & \text{if } T = \{v\} \text{ and } w|v \end{cases}.$$

Remark 8.1.2. Let $\mathfrak{p}$ be a finite prime of k which splits completely in K and let $S = T \cup \{\mathfrak{p}\}$. Consider the Stark conjecture $\mathrm{St}(K/k, S)$ with $v = \mathfrak{p}$ and $w = \mathfrak{P}$ a prime of K lying above $\mathfrak{p}$. Note that $\mathrm{N}\mathfrak{P} = \mathrm{N}\mathfrak{p}$ since $\mathfrak{p}$ splits completely in K. Let $\epsilon \in K$ be a Stark unit for this case, so ϵ is a unit away from the primes dividing $\mathfrak{p}$ and we can write

$$(\epsilon) = \prod_{\sigma \in G} (\mathfrak{P}^\sigma)^{n_\sigma} \qquad \text{where} \qquad n_\sigma = -\frac{\log |\epsilon|_{\mathfrak{P}^\sigma}}{\log \mathrm{N}\mathfrak{p}} \in \mathbf{Z}.$$

Define $\lambda = \sum n_\sigma \sigma \in \mathbf{Z}[G]$, so $(\epsilon) = \mathfrak{P}^\lambda$, and (since ϵ is a Stark unit with respect to S)

$$\lambda = \sum_{\sigma \in G} -\frac{\log |\epsilon|_{\mathfrak{P}^\sigma}}{\log \mathrm{N}\mathfrak{p}} \sigma = \sum_{\sigma \in G} \frac{e\zeta_S'(0, \sigma^{-1})}{\log \mathrm{N}\mathfrak{p}} \sigma = \frac{e\theta_S'(0)}{\log \mathrm{N}\mathfrak{p}}.$$

By our hypothesis that $\mathfrak{p}$ splits completely in K, $\sigma_\mathfrak{p}$ is trivial and Corollary 6.1.4 yields $\theta_T(0) = \theta_S'(0)/\log \mathrm{N}\mathfrak{p}$. The fact that ϵ is a Stark unit can thus be summarized by the statements below:

- $e\theta_T(0) \in \mathbf{Z}[G]$.
- $(\epsilon) = \mathfrak{P}^{e\theta_T(0)}$.
- $\epsilon \in K_T$.
- $K(\epsilon^{1/e})/k$ is an abelian extension.

Note in particular that $e\theta_T(0) \in \mathbf{Z}[G]$ follows from our calculations, assuming the existence of a Stark unit. In fact, a more general statement due to Deligne and Ribet [5] is true (unconditionally).

Theorem 8.1.3 (Deligne-Ribet). *Let A be the annihilator of the $\mathbf{Z}[G]$-module $\mu(K)$. Then*

$$A \cdot \theta_T(0) \subset \mathbf{Z}[G].$$

In particular, $e\theta_T(0) \in \mathbf{Z}[G]$.

Conjecture 8.1.4. *Let $\mathfrak{A}$ be a fractional ideal of K. Then there exists an $\epsilon \in K_T$ such that $(\epsilon) = \mathfrak{A}^{e\theta_T(0)}$ and $K(\epsilon^{1/e})/k$ is an abelian extension.*

Remark 8.1.5. The idea that $e\theta_T(0)$ annihilates the ideal class group of $\mathcal{O}_K$ is due to Brumer and generalizes the result of Stickelberger (see [11] for the classical Stickelberger result with Gauss sums). The assertion that $K(\epsilon^{1/e})/k$ is abelian is motivated by Stark's conjecture, and generalizes the fact that Stickelberger's Gauss sums lie in cyclotomic fields. For this reason, Tate has called Conjecture 8.1.4 the Brumer-Stark conjecture. We denote it $\mathrm{BS}(K/k,T)$.

We write $I^* = I^*_{K/k,T}$ for the subgroup of I_K consisting of those $\mathfrak{U}$ which satisfy the Conjecture 8.1.4. It is clear that I^* is stable under the action of G. With the notation of Remark 8.1.2, the Stark conjecture $\mathrm{St}(K/k,S)$ is equivalent to the statement $\mathfrak{P} \in I^*$.

Theorem 8.1.6. *I^* contains the set of principal ideals $P_K \subset I_K$.*

Proof. Let $(\gamma) \in I_K$ be a principal ideal. For each $\sigma \in G$ choose a $\mathfrak{p} \notin T$ such that $\sigma = \sigma_{\mathfrak{p}}$. Let $n_\sigma = \mathrm{N}\mathfrak{p}$ and define

$$\alpha_\sigma = \gamma^{(\sigma - n_\sigma)\theta_T(0)}, \qquad \epsilon = \gamma^{e\theta_T(0)}.$$

Note that the exponents lie in $\mathbf{Z}[G]$ by Theorem 8.1.3, so the definitions of α_σ, ϵ make sense. Furthermore, for any $\alpha \in K$ we have

$$|\alpha|_w^{|G_w|} = \prod_{\sigma \in G_w} |\alpha|_w = \prod_{\sigma \in G_w} |\alpha|_{\sigma w} = \prod_{\sigma \in G_w} |\alpha^{\sigma^{-1}}|_w = |\alpha^{\mathrm{N}G_w}|_w.$$

Hence if $\alpha^{\mathrm{N}G_w} = 1$ for all $w \in T$ then $\alpha \in K_T$. Proposition 6.4.2 with $n = 1$ now shows that $\epsilon, \alpha_\sigma \in K_T$. Furthermore, the α_σ and ϵ satisfy

$$\alpha_\sigma^e = \epsilon^{\sigma - n_\sigma} \text{ and } \alpha_\sigma^{\sigma' - n_{\sigma'}} = \alpha_{\sigma'}^{\sigma - n_\sigma}.$$

These were precisely the relations used in the proof of the implication (d) $\implies$ (a) of Proposition 6.2.3 to show that $K(\epsilon^{1/e})/k$ is abelian. □

Remark 8.1.7. Let $\mathcal{P}$ be a set of finite primes of K satisfying

- if $\mathfrak{P} \in \mathcal{P}$ lies over $\mathfrak{p}$, then $\mathfrak{p} \notin T$ and $\mathfrak{p}$ splits completely in K, and
- the $\mathfrak{P} \in \mathcal{P}$ generate the ideal class group I_K/P_K over G.

Theorem 8.1.6 along with Remark 8.1.5 shows that the conjecture $\mathrm{BS}(K/k,T)$ is true if and only if $\mathrm{St}(K/k, T \cup \{\mathfrak{p}\})$ is true for all $\mathfrak{p}$ lying below primes $\mathfrak{P} \in \mathcal{P}$. In particular, $\mathrm{BS}(K/k,T)$ is true if and only if $\mathrm{St}(K/k, T \cup \{\mathfrak{p}\})$ is true for all $\mathfrak{p} \notin T$ which split completely in K (see A.3.10).

Proposition 8.1.8. *With notation as above, we have*

- *If $T \subset T'$ then* $\mathrm{BS}(K/k,T)$ *implies* $\mathrm{BS}(K/k,T')$.
- *If $k \subset K' \subset K$, then* $\mathrm{BS}(K/k,T)$ *implies* $\mathrm{BS}(K'/k,T)$.
- $\mathrm{BS}(K/k,T)$ *is true if k is not totally real or if K is not totally complex.*
- *If $|T| = 1$ then* $\mathrm{BS}(K/k,T)$ *is true.*

Proof. Using Remark 8.1.7, the assertions follow from 4.3.7, 4.3.8, 4.3.4, and 4.3.11, respectively. □

8.2 Galois groups of exponent 2

We now present Sands' proof [14] of the Brumer-Stark conjecture for extensions K/k with Galois group G of exponent 2 (and satisfying one extra condition). Recall the definition of $S(K/k)$ from section 7.1. By Proposition 8.1.8, we may assume that $|S(K/k)| \geq 2$, k is totally real, and K is totally complex. Let K° be the subgroup of K^* consisting of all elements which have valuation 1 at the infinite primes of K. Note that if $\mathfrak{P} \in T_K$, then by Proposition 6.4.2 with $n = 2$ we have

$$\left(\mathfrak{P}^{e\theta_T(0)}\right)^{|G_{\mathfrak{P}}|} = \left(\mathfrak{P}^{e\theta_T(0)}\right)^{\mathrm{N}G_{\mathfrak{P}}} = (1),$$

and hence $\mathfrak{P}^{e\theta_T(0)} = (1)$. Therefore, for any $\mathfrak{U} \in I_K$ and $\epsilon \in K^\circ$, if $\mathfrak{U}^{e\theta_T(0)} = (\epsilon)$ then in fact $\epsilon \in K_T$.

Definition 8.2.1. Suppose $\gamma \in \mathbf{Z}[G]$ and $\mathfrak{U} \in I_K$. We say that γ is a *BS-annihilator* for $\mathfrak{U}$ if there exists $\epsilon \in K^\circ$ such that $\mathfrak{U}^\gamma = (\epsilon)$ and $K(\epsilon^{1/e})/k$ is abelian.

Note that the ideals for which a given γ is a BS-annihilator form a subgroup of I_K; γ is a called a BS-annihilator for this subgroup. Also, the γ which are BS-annihilators for a given $\mathfrak{U}$ form an ideal of $\mathbf{Z}[G]$. The conjecture $\mathrm{BS}(K/k, T)$ for $|T| \geq 2$ is that $e\theta_T(0)$ is a BS-annihilator for I_K.

8.2.1 Relative quadratic extensions

We now assume that $[K : k] = 2$ with $G = \{1, \tau\}$ and maintain our assumptions that $|S(K/k)| \geq 2$, k is totally real, and K is totally complex.

Lemma 8.2.2. *Let d be the number of finite primes of k which ramify in K, and let A_K be the set of ambiguous ideals $A_K = \{\mathfrak{U} \in I_K : \mathfrak{U}^\tau = \mathfrak{U}\}$. Then we have:*

- $K^\circ = (K^*)^{1-\tau}$;
- *if $\epsilon \in K^\circ$, then $K(\epsilon^{1/e})/k$ is an abelian Galois extension;*
- $e\theta_{S(K/k)}(0) = 2^{n+d-2}[I_K : P_K A_K](1 - \tau)$.

Proof. For the first assertion, note that since K is totally complex and k is totally real, τ represents complex conjugation in each archimedean place of K. Thus $|a|_w = |a^\tau|_w$ for each $a \in K$ and complex place w of K, so $K^\circ \supset (K^*)^{1-\tau}$. For the reverse inclusion, note that $\mathrm{N}_{K/k}a = a^\tau a = |a|_w = 1$ (in k_v, hence in k) for $a \in K^\circ$ and w a complex place of K. Thus Hilbert's Theorem 90 gives the desired result.

The second assertion follows from Lemma 7.1.5 and the identification of τ with complex conjugation at each archimedean place of K.

The final assertion is derived in an analogous manner to the methods of section 7, as we now explain. The calculation is actually easier since $U^- = \{u : u^{1+\tau} = 1\}$ equals $\mu(K)$. Indeed, U^- is the kernel of the norm map $\mathrm{N}\colon U_K \to U_k$ and the image of N has rank equal to that of both U_K and U_k, namely $r - 1$, so U^- is a finite group. Since U^- contains $\mu(K)$, we obtain $U^- = \mu(K)$.

To summarize the argument for the proof of the final assertion, we note that for the non-trivial character χ of $G = \{1, \tau\}$,

$$L_{S(K/k)}(0, \chi) = \lim_{s \to 0} \frac{\zeta_K(s)/s^{r-1}}{\zeta_k(s)/s^{r-1}} = \frac{e_k \cdot h_K \cdot R_K}{e_K \cdot h_k \cdot R_k} = \frac{2 \cdot h_K \cdot R_K}{e_K \cdot h_k \cdot R_k}.$$

The method of proof of Lemma 7.2.1 shows that

$$\frac{R_K}{R_k} = \frac{2^{r-1}}{[U_K : \mu(K)U_k]},$$

and the method of proof of Lemma 7.2.3 applies without change to give

$$\frac{h_K}{h_k} = 2^{d-1}[I_K : P_K A_K][U_K : \mu(K)U_k].$$

Since $|S(k/k)| \geq 2$, we have $L_{S(K/k)}(0, 1_G) = 0$, so from the definition of θ we find

$$\theta_{S(K/k)}(0) = L_{S(K/k)}(0, \chi) \cdot \frac{1-\tau}{2} = \frac{2}{e_K} 2^{r+d-2}[I_K : P_K A_K] \cdot \frac{1-\tau}{2}$$

as desired. □

We are now in a position to prove $\mathrm{BS}(K/k, T)$ for general $T \supset S(K/k)$ in the relative quadratic case.

Proposition 8.2.3. *Suppose L is an everywhere unramified extension of K with L/k Galois of degree 2^{h+1}, such that $\mathrm{Gal}(L/k)$ is abelian and has exponent 2. Let H be the kernel of the Artin map $I_K \to \mathrm{Gal}(L/K)$. Then we have*

- $\frac{e_K}{2^{|T|-2}}\theta_{T,K/k}(0) \in \mathbf{Z}[G]$ *and is a BS-annihilator for I_K. In particular,* $\mathrm{BS}(K/k,T)$ *is true.*
- $\frac{e_K}{2^{|T|+h-d-2}}\theta_{T,K/k}(0) \in \mathbf{Z}[G]$ *and is a BS-annihilator for H.*

Proof. Lemma 8.2.2 implies $\frac{e_K}{2^{r+d-2}}\theta_{S(K/k)}(0) = [I_K : P_K A_K](1-\tau)$. By Corollary 6.1.4, we see that

$$\begin{aligned}
\theta_T(0) &= \theta_{S(K/k)}(0) \prod_{\mathfrak{p} \in T - S(K/k)} (1 - \sigma_{\mathfrak{p}}^{-1}) \\
&= \theta_{S(K/k)}(0) \cdot \begin{cases} 0 & \text{if there exists } \mathfrak{p} \in T \text{ such that } \mathfrak{p} \text{ splits completely} \\ (1-\tau)^{|T - S(K/k)|} & \text{otherwise} \end{cases}.
\end{aligned}$$

In the first case, $\theta_T(0) = 0$ and both assertions follow trivially. In the second case, recall that $|S(K/k)| = r + d$ and note that $(1-\tau)^2 = 2(1-\tau)$, so the above expression for $\theta_T(0)$ yields

$$\theta_T(0) = \frac{2^{r+d-2}}{e_K}[I_K : P_K A_K](1-\tau) \cdot 2^{|T|-r-d},$$

or equivalently,

$$\frac{e_K}{2^{|T|-2}}\theta_{T,K/k}(0) = [I_K : P_K A_K](1-\tau) \in \mathbf{Z}[G].$$

For notational purposes let $t = [I_K : P_K A_K]$. For any $\mathfrak{U} \in I_K$, we clearly have $\mathfrak{U}^t \in P_K A_K$ and hence (by definition of A_K) $\mathfrak{U}^{t(1-\tau)} \in P_K^{1-\tau}$. The first assertion of the proposition now follows from Lemma 8.2.2.

For the second assertion, note that

$$\frac{e_K}{2^{|T|+h-d-2}}\theta_{T,K/k}(0) = \frac{t}{2^{h-d}}(1-\tau).$$

By Proposition 7.1.4, $t/2^{h-d}$ is an integer and $\mathfrak{U}^{t/2^{h-d}} \in P_K A_K$ for all $\mathfrak{U} \in H \subset HA_K$. The result follows as above. □

8.2.2 General K/k of exponent 2

We now let K/k have Galois group G of exponent 2 and order 2^m. As usual, we may assume that $|S(K/k)| \geq 2$, k is totally real, and K is totally complex. To derive the Brumer-Stark conjecture for this case from the relative quadratic case, we will use the same method as in section 7. That is, we let $\tau \in G$ denote complex conjugation in some complex place of K lying above a real place v of k. Let K_i for $i = 1, \ldots, 2^m - 1$ be the quadratic extensions of k contained in K, ordered so that τ fixes K_i if and only if $i > 2^{m-1}$. We will prove $\mathrm{BS}(K/k, T)$ from the statements $\mathrm{BS}(K_i/k, T)$.

For each i, let $G_i = \mathrm{Gal}(K/K_i)$ and $\mathrm{N}G_i = \sum_{\sigma \in G_i} \sigma \in G$. We also write e_i for $e_{K_i} = |\mu(K_i)|$. Let $\theta = \theta_{T,K/k}(0)$ and $\theta_i = \theta_{T,K_i/k}(0)$. Finally, for each i choose $\widetilde{\theta}_i \in \mathbf{Q}[G]$ whose image under $\pi_i \colon \mathbf{Q}[G] \to \mathbf{Q}[G/G_i]$ is $\theta_i \in \mathbf{Q}[G/G_i]$.

Lemma 8.2.4. *In $\mathbf{Q}[G]$, we have*

$$2^{m-1}\theta = \sum_{i=1}^{2^{m-1}} \widetilde{\theta}_i \,\mathrm{N}G_i.$$

Proof. For $x \in \mathbf{Q}[G]$ such that $\pi_i(x) = 0 \in \mathbf{Q}[G/G_i]$, obviously $x\,\mathrm{N}G_i = 0$. Thus the right-hand side of the equation in the lemma is independent of choices of $\widetilde{\theta}_i$'s. Furthermore, Proposition 6.1.7 shows that we may take $\widetilde{\theta}_i = \theta$. By Remark 6.1.1 and Proposition 6.1.3, it therefore suffices to show

$$2^{m-1}L_{T,K/k}(0,\overline{\chi}) = \chi\left(\sum_{i=1}^{2^{m-1}} \theta\,\mathrm{N}G_i\right)$$

for all characters χ of G. Since χ is an algebra homomorphism $\mathbf{C}[G] \to \mathbf{C}$, we have

$$\chi\left(\sum_{i=1}^{2^{m-1}} \theta\,\mathrm{N}G_i\right) = \chi(\theta)\sum_{i=1}^{2^{m-1}} \chi(\mathrm{N}G_i) = L_{T,K/k}(0,\overline{\chi})\sum_{i=1}^{2^{m-1}} \chi(\mathrm{N}G_i).$$

As we noted in the proof of Theorem 7.3.2, since G has exponent 2 it follows that $\chi = 1_G$ or $\mathrm{Ker}\,\chi = G_i$ for some i. If $\chi = 1_G$ or $\mathrm{Ker}\,\chi = G_i$ for some $i > 2^{m-1}$, then χ can be viewed as a character of $\mathrm{Gal}(K_i/k)$, and $\chi(\tau) = 1$, so the real place v of k corresponding to τ splits completely in K_i. In this case,

$$L_{T,K/k}(0,\overline{\chi}) = L_{T,K_i/k}(0,\overline{\chi}) = 0$$

by Proposition A.7.2, and the desired equation follows.

If $\mathrm{Ker}\,\chi = G_i$ for some $i \leq 2^{m-1}$, then $\chi(\mathrm{N}G_i) = 2^{m-1}$ and $\chi(\mathrm{N}G_j) = 0$ and $j \neq i$. The result follows. ☐

Lemma 8.2.5. *Suppose that $M \in \mathbf{Z}$ such that for each $i \leq 2^{m-1}$, $\frac{e_i}{M}\theta_i$ is a BS-annihilator for $\mathrm{N}_{K/K_i} I_K \subset I_{K_i}$. Then $\frac{2^{m-1}e}{N}\theta \in \mathbf{Z}[G]$ and is a BS-annihilator for I_K.*

Proof. Since $\frac{e_i}{M}\theta_i \in \mathbf{Z}[G/G_i]$, we may choose $\widetilde{\theta}_i \in \mathbf{Q}[G]$ lifting θ_i so that $\frac{e_i}{M}\widetilde{\theta}_i \in \mathbf{Z}[G/G_i]$. By Lemma 8.2.4, we have

$$\frac{2^{m-1}e}{M}\theta = \sum_{i=1}^{2^{m-1}} \frac{e}{e_i}\frac{e_i}{M}\widetilde{\theta}_i\,\mathrm{N}G_i \in \mathbf{Z}[G].$$

From Proposition 6.1.7, we see that if $\mathfrak{a} \in I_{K_i}$ and $\pi_i \colon \mathbf{C}[G] \to \mathbf{C}[G/G_i]$ is the projection map, then $\mathfrak{a}^{n_i\theta_i}\mathcal{O}_K = \mathfrak{a}^{n_i\pi\theta}\mathcal{O}_K = (\mathfrak{a}\mathcal{O}_K)^{n_i\theta}$ for any $n_i \in \mathbf{Q}$ such that $n_i\theta \in \mathbf{Z}[G]$. Hence for any $\mathfrak{U} \in I_K$ we have

$$\begin{aligned}
\mathfrak{U}^{\frac{2^{m-1}e}{M}\theta} &= \prod_{i=1}^{2^{m-1}} \mathfrak{U}^{\frac{e}{M}\theta_i\,\mathrm{N}G_i} \\
&= \prod_{i=1}^{2^{m-1}} ((\mathrm{N}_{K/K_i}\mathfrak{U})\mathcal{O}_K)^{\frac{e}{M}\theta_i} \\
&= \prod_{i=1}^{2^{m-1}} \left((\mathrm{N}_{K/K_i}\mathfrak{U})^{\frac{e_i}{M}\theta_i}\right)^{e/e_i} \mathcal{O}_K \\
&= \prod_{i=1}^{2^{m-1}} (\epsilon_i)^{e/e_i} \\
&= (\epsilon)
\end{aligned}$$

where $\epsilon = \prod \epsilon_i^{e/e_i}$ and $\epsilon_i \in K_i^\circ$ with $K_i(\epsilon_i^{1/e_i})/k$ abelian. Thus $\epsilon \in K^\circ$ and $\prod \epsilon_i^{1/e_i}$ is an eth root of ϵ. Since K is the compositum of the K_i, we see that $K(\epsilon^{1/e})$ is contained in the compositum of the $K_i(\epsilon^{1/e_i})$ and is thus an abelian extension of k. $\square$

Theorem 8.2.6 (Sands). *Assume that $k \neq \mathbf{Q}$ (so $r \geq 2$). Let K/k be an abelian extension with $\mathrm{Gal}(K/k)$ of exponent 2 and order 2^m, and let $T = S(K/k)$. If (1) $|T| > m+1$ or (2) K/k is tame, then $\mathrm{BS}(K/k, T)$ is true. In case (1) let $N = 2^{|T|-m-1}$ and in case (2) let $N = 2^{r-2}$. Then $\frac{e}{N}\theta_T(0)$ is a BS-annihilator for I_K.*

Proof. Without loss of generality, $|T| \geq 2$. Proposition 8.2.3 shows that $\frac{e_i}{2^{|T|-2}}\theta_i$ is a BS-annihilator for I_{K_i}. Therefore, Lemma 8.2.5 shows that $\frac{e}{2^{|T|-m-1}}\theta$ is a BS-annihilator for I_K. This gives the result in case (1).

In case (2), the Lemma 8.2.5 implies that it suffices to show that for each $i \leq 2^{m-1}$, $\frac{e_i}{2^{m+r-3}}\theta_i \in \mathbf{Z}[G/G_i]$ and is a BS-annihilator for $\mathrm{N}_{K/K_i}I_K$. Let F be a fixed K_i and write $\theta_F = \theta_i$ and $e_F = e_i$; we may assume that F is totally complex, or else the result is trivial. As in Lemma 7.3.1 we consider the subfield L of K fixed by the inertia groups of the $s = |S| - |S(F/k)|$ primes of k which ramify in K but not F. As in that lemma, we find that L/F is unramified and $[L : F] \geq 2^{m-1-s}$. Proposition 8.2.3 now applies to show that $(e_F/2^{m+r-3})\theta_F$ is a BS-annihilator for H, the kernel of the Artin map $I_F \to \mathrm{Gal}(L/F)$. Since $H \supset N_{L/F}I_L \supset N_{K/F}I_K$, the result follows. $\square$

Since $\mathrm{BS}(K/k, T)$ is known to be true when $k = \mathbf{Q}$ as well, we have

Corollary 8.2.7. *If $G = \mathrm{Gal}(K/k)$ is of exponent 2 and K/k is unramified at primes dividing 2, then $\mathrm{BS}(K/k, T)$ is true.*

Corollary 8.2.8. $\mathrm{BS}(K/k, T)$ *is true if G is of type $(2, 2)$.*

Proof. We can assume that $k \neq \mathbf{Q}$ and $|T| > 1$ since $\mathrm{BS}(K/k, T)$ is known in these cases. Case (1) of Theorem 8.2.6 handles the present situation unless $T = |2|$ and no finite primes of k ramify in K. Case (2) of Theorem 8.2.6 handles this possibility. $\square$

9 Proof of Stark's Conjecture for rational characters

In this final chapter of the thesis, we present the difficult proof of Stark's non-abelian conjecture for characters which assume only rational values. The focus so far has been the existence of Stark units, so our main tool has been class field theory. In this section, we will see how algebraic methods, particularly the use of cohomology of non-abelian Galois groups, can be used in the study of Stark's conjecture. The flavor of our exposition will therefore be different from that of the previous sections.

Let χ be the character of a representation of G over $\mathbf{C}$ such that the values of χ lie in $\mathbf{Q}$. We call such a χ a *rational character* of G. This is not to be confused with the character of a $\mathbf{Q}[G]$-module (i.e. a representation with a rational character might not admit a realization over $\mathbf{Q}$). We show that $A(\chi, f) \in \mathbf{Q}$ for rational characters χ of G, thereby proving Stark's conjecture for this case. For example, this will prove Stark's conjecture for Galois extensions K/k with $\mathrm{Gal}(K/k) = S_n$, the symmetric group on n letters.

The tool from representation theory which we will use is that a rational character of G has an integer multiple which can be expressed as an integer linear combination of induced *trivial* characters. This result will imply that a *power* of $A(\chi, f)$ is rational. To actually show that $A(\chi, f)$ is rational, we will need to give an explicit formula for its value. This will require constructing an algebraic invariant $q(V, f) \in \mathbf{Q}$ which depends on f and on algebraic properties of a realization V of χ. Then we will prove that the ratio $B(V, f) = \frac{A(\chi,f)}{q(V,f)}$ is a real number which "behaves well" under induction and linear combinations. By proving the simplest case $B(1_G, f) = \pm 1$ — which essentially boils down to the calculation we have already done for $A(1_G, f)$ in Proposition 3.7.4 — we will have that $B(\chi, f) = \pm 1$ for general rational characters, completing the proof. Unfortunately, this description of the proof is not entirely accurate, because we will not be able to define the invariant $q(V, f)$ for general representations V. We can only define it when V is of the form $\mathbf{C}M$, where M is a G-module that is $\mathbf{Z}$-free of finite type. To deduce the result for general rational characters χ from these specific characters, we will use a theorem on maximal orders in central division algebras over number fields.

By the analysis of the behavior of A under a change of f (in section 3.6), we may assume that f arises from an injection of G-modules $X \hookrightarrow U$. For the remainder of this section, we write f for both the injection $X \hookrightarrow U$ and the induced $\mathbf{Q}[G]$-module isomorphism $\mathbf{Q}X \to \mathbf{Q}U$.

9.1 A decomposition theorem for rational characters

In this section we present a decomposition for rational characters as a $\mathbf{Q}$-linear combination of induced trivial representations. As a corollary, we will find that $A(\chi, f)$ is an mth root of a rational number, for some $m \in \mathbf{Z}$.

Theorem 9.1.1. *Let G be a finite group and let χ be a virtual character of G which assumes only rational values. Then there exists a positive integer m and a $\mathbf{Z}$-linear combination*

$$\sum_H n_H \operatorname{Ind}_H^G 1_H = m\chi,$$

where H ranges over the subgroups of G.

Proof. Let n be the exponent of the group G and let $\rho\colon G \to \mathrm{GL}(V)$ be any finite-dimensional complex representation of G. Since the eigenvalues of the automorphism $\rho(g)$ are nth roots of unity, we see that $\chi_V = \mathrm{Tr} \circ \rho$ assumes values in $\mathbf{Q}(\zeta_n)$ for all characters χ_V of G, where ζ_n is a primitive nth root of unity. Let a be an integer relatively prime to n and write σ_a for the element of $\mathrm{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q})$ such that $\sigma_a(\zeta) = \zeta^a$ for nth roots of unity $\zeta = \zeta_n^i \in \mathbf{Q}(\zeta_n)$. Since χ assumes only rational values, we obtain

$$\chi(g^a) = \chi(g)^{\sigma_a} = \chi(g).$$

Thus, χ lies in the subspace of the space C of central functions $G \to \mathbf{C}$ defined by

$$\begin{aligned} E &= \{\theta \in C : \theta(g) = \theta(g^a) \text{ for all } g \in G \text{ and } a \text{ relatively prime to } n\} \\ &= \{\theta \in C : \theta \text{ is constant on the generators of each cyclic subgroup of } G\}. \end{aligned}$$

The characters of the form $\mathrm{Ind}_H^G 1_H$ lie in E, since

$$\mathrm{Ind}_H^G 1_H(g) = \sum_{\substack{l \in G \\ lgl^{-1} \in H}} 1_H(g) = |\{l \in G : lgl^{-1} \in H\}|,$$

and $lgl^{-1} \in H$ if and only if $lg^al^{-1} \in H$ for a relatively prime to n. In fact, the elements $\mathrm{Ind}_H^G 1_H$ span the inner product space $E \subset C$. For if not, there exists a non-zero $\theta \in E$ orthogonal to all the $\mathrm{Ind}_H^G 1_H$. Suppose this is the case, and let $g \in G$ be an element of minimal order in G such that $\theta(g) \neq 0$. Then if H is the cyclic subgroup of G generated by g and φ denotes the Euler φ-function,

$$0 = \langle \theta, \mathrm{Ind}_H^G 1_H \rangle_G = \langle \theta|_H, 1_H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \theta(h) = \frac{1}{|H|} \sum_{h \text{ gen } H} \theta(h) = \theta(g) \frac{\varphi(|H|)}{|H|},$$

since any $h \in H$ which does not generate H has order less than that of g. Thus $\theta(g) = 0$, contradicting our assumption, and the $\mathrm{Ind}_H^G 1_H$ span E. We can therefore write

$$\chi = \sum_{H \subset G} \alpha_H \, \mathrm{Ind}_H^G 1_H$$

for suitable $\alpha_H \in \mathbf{C}$. Letting $p \colon \mathbf{C} \to \mathbf{Q}$ be any $\mathbf{Q}$-linear projection of the $\mathbf{Q}$-vector space $\mathbf{C}$ onto its subspace $\mathbf{Q}$, we see that the α_H can be replaced by $p(\alpha_H) \in \mathbf{Q}$ in the equation above, since χ and $\mathrm{Ind}_H^G 1_H$ assume rational values. Multiplying by a common denominator of the $p(\alpha_H)$, we achieve the desired result. $\square$

Since we have proven that $A(1_G, f) \in \mathbf{Q}$ and that A behaves well under linear combinations and induction (Propositions 3.7.4 and 3.7.1), we immediately obtain the following corollary.

Corollary 9.1.2 (Stark). *If χ assumes rational values, then $A(\chi, f)^m \in \mathbf{Q}$ for some positive integer m.*

9.2 Definition of the invariant $q(M, f)$

Let $\mathcal{M}_G$ be the category of G-modules which are $\mathbf{Z}$-free of finite type. Our goal is to give an explicit formula for $A(\chi, f)$ when χ is the character of a representation $\mathbf{C}M$, with $M \in \mathcal{M}_G$. We will deduce our desired result for general rational characters χ from this formula. In this subsection, we define the algebraic invariant $q(M, f)$ which will turn out to be $\pm A(\chi, f)$ when χ is the character of a representation $\mathbf{C}M$ for $M \in \mathcal{M}_G$. There are two difficult steps in proving $A(\chi, f) = \pm q(M, f)$. The first is demonstrating that $q(M, f)$ depends only on the representation $\mathbf{C}M$ and not the actual module M. The second is relating the Tate cohomology of U and X, as these arise in the computation of the invariant $q(M, f)$.

We first give precise statements for these "difficult steps" and then use these assertions to prove $A(\chi, f) = \pm q(M, f)$ for $M \in \mathcal{M}_G$. In the next section, we will show how to deduce $A(\chi, f) \in \mathbf{Q}$ for characters $\mathbf{Q}$-valued χ of arbitrary representations using this result. We then go back and prove the two "difficult steps."

Definition 9.2.1. If φ is any homomorphism of abelian groups with finite kernel and cokernel, the *Herbrand quotient* is defined to be

$$q(\varphi) = \frac{|\operatorname{Coker}\varphi|}{|\operatorname{Ker}\varphi|}.$$

If $f\colon X \to U$ is an injection of $\mathbf{Z}[G]$-modules, we define

$$f_M\colon \operatorname{Hom}(M, X)_G \to \operatorname{Hom}(M, U)^G$$

to be the composition of the norm

$$\widetilde{\mathrm{N}G}\colon \operatorname{Hom}(M, X)_G \to \operatorname{Hom}(M, X)^G$$

with the map $\operatorname{Hom}(M, X)^G \to \operatorname{Hom}(M, U)^G$ induced by f. The $\mathbf{Q}[G]$-module map obtained from f_M by $\mathbf{Q}$-linearizing is an isomorphism, so f_M has finite kernel and cokernel. The invariant $q(M, f)$ discussed above is defined to be the rational number $q(f_M)$.

We now state two theorems whose proofs we will deter until section 9.5.

Definition 9.2.2. A G-module A is said to be *cohomologically trivial* if $\widehat{H}^r(H, A) = 0$ for all $r \in \mathbf{Z}$ and all subgroups $H \subset G$.

Theorem 9.2.3. *If S contains the ramified places of K/k and the class number $h_{K,S_K} = 1$, then there exist two cohomologically trivial G-modules of finite type over $\mathbf{Z}$, A and B, which fit into an exact sequence*

$$0 \longrightarrow U \longrightarrow A \longrightarrow B \longrightarrow X \longrightarrow 0.$$

Theorem 9.2.4. *If $M, M' \in \mathcal{M}_G$ and $\mathbf{C}M \cong \mathbf{C}M'$ as $\mathbf{C}[G]$-modules, then $q(M, f) = q(M', f)$.*

These two theorems are needed in the proof of the following result, which is where the property of rational characters from Theorem 9.1.1 is used.

Theorem 9.2.5. *Suppose $M \in \mathcal{M}_G$. If S satisfies the conditions of Theorem 9.2.3, then*

$$A(\mathbf{C}M, f) = \pm q(M, f).$$

Proof. For $M \in \mathcal{M}_G$, define

$$B(M) = \frac{A(\mathbf{C}M, f)}{q(M, f)}.$$

Theorem 9.2.4 shows that $B(M)$ depends only on the representation $W = \mathbf{C}M$. We want to show $B(M) = \pm 1$ for $M \in \mathcal{M}_G$.

The character of W takes on only rational values, since $\mathbf{C}M = \mathbf{C} \otimes_{\mathbf{Q}} \mathbf{Q}M$ is realizable over $\mathbf{Q}$. Thus, Theorem 9.1.1 implies that we can write

$$mW = \sum_{H \subset G} n_H \operatorname{Ind}_H^G \mathbf{C}$$

for $n_H \in \mathbf{Z}$ and some positive $m \in \mathbf{Z}$. Since $B(-)$ behaves multiplicatively under direct sums by Propositions 3.7.1 and A.10.5, and B is real-valued (Proposition 3.7.1), we are reduced to showing that

$$B(\operatorname{Ind}_H^G M) = B(M) \tag{29}$$

for $M \in \mathcal{M}_H$ and

$$B(\mathbf{Z}) = \pm 1. \tag{30}$$

The validity of (29) is a consequence of the corresponding equality for A (see 3.7.1) and the equality $q(M, f) = q(\operatorname{Ind}_H^G M, f)$, which follows from the fact that the vertical arrows in the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(\operatorname{Ind}_H^G M, X)_G \arrow[r, "\widetilde{\mathrm{N}G}"] \arrow[d] & \operatorname{Hom}(\operatorname{Ind}_H^G M, X)^G \arrow[r, "f"] \arrow[d] & \operatorname{Hom}(\operatorname{Ind}_H^G M, U)^G \arrow[d] \\
\operatorname{Hom}(M, X)_H \arrow[r, "\widetilde{\mathrm{N}G}"] & \operatorname{Hom}(M, X)^H \arrow[r, "f"] & \operatorname{Hom}(M, U)^H
\end{tikzcd}$$

are obviously isomorphisms, so $q(f_{\operatorname{Ind}_H^G M}) = q(f_M)$.

To show $B(\mathbf{Z}) = \pm 1$, we recall the explicit calculation from Proposition 3.7.4:

$$A(1_G, f) = \pm \frac{[U_k : f(X_k)]}{h_{k,S}} \in \mathbf{Q}.$$

The function $f_{\mathbf{Z}}$ is the composite of the maps

$$\operatorname{Hom}(\mathbf{Z}, X)_G = X_G \xrightarrow{\widetilde{\mathrm{N}G}} \operatorname{Hom}(\mathbf{Z}, X)^G = X^G \xrightarrow{\operatorname{Hom}(\mathbf{Z}, f)} \operatorname{Hom}(\mathbf{Z}, U)^G = U^G = U_k.$$

With X_k embedded in X as in 3.2.1, recall that $\mathrm{N}G \cdot X = X_k$, so $\operatorname{Coker} f_{\mathbf{Z}} = U_k/f(X_k)$. Also, $\operatorname{Ker} f_{\mathbf{Z}} = \operatorname{Ker} \mathrm{N}G = \widehat{H}^{-1}(G, X)$ since f is injective. By decomposing the complex in Theorem 9.2.3 into a "composite" of short exact sequences, the long exact sequences in Tate cohomology give $\widehat{H}^{-1}(G, X) \cong \widehat{H}^1(G, U)$ (see A.9.1). Thus,

$$B(\mathbf{Z}) = \pm \frac{|\widehat{H}^1(G, U)|}{h_{k,S}}.$$

It remains to show that $\widehat{H}^1(G, U)$ has size $h_{k,S}$.

Using that $h_{K,S_K} = 1$, we have an exact sequence

$$0 \longrightarrow U \longrightarrow K^* \longrightarrow I_{K,S_K} \longrightarrow 0,$$

from which we get the long exact sequence

$$0 \longrightarrow U_k \longrightarrow k^* \longrightarrow I_{K,S_K}^G \longrightarrow H^1(G, U) \longrightarrow H^1(G, K^*) = 0.$$

If ideals of $I_{k,S}$ are identified with their extensions to I_{K,S_K}, then $I_{K,S_K}^G = I_{k,S}$, since S contains the primes of k ramifying in K. As $\widehat{H}^1(G, U) = H^1(G, U)$ we now conclude $|\widehat{H}^1(G, U)| = [I_{k,S} : k^*] = h_{k,S}$, completing the proof. □

9.3 Proof of Stark's Conjecture for rational characters

We retain the assumption that our $\mathbf{Q}[G]$-module isomorphism $f\colon \mathbf{Q}X \cong \mathbf{Q}U$ comes from an injection of G-modules $f\colon X \to U$. Let θ be an arbitrary irreducible character of G. Let $\psi = \operatorname{Tr}_{\mathbf{Q}(\theta)/\mathbf{Q}} \theta$ and $\Gamma = \operatorname{Gal}(\mathbf{Q}(\theta)/\mathbf{Q})$. Stark's conjecture implies that

$$A(\psi, f) = \prod_{\sigma \in \Gamma} A(\theta^\sigma, f) = \prod_{\sigma \in \Gamma} A(\theta, f)^\sigma = \mathrm{N}_{\mathbf{Q}(\theta)/\mathbf{Q}} A(\theta, f).$$

Chinburg has observed that it may be possible to prove the following special case of this consequence of Stark's conjecture.

Conjecture 9.3.1 (Chinburg). *With the notation as above, $A(\psi, f)$ is the norm in $\mathbf{Q}$ of an element of $\mathbf{Q}(\theta)$.*

Only an approximation to this conjecture is currently known, but we will see that this is sufficient to deduce Stark's conjecture for rational characters.

Theorem 9.3.2. *Suppose that S satisfies the conditions of Theorem 9.2.3 and that f is as above. Let θ be the character of an irreducible representation of G over $\mathbf{C}$ and define $\psi = \operatorname{Tr}_{\mathbf{Q}(\theta)/\mathbf{Q}} \theta$. Then there exists a fractional ideal $\mathfrak{U}$ of $\mathbf{Q}(\theta)$ such that*

$$A(\psi, f) = \pm \operatorname{N}_{\mathbf{Q}(\theta)/\mathbf{Q}} \mathfrak{U}.$$

Proof. Since θ is irreducible over $\mathbf{C}$ we can apply Lemma A.12.4 with $F = \mathbf{Q}$, so there is a positive integer m such that $\varphi = m\psi$ is the character of an irreducible representation W of G over $\mathbf{Q}$ and $D = \operatorname{End}_{\mathbf{Q}[G]} W$ is a division algebra with center $E \cong \mathbf{Q}(\theta)$ and $[D : E] = m^2$. Since ψ takes on real values, $A(\psi, f)$ is a real number, by Proposition 3.7.1. Again using Proposition 3.7.1, $A(\psi, f)^m = A(\varphi, f)$, so it remains to show that there exists a fractional ideal $\mathfrak{U}$ of $\mathbf{Q}(\theta)$ such that $A(\varphi, f) = \pm(\operatorname{N}_{E/\mathbf{Q}} \mathfrak{U})^m$.

Let Λ be a maximal $\mathcal{O}_E$-order of the algebra D. Note that D acts on the right on W and G acts on the left. Let M_0 be any $\mathbf{Z}$-lattice in W and let $M = \mathbf{Z}[G] \cdot M_0 \cdot \Lambda$. Then M is a lattice in W which is stable under the action of G and the action of Λ, so $\mathbf{C}W \cong \mathbf{C}M$. Thus, the $\mathbf{C}[G]$-modules $\mathbf{C}W$ and $\mathbf{C}M$ have the same character φ, so we conclude from Theorem 9.2.5 that

$$A(\varphi, f) = \pm \frac{|\operatorname{Coker} f_M|}{|\operatorname{Ker} f_M|}.$$

Since $\operatorname{Coker} f_M$ and $\operatorname{Ker} f_M$ are Λ-modules of finite length, and the result follows from Theorem 9.3.3 below with $R = \mathcal{O}_E$, $K = E$, and $A = D$. □

Theorem 9.3.3. *Let R be a Dedekind domain with quotient field K. Let Λ be a maximal R-order in a finite-dimensional central simple K-algebra A. Let T be a simple Λ-module and let $\mathfrak{p}$ be a nonzero prime ideal of R which annihilates T (such $\mathfrak{p}$ obviously exists). If $\dim_K A = m^2$ and $R/\mathfrak{p}$ is finite, then $|T| = |R/\mathfrak{p}|^m = (\operatorname{N}_{K/\mathbf{Q}}\mathfrak{p})^m$.*

Proof. See [21], Theorem 7.1. □

We can now obtain the desired result concerning Stark's conjecture, via the special case in Theorem 9.3.2:

Corollary 9.3.4 (Tate). *If χ is a character of G with values in $\mathbf{Q}$, then Stark's conjecture is true for χ.*

Proof. By Proposition 3.7.2, we may assume that the conditions of Theorem 9.2.3 hold. Write χ uniquely as a sum of distinct irreducible characters of G over $\mathbf{C}$,

$$\chi = \sum n_\theta \theta,$$

with positive integers n_θ. For a given θ and $\sigma \in \operatorname{Gal}(\mathbf{Q}(\theta)/\mathbf{Q})$, θ^σ is an irreducible character of G over $\mathbf{C}$. The fact that χ assumes only rational values implies that $n_\theta = n_{\theta^\sigma}$. For $\psi_\theta = \operatorname{Tr}_{\mathbf{Q}(\theta)/\mathbf{Q}} \theta$, we deduce

$$\chi = \sum n_\theta \psi_\theta,$$

where the sum is taken over the distinct characters ψ_θ. Thus, χ is a linear combination with positive integer coefficients of characters of the type ψ described in Theorem 9.3.2. By Proposition 3.7.1 we get $A(\chi, f) \in \mathbf{Q}$, so Stark's conjecture follows for χ. □

9.4 Facts from group cohomology

We now introduce the cohomological background which is needed prove Theorems 9.2.3 and 9.2.4. Throughout this discussion, G denotes a *finite* group.

Theorem 9.4.1 (Nakamaya). *For a G-module A of finite type over $\mathbf{Z}$, the following are equivalent:*

- *A is $\mathbf{Z}$-free and cohomologically trivial.*
- *A is $\mathbf{Z}[G]$-projective.*

Proof. See [19, IX.V, Theorems 7 and 8]. □

Throughout our work, we will only be considering G-modules of finite type. Since G is finite, the notion of finite type is the same over $\mathbf{Z}[G]$ and over $\mathbf{Z}$.

Theorem 9.4.2 (Swan). *Let A be a projective $\mathbf{Z}[G]$-module of finite type and let m be a non-zero integer. There exists an integer r and an ideal $\mathfrak{U}$ in $\mathbf{Z}[G]$ with finite index relatively prime to m such that $A \cong \mathbf{Z}[G]^r \oplus \mathfrak{U}$ as $\mathbf{Z}[G]$-modules.*

Proof. This is a special case of [20], Theorem A for the Dedekind domain $\mathbf{Z}$. □

To prove Theorem 9.2.3, we begin by introducing some natural isomorphisms which will be used in the proof.

Proposition 9.4.3. *Let H be a subgroup of G. There is a unique isomorphism of δ-functors on the category of H-modules*

$$\widehat{H}^*(G, \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} A) \xrightarrow{\text{shap}} \widehat{H}^*(H, A) \tag{31}$$

which coincides with the inverse of the isomorphism $A^H \cong (\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} A)^G$ given by $a \mapsto 1 \otimes a$ for $r = 0$. The action of G on $\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} A$ is by left multiplication on $\mathbf{Z}[G]$.

Proof. (Sketch.) Recall that induced and coinduced modules are cohomologically trivial for finite groups. Using this, one shows the left side of (31) is erasable for $* \geq 0$ and coerasable for $* \leq 0$, so Grothendieck's method of universal δ-functors can be used. □

Proposition 9.4.4. *Let H be a subgroup of G. There is a unique isomorphism of δ-functors on the category of G-modules*

$$\widehat{H}^*(G, \mathrm{Hom}(\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} \mathbf{Z}, A)) \xrightarrow{\text{groth}} \widehat{H}^*(H, A)$$

which coincides with the isomorphism $\mathrm{Hom}_G(\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} \mathbf{Z}, A) \cong A^H$ given by $\varphi \mapsto \varphi(1 \otimes 1)$. The action of G on $\mathrm{Hom}(\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} \mathbf{Z}, A)$ is the standard action $(g\varphi)(x) = g\varphi(g^{-1}x)$.

Proof. The method is the same as the proof of Proposition 9.4.3. □

Definition 9.4.5. Let $(\underline{A})$ and $(\underline{B})$ be two short exact sequences of G-modules. We define

$$\mathrm{Hom}((\underline{A}), (\underline{B}))$$

to be the G-module consisting of all $\mathbf{Z}$-homomorphisms between the complexes $(\underline{A})$ and $(\underline{B})$; i.e. all commutative diagrams of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \xrightarrow{\pi_A} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \xrightarrow{\pi_B} & B_3 & \longrightarrow & 0 \end{array}$$

The G-module structure on $\mathrm{Hom}((\underline{A}),(\underline{B}))$ is defined in the obvious way.

Proposition 9.4.6. *Let the notation be as in* 9.4.5*, and assume that A_2 is $\mathbf{Z}$-free. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Hom}((\underline{A}),(\underline{B})) \longrightarrow \mathrm{Hom}(A_2,B_2)\times\mathrm{Hom}(A_3,B_3) \xrightarrow{\pi_B-\pi_A} \mathrm{Hom}(A_2,B_3) \longrightarrow 0,$$

where $\pi_B-\pi_A\colon \mathrm{Hom}(A_2,B_2)\times\mathrm{Hom}(A_3,B_3)\to\mathrm{Hom}(A_2,B_3)$ is given by $(\varphi_2,\varphi_3)\mapsto \pi_B\circ\varphi_2-\varphi_3\circ\pi_A$.

Proof. One checks exactness at each point in the sequence. The fact that A_2 is $\mathbf{Z}$-free ensures exactness on the right. □

9.5 Proof of Theorem 9.2.3

The following results from the cohomological formulation of local and global class field theory can be found in [1, VI and VII]:

Proposition 9.5.1. *Let K/k be a finite Galois extension of number fields with Galois group G. Let C_K be the idele class group of K.*

1. *The natural map $C_k\to H^0(G,C_K)$ is an isomorphism and $\widehat{H}^1(G,C_K)=0$.*
2. *$\widehat{H}^2(G,C_K)$ is cyclic of the same order as G and has a canonical generator α_2 called the global fundamental class.*
3. *Cup product with α_2 induces an isomorphism $\widehat{H}^r(G,\mathbf{Z})\to\widehat{H}^{r+2}(G,C_K)$ for all $r\in\mathbf{Z}$.*
4. *If $H\subset G$, then $\mathrm{res}\colon \widehat{H}^2(G,C_K)\to\widehat{H}^2(H,C_K)$ takes the fundamental class for the extension K/k to the fundamental class for K/K^H.*
5. *For every place w of K, $\widehat{H}^2(G_w,K_w^*)$ is cyclic of the same order as G_w and has a canonical generator $\alpha_{2,w}$ called the local fundamental class.*
6. *Cup product with $\alpha_{2,w}$ induces an isomorphism $\widehat{H}^r(G_w,\mathbf{Z})\to\widehat{H}^{r+2}(G_w,K_w^*)$ for all $r\in\mathbf{Z}$.*
7. *The local fundamental classes are related as follows. Suppose $\sigma w=w'$ for $\sigma\in G$. Consider the maps $G_{w'}\to G_w$ and $K_w^*\to K_{w'}^*$ defined by $g\mapsto\sigma g\sigma^{-1}$ and $x\mapsto x^\sigma$. These maps induce a map $\widetilde{\sigma}\colon\widehat{H}^2(G_w,K_w^*)\to\widehat{H}^2(G_{w'},K_{w'}^*)$. We then have $\widetilde{\sigma}(\alpha_{2,w})=\alpha_{2,w'}$*
8. *When $G=G_w$, or equivalently when w is the only prime lying above v, the image of the local fundamental class $\alpha_{2,w}\in\widehat{H}^2(G_w,K_w^*)=\widehat{H}^2(G,K_w^*)$ under the map*

   $$\widehat{H}^2(G,K_w^*)\to\widehat{H}^2(G,C_K)$$

   induced by $K_w^*\to C_K$ is the global fundamental class α_2.

Property 8 of the proposition follows from the *construction* of the global fundamental class in terms of local fundamental classes. We use Proposition 9.5.1 in order to prove Theorem 9.2.3, whose statement we now recall.

Theorem 9.5.2. *If S contains the ramified places of K/k and the class number h_{K,S_K} equals 1, then there exist two cohomologically trivial G-modules of finite type, A and B, and an exact sequence of $\mathbf{Z}[G]$-modules*

$$0 \longrightarrow U \longrightarrow A \longrightarrow B \longrightarrow X \longrightarrow 0.$$

Proof. Consider the S_K ideles of K,

$$J_S = J_{K,S_K} = \prod_{w \in S_K} K_w^* \times \prod_{w \notin S_K} \mathcal{O}_w^*.$$

Since $h_{K,S_K} = 1$ and S_K contains the infinite primes, the map $\pi\colon J_S \to C_K$ is surjective and we obtain an exact sequence of $\mathbf{Z}[G]$-modules (denoted ($\underline{U}$)):

$$0 \longrightarrow U \longrightarrow J_S \xrightarrow{\ \pi\ } C_K \longrightarrow 0.$$

Recall also the exact sequence of $\mathbf{Z}[G]$-modules (denoted ($\underline{X}$)):

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\ \epsilon\ } \mathbf{Z} \longrightarrow 0,$$

where $\epsilon\colon Y \to \mathbf{Z}$ is the augmentation map.

For each $v \in S$, choose a fixed place $w \in S_K$ over v. We can write

$$Y = \bigoplus_{v \in S} \mathrm{Ind}_{G_w}^G \mathbf{Z} = \bigoplus_{v \in S} \mathbf{Z}[G] \otimes_{\mathbf{Z}[G_w]} \mathbf{Z}, \tag{32}$$

where the action of G_w on $\mathbf{Z}$ is trivial.

Note that for each v, $\prod_{w'|v} K_{w'}^* \cong \mathrm{Ind}_{G_w}^G K_w^*$ as $\mathbf{Z}[G]$-modules, so we have a decomposition

$$\prod_{w'|v} K_{w'}^* = K_w^* \times \prod_{\substack{w'|v \\ w' \neq w}} K_{w'}^*$$

as $\mathbf{Z}[G_w]$-modules. We can define a unique element

$$\alpha_{2,v} \in \widehat{H}^2(G, \mathrm{Hom}(\mathbf{Z}[G] \otimes_{\mathbf{Z}[G_w]} \mathbf{Z}, \prod_{w'|v} K_{w'}^*))$$

such that the projection of

$$\mathrm{groth}(\alpha_{2,v}) \in \widehat{H}^2(G_w, \prod_{w'|v} K_{w'}^*)$$

onto $\widehat{H}^2(G_w, K_w^*)$ is $\alpha_{2,w}$ and the projection onto

$$\widehat{H}^2(G_w, \prod_{\substack{w'|v \\ w' \neq w}} K_{w'}^*)$$

is zero. One checks from property (7) of Proposition 9.5.1 that $\alpha_{2,v}$ is independent of the choice of $w|v$. By a universal δ-functor argument, one checks the diagram

$$\begin{array}{ccc} \widehat{H}^r(G, \mathbf{Z}[G] \otimes_{\mathbf{Z}[G_w]} \mathbf{Z}) & \xrightarrow{\text{shap}} & \widehat{H}^r(G_w, \mathbf{Z}) \\ \downarrow{\scriptstyle \cup\alpha_{2,v}} & & \downarrow{\scriptstyle \cup\alpha_{2,w}} \\ \widehat{H}^{r+2}\left(G, \prod_{w'|v} K_{w'}^*\right) & \xrightarrow{\text{shap}} & \widehat{H}^{r+2}(G_w, K_w^*) \end{array} \tag{33}$$

commutes for each $v \in S$ and $r \in \mathbf{Z}$, where shap is the isomorphism from Proposition 9.4.3. The maps shap and $\cup\alpha_{2,w}$ are isomorphisms, so it follows that $\cup\alpha_{2,v}$ is an isomorphism.

Now

$$\widehat{H}^r(G_w, \mathcal{O}_w^*) = 0 \tag{34}$$

for $w \notin S_K$, since such w are unramified over k, so we easily compute

$$\widehat{H}^r(G, J_S) \cong \bigoplus_{v \in S} \widehat{H}^r(G_w, K_w^*) \tag{35}$$

by means of the isomorphism shap and the compatibility of Tate cohomology with respect to direct products. Taking the direct sum of the commutative diagrams (33) for all $v \in S$, we obtain the commutative diagram

$$\begin{array}{ccc} \widehat{H}^r(G, Y) & \xrightarrow{\oplus\,\text{shap}} & \bigoplus_{v \in S} \widehat{H}^r(G_w, \mathbf{Z}) \\ \downarrow{\scriptstyle \cup\alpha_2'} & & \downarrow{\scriptstyle \cup \bigoplus_{v\in S} \alpha_{2,w}} \\ \widehat{H}^r(G, J_S) & \xrightarrow{\oplus\,\text{shap}} & \bigoplus_{v \in S} \widehat{H}^{r+2}(G_w, K_w^*) \end{array} \tag{36}$$

where, by (32) and (35), we can view $\alpha_2' = \oplus\alpha_{2,v}$ as an element of $\widehat{H}^2(G, \operatorname{Hom}(Y, J_S))$. All the maps in (36) are isomorphisms, since they are defined as a "direct sum" of isomorphisms.

We now claim that the theory of global and local fundamental classes gives the compatibility $\epsilon(\alpha_2) = \pi(\alpha_2')$ in $\widehat{H}^2(G, \operatorname{Hom}(Y, C_K))$, where we recall $\epsilon\colon Y \to \mathbf{Z}$ and $\pi\colon J_S \to C_K$ are the natural $\mathbf{Z}[G]$-module projections. Consider the diagram

$$\begin{array}{ccc} & & \widehat{H}^2(G, C_K) \\ & & \downarrow{\scriptstyle \epsilon} \\ \widehat{H}^2(G, \operatorname{Hom}(Y, J_S)) & \xrightarrow{\pi} & \widehat{H}^2(G, \operatorname{Hom}(Y, C_K)) \\ \downarrow{\scriptstyle \text{groth}} & & \downarrow{\scriptstyle \text{groth}} \\ \prod_{v \in S} \widehat{H}^2(G_w, K_w^*) & \longrightarrow & \prod_{v \in S} \widehat{H}^2(G_w, C_K) \\ \downarrow & & \downarrow \\ \widehat{H}^2(G_w, K_w^*) & \longrightarrow & \widehat{H}^2(G_w, C_K) \end{array} \tag{37}$$

In the bottom square, the horizontal maps are induced from the canonical map $K_w^* \to C_K$, and the vertical arrows are projection onto some factor (fix a place $v \in S$). This bottom square is readily seen to be commutative. Since the maps labeled groth are isomorphisms, to show that $\pi(\alpha_2') = \epsilon(\alpha_2)$, it suffices to show that their images in the lower right corner $\widehat{H}^2(G_w, C_K)$ are the same for each $v \in S$.

A standard universal δ-functor argument shows that the composition of the vertical maps in the right column is simply the restriction $\widehat{H}^2(G, C_K) \to \widehat{H}^2(G_w, C_K)$. Thus, by statement (4) of Proposition 9.5.1, the image of α_2 under this composition is the global fundamental class for K/K^{G_w} in $\widehat{H}^2(G_w, C_K)$. By replacing k by K^{G_w} and using statement (8) of Proposition 9.5.1, the image of α_2' in $\widehat{H}^2(G_w, C_K)$ (given by the diagram (37)) is the global fundamental class for K/K^{G_w} as well. This proves our claim that $\pi(\alpha_2') = \epsilon(\alpha_2)$.

From Proposition 9.4.6, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}((\underline{X}), (\underline{U})) \longrightarrow \mathrm{Hom}(Y, J_S) \times C_K \xrightarrow{\pi-\epsilon} \mathrm{Hom}(Y, C_K) \longrightarrow 0 \tag{38}$$

since Y is $\mathbf{Z}$-free.

By (32) and Proposition 9.4.4, we have an isomorphism

$$\widehat{H}^1(G, \mathrm{Hom}(Y, C_K)) \cong \bigoplus_{v \in S} \widehat{H}^1(G_w, C_K).$$

As G_w can be viewed as the Galois group of K/K^{G_w}, this vanishes (see (1) in Proposition 9.5.1).

The long exact sequence in cohomology corresponding to the exact sequence (38) therefore gives

$$\begin{aligned} 0 \longrightarrow \widehat{H}^2(G, \mathrm{Hom}((\underline{X}), (\underline{U}))) &\longrightarrow \widehat{H}^2(G, C_K) \times \widehat{H}^2(G, \mathrm{Hom}(Y, J_S)) \\ \xrightarrow{\pi-\epsilon} \widehat{H}^2(G, \mathrm{Hom}(Y, C_K)). \end{aligned}$$

Since $\epsilon(\alpha_2) = \pi(\alpha_2')$, we conclude the existence of $\overline{\alpha}_3 \in \widehat{H}^2(G, \mathrm{Hom}((\underline{X}), (\underline{U})))$ such that

$$\overline{\alpha}_3 \mapsto (\alpha_2, \alpha_2') \in \widehat{H}^2(G, C_K) \times \widehat{H}^2(G, \mathrm{Hom}(Y, J_S)).$$

Let $\alpha_3 \in \widehat{H}^2(G, \mathrm{Hom}(X, U))$ be the image of $\overline{\alpha}_3$ under the canonical map

$$\mathrm{Hom}((\underline{X}), (\underline{U}))) \to \mathrm{Hom}(X, U).$$

We then have a diagram

$$\begin{tikzcd} \cdots \arrow[r] & \widehat{H}^r(G, X) \arrow[r] \arrow[d, "\cup\alpha_3"] & \widehat{H}^r(G, Y) \arrow[r] \arrow[d, "\cup\alpha_2'"] & \widehat{H}^r(G, \mathbf{Z}) \arrow[r] \arrow[d, "\cup\alpha_2"] & \widehat{H}^{r+1}(G, X) \arrow[r] \arrow[d, "\cup\alpha_3"] & {} \\ \cdots \arrow[r] & \widehat{H}^{r+2}(G, U) \arrow[r] & \widehat{H}^{r+2}(G, J_S) \arrow[r] & \widehat{H}^{r+2}(G, C_K) \arrow[r] & \widehat{H}^{r+3}(G, U) \arrow[r] & {} \end{tikzcd}$$

whose commutativity is not difficult to verify. Since the maps labeled $\cup\alpha_2$ and $\cup\alpha_2'$ are isomorphisms for all $r \in \mathbf{Z}$, it follows from the "five lemma" that $\cup\alpha_3 \colon \widehat{H}^r(G, X) \to \widehat{H}^{r+2}(G, U)$ is an isomorphism as well.

Now let

$$0 \longrightarrow X' \longrightarrow B' \longrightarrow B \longrightarrow X \longrightarrow 0$$

be an exact sequence of $\mathbf{Z}[G]$-modules with B and B' free $\mathbf{Z}[G]$-modules of finite type. Since all the modules of the sequence are free over $\mathbf{Z}$, the induced sequence

$$0 \longrightarrow \mathrm{Hom}(X, U) \longrightarrow \mathrm{Hom}(B, U) \longrightarrow \mathrm{Hom}(B', U) \longrightarrow \mathrm{Hom}(X', U) \longrightarrow 0$$

is exact. Furthermore, since B and B' are free $\mathbf{Z}[G]$-modules of finite type, $\mathrm{Hom}(B, U)$ and $\mathrm{Hom}(B', U)$ are induced and therefore cohomologically trivial (see [1, IV.1]). Lemma A.9.1 now shows that we have natural isomorphisms

$$\widehat{H}^r(G, X) \cong \widehat{H}^{r+2}(G, X')$$

and
$$\widehat{H}^r(G, \mathrm{Hom}(X', U)) \cong \widehat{H}^{r+2}(G, \mathrm{Hom}(X, U)) \tag{39}$$
for all $r \in \mathbf{Z}$.

Let $\alpha \in \mathrm{Hom}_G(X', U) = (\mathrm{Hom}(X', U))^G$ be a representative of the inverse image of $\alpha_3 \in \widehat{H}^2(G, \mathrm{Hom}(X, U))$ in the isomorphism (39) for $r = 0$. By the commutativity of the diagram

$$\begin{tikzcd}
\widehat{H}^{r+2}(G, X') \otimes \widehat{H}^0(G, \mathrm{Hom}(X', U)) \arrow[rr] \arrow[dr, "\cup"'] & & \widehat{H}^r(G, X) \otimes \widehat{H}^2(G, \mathrm{Hom}(X, U)) , \arrow[dl, "\cup"] \\
& \widehat{H}^{r+2}(G, U) &
\end{tikzcd}$$

we see that cup product with α gives an isomorphism $\widehat{H}^r(G, X') \to \widehat{H}^r(G, U)$ for all $r \in \mathbf{Z}$.

We are now close to deriving our desired exact sequence
$$0 \longrightarrow U \longrightarrow A \longrightarrow B \longrightarrow X \longrightarrow 0$$
with A and B cohomologically trivial. If $\alpha \colon X' \to U$ is not surjective, we may replace X' and B' by $X' \oplus L$ and $B' \oplus L$ where L is a free $\mathbf{Z}[G]$-module of finite type in the exact sequence
$$0 \longrightarrow X' \longrightarrow B' \longrightarrow B \longrightarrow X \longrightarrow 0.$$
Thus we can assume that α is surjective, and we have an exact sequence
$$0 \longrightarrow \mathrm{Ker}\,\alpha \longrightarrow X' \xrightarrow{\alpha} U \longrightarrow 0. \tag{40}$$

Since the maps $\widehat{H}^r(G, X') \to \widehat{H}^r(G, U)$ are isomorphisms, the long exact sequence in cohomology corresponding to (40) implies that $\widehat{H}^r(G, \mathrm{Ker}\,\alpha) = 0$ for all $r \in \mathbf{Z}$. Furthermore, B and B' are cohomologically trivial since they are free $\mathbf{Z}[G]$-modules, and hence $\widehat{H}^r(G, B'/\,\mathrm{Ker}\,\alpha) = 0$ for all $r \in \mathbf{Z}$ as well.

With $A = B'/\,\mathrm{Ker}\,\alpha$ and $U \cong X'/\,\mathrm{Ker}\,\alpha$, we have the exact sequence
$$0 \longrightarrow U \longrightarrow A \longrightarrow B \longrightarrow X \longrightarrow 0$$
with $\widehat{H}^r(G, A) \cong 0$ for all $r \in \mathbf{Z}$. The above arguments are compatible with replacing G by any of its subgroups, so it is easily seen that A is cohomologically trivial. This gives the desired result. □

9.6 The category $\mathcal{M}_G$.

In this section, we will prove Theorem 9.2.4, thereby completing the proof of Stark's non-abelian conjectures for rational characters. Let $M \in \mathcal{M}_G$ so $\mathrm{Hom}(M, -)$ is an exact functor. From the exact sequence in 9.2.3, we obtain an exact sequence of $\mathbf{Z}[G]$-modules
$$0 \longrightarrow \mathrm{Hom}(M, U) \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, X) \longrightarrow 0 \tag{41}$$
where A and B are as in Theorem 9.2.3. Furthermore, $\mathrm{Hom}(M, A)$ and $\mathrm{Hom}(M, B)$ are also cohomologically trivial since M is $\mathbf{Z}$-free (see [19, IX.5, Theorem 9]). In particular, $\widehat{H}^{-1}(G, \mathrm{Hom}(M, A)) = \mathrm{Ker}\,\widetilde{\mathrm{N}G}$ and $\widehat{H}^0(G, \mathrm{Hom}(M, A)) = \mathrm{Coker}\,\widetilde{\mathrm{N}G}$ are trivial, so
$$\widetilde{\mathrm{N}G} \colon \mathrm{Hom}(M, A)_G \to \mathrm{Hom}(M, A)^G$$

is an isomorphism, and the same is true when A is replaced by B. Since taking co-invariants (resp. invariants) is a right exact (resp. left exact) functor, the exact sequence (41) yields the two exact sequences given below. The isomorphisms given by $\widetilde{\mathrm{N}G}$ allow us to connect the sequences in a commutative diagram as shown.

$$\begin{array}{ccccccccc} & & & & \mathrm{Hom}(M,A)_G & \longrightarrow & \mathrm{Hom}(M,B)_G & \longrightarrow & \mathrm{Hom}(M,X)_G & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle\widetilde{\mathrm{N}G}} & & \big\downarrow{\scriptstyle\widetilde{\mathrm{N}G}} & & & & \\ 0 & \longrightarrow & \mathrm{Hom}(M,U)^G & \longrightarrow & \mathrm{Hom}(M,A)^G & \longrightarrow & \mathrm{Hom}(M,B)^G & & & & \end{array}$$

We can therefore combine the sequences into one exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(M,U)^G &\longrightarrow \mathrm{Hom}(M,A)^G \\ \longrightarrow \mathrm{Hom}(M,B)_G &\longrightarrow \mathrm{Hom}(M,X)_G \longrightarrow 0. \end{aligned}$$

If $M' \subset M$ is a fixed $\mathbf{Z}[G]$-submodule of M, then $M' \in \mathcal{M}_G$ and we have the restriction functor

$$\varphi\colon\ \mathrm{Hom}(M,-) \to \mathrm{Hom}(M',-).$$

If C is a G-module, then we get induced maps

$$\varphi^C\colon\ \mathrm{Hom}(M,C)^G \to \mathrm{Hom}(M',C)^G, \qquad \varphi_C\colon\ \mathrm{Hom}(M,C)_G \to \mathrm{Hom}(M',C)_G.$$

These fit into the commutative diagram below, with horizontal lines exact:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Hom}(M,U)^G & \longrightarrow & \mathrm{Hom}(M,A)^G & \longrightarrow & \mathrm{Hom}(M,B)_G & \longrightarrow & \mathrm{Hom}(M,X)_G & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\varphi^U} & & \big\downarrow{\scriptstyle\varphi^A} & & \big\downarrow{\scriptstyle\varphi_B} & & \big\downarrow{\scriptstyle\varphi_X} & & \\ 0 & \longrightarrow & \mathrm{Hom}(M',U)^G & \longrightarrow & \mathrm{Hom}(M',A)^G & \longrightarrow & \mathrm{Hom}(M',B)_G & \longrightarrow & \mathrm{Hom}(M',X)_G & \longrightarrow & 0 \end{array} \tag{42}$$

Lemma 9.6.1. *Suppose that M' is a G-submodule of finite index in $M \in \mathcal{M}_G$. If C is a cohomologically trivial G-module of finite type, then*

$$q(\varphi^C) = q(\varphi_C) = [M : M']^{(\mathrm{Rank}_{\mathbf{Z}} C)/|G|}.$$

Proof. Since C is cohomologically trivial, the norm map $\widetilde{\mathrm{N}G}\colon\ \mathrm{Hom}(M,C)_G \to \mathrm{Hom}(M,C)^G$ is an isomorphism. Thus the vertical arrows in the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(M,C)_G & \xrightarrow{\ \varphi^C\ } & \mathrm{Hom}(M,C)_G \\ \big\downarrow{\scriptstyle\widetilde{\mathrm{N}G}} & & \big\downarrow{\scriptstyle\widetilde{\mathrm{N}G}} \\ \mathrm{Hom}(M',C)^G & \xrightarrow{\ \varphi_C\ } & \mathrm{Hom}(M',C)_G \end{array}$$

are isomorphisms, and we conclude that $q(\varphi^C) = q(\varphi_C)$. Both $q(\varphi_C)$ and $[M : M']^{(\mathrm{Rank}_{\mathbf{Z}} C)/|G|}$ are multiplicative under direct sums, so by Theorems 9.4.1 and 9.4.2 it suffices to prove that $q(\varphi_C) = [M : M']^{(\mathrm{Rank}_{\mathbf{Z}} C)/|G|}$ for $C = \mathfrak{U}$ an ideal of $\mathbf{Z}[G]$ of finite index relatively prime to $[M : M']$. Since $\mathbf{Z}[G]$ and $\mathfrak{U}$ are cohomologically trivial, $\mathbf{Z}[G]/\mathfrak{U}$ is also cohomologically trivial.

The norm map $\widetilde{\mathrm{N}G}$ is an isomorphism for cohomologically trivial modules, so the exact sequences resulting from taking the invariants and coinvariants of

$$0 \longrightarrow \mathrm{Hom}(M,\mathfrak{U}) \longrightarrow \mathrm{Hom}(M,\mathbf{Z}[G]) \longrightarrow \mathrm{Hom}(M,\mathbf{Z}[G]/\mathfrak{U}) \longrightarrow 0$$

can be spliced together in the commutative diagram below.

$$\begin{array}{ccccccc} & & \mathrm{Hom}(M,\mathfrak{U})_G & \longrightarrow & \mathrm{Hom}(M,\mathbf{Z}[G])_G & \longrightarrow & \mathrm{Hom}(M,\mathbf{Z}[G]/\mathfrak{U})_G & \longrightarrow & 0 \\ & & \big\downarrow \widetilde{\mathrm{N}G} & & \big\downarrow \widetilde{\mathrm{N}G} & & \big\downarrow \widetilde{\mathrm{N}G} & & \\ 0 & \longrightarrow & \mathrm{Hom}(M,\mathfrak{U})^G & \longrightarrow & \mathrm{Hom}(M,\mathbf{Z}[G])^G & \longrightarrow & \mathrm{Hom}(M,\mathbf{Z}[G]/\mathfrak{U})^G & & \end{array}$$

Doing this for M and M', we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(M,\mathfrak{U})_G & \longrightarrow & \mathrm{Hom}(M,\mathbf{Z}[G])_G & \longrightarrow & \mathrm{Hom}(M,\mathbf{Z}[G]/\mathfrak{U})_G & \longrightarrow & 0 \\ & & \big\downarrow \varphi_{\mathfrak{U}} & & \big\downarrow \varphi_{\mathbf{Z}[G]} & & \big\downarrow \varphi_{\mathbf{Z}[G]/\mathfrak{U}} & & \\ 0 & \longrightarrow & \mathrm{Hom}(M',\mathfrak{U})_G & \longrightarrow & \mathrm{Hom}(M',\mathbf{Z}[G])_G & \longrightarrow & \mathrm{Hom}(M',\mathbf{Z}[G]/\mathfrak{U})_G & \longrightarrow & 0. \end{array} \tag{43}$$

Also, the exact sequence $0 \to M \to M \to M/M' \to 0$ yields the exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}(M/M',\mathbf{Z}[G]/\mathfrak{U}) & \longrightarrow & \mathrm{Hom}(M,\mathbf{Z}[G]/\mathfrak{U}) & \longrightarrow & \mathrm{Hom}(M',\mathbf{Z}[G]/\mathfrak{U}) \\ & & & \longrightarrow & \mathrm{Ext}^1_{\mathbf{Z}}(M/M',\mathbf{Z}[G]/\mathfrak{U}) \end{array},$$

and since $[M : M']$ and $[\mathbf{Z}[G] : \mathfrak{U}]$ are relatively prime, the outer two objects in this sequence are zero. Thus, $\mathrm{Hom}(M,\mathbf{Z}[G]/\mathfrak{U}) \cong \mathrm{Hom}(M',\mathbf{Z}[G]/\mathfrak{U})$, and so $\varphi_{\mathbf{Z}[G]/\mathfrak{U}}$ is an isomorphism as well. The lemma is therefore satisfied for $\mathbf{Z}[G]/\mathfrak{U}$, since

$$q(\varphi_{\mathbf{Z}[G]/\mathfrak{U}}) = 1 = [M : M]^{(\mathrm{Rank}_{\mathbf{Z}}\, \mathbf{Z}[G]/\mathfrak{U})/|G|}.$$

By the commutative diagram (43), it remains to prove the lemma for $\mathbf{Z}[G]$ in order to prove it for $\mathfrak{U}$ (see Proposition A.10.5). We can verify the lemma explicitly in this case. Note that the projection map $\mathbf{Z}[G] \to \mathbf{Z}$ given by $\sum a_g g \mapsto a_1$ induces an isomorphism of abelian groups

$$\mathrm{Hom}(M,\mathbf{Z}[G])^G = \mathrm{Hom}_G(M,\mathbf{Z}[G]) \cong \mathrm{Hom}(M,\mathbf{Z}).$$

We have the same isomorphism for M', and under this identification the map $\varphi^{\mathbf{Z}[G]}$ becomes the forgetful map $\mathrm{Hom}(M,\mathbf{Z}) \to \mathrm{Hom}(M',\mathbf{Z})$. We thus obtain an exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}(M/M',\mathbf{Z}) & \longrightarrow & \mathrm{Hom}(M,\mathbf{Z}) & \xrightarrow{\varphi^{\mathbf{Z}[G]}} & \mathrm{Hom}(M',\mathbf{Z}) \\ & \longrightarrow & \mathrm{Ext}^1_{\mathbf{Z}}(M/M',\mathbf{Z}) & \longrightarrow & \mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Z}). \end{array}$$

As $[M : M']$ is finite, $\mathrm{Hom}(M/M',\mathbf{Z}) = 0$ and $|\,\mathrm{Ext}^1_{\mathbf{Z}}(M/M',\mathbf{Z})| = [M : M']$. Since M is $\mathbf{Z}$-free, $\mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Z}) = 0$ and we conclude

$$q\left(\varphi^{\mathbf{Z}[G]}\right) = [M : M'] = [M : M']^{(\mathrm{Rank}_{\mathbf{Z}}\, \mathbf{Z}[G])/|G|},$$

as desired. ☐

Lemma 9.6.2. *If $M \in \mathcal{M}_G$ and M' is a G-submodule of finite index in M, then*

$$q(\varphi_X) = q(\varphi_U).$$

Proof. By breaking up the rows in (42) into two pieces as in Lemma A.9.1, and applying Proposition A.10.5, we obtain $q(\varphi_X)q(\varphi^A) = q(\varphi_B)q(\varphi^U)$. Furthermore, the exact sequence

$$0 \longrightarrow U \longrightarrow A \longrightarrow B \longrightarrow X \longrightarrow 0$$

shows that $\mathrm{Rank}_{\mathbf{Z}}\, A = \mathrm{Rank}_{\mathbf{Z}}\, B$, since $\mathrm{Rank}_{\mathbf{Z}}\, U = \mathrm{Rank}_{\mathbf{Z}}\, X$. The result therefore follows from Lemma 9.6.1. ☐

We now have assembled all of the tools needed to prove Theorem 9.2.4, whose statement we now recall:

Theorem 9.6.3. *If $M, M' \in \mathcal{M}_G$ and $\mathbf{C}M \cong \mathbf{C}M'$, then $q(M, f) = q(M', f)$.*

Proof. Since the characters of the $\mathbf{C}[G]$-modules $\mathbf{C}M$ and $\mathbf{C}M'$ are equal, the same is true of the $\mathbf{Q}[G]$-modules $\mathbf{Q}M$ and $\mathbf{Q}M'$. Therefore, there is a (non-canonical) $\mathbf{Q}[G]$-module isomorphism $\psi\colon \mathbf{Q}M' \to \mathbf{Q}M$. Let $M'' = M \cap \psi(M') \subset \mathbf{Q}M$. Since $\mathbf{Q}M'' = \mathbf{Q}M = \mathbf{Q}M'$, we have reduced to the case where $\mathbf{C}M \cong \mathbf{C}M'$ is induced by an inclusion $M' \hookrightarrow M$ with finite index. This case follows from Proposition A.10.4, Lemma 9.6.2, and the commutative diagram below

$$\begin{array}{ccc} \mathrm{Hom}(M, X)_G & \xrightarrow{f_M} & \mathrm{Hom}(M, U)^G \\ \Big\downarrow{\scriptstyle \varphi_X} & & \Big\downarrow{\scriptstyle \varphi^U} \\ \mathrm{Hom}(M'', X)_G & \xrightarrow{f_{M''}} & \mathrm{Hom}(M'', U)^G. \end{array}$$

∎

A Background

In this section, we present background work to be used in the main body of the text. All of the material is standard, but we have presented it here in order to maintain consistency of notation and to add a sense of self-containment to the remainder of the work. In each subsection below, we have given references for detailed exposition of the subjects discussed.

A.1 The Dedekind zeta-function

Definition A.1.1. Let k be a number field, and let S be a finite set of primes of k containing the set S_∞ of infinite primes. Define the Dedekind zeta function for $\mathrm{Re}(s) > 1$ by

$$\zeta_k(s) = \zeta_{k,S_\infty}(s) = \sum_{\mathfrak{U}} \frac{1}{(\mathrm{N}\mathfrak{U})^s}$$

and more generally

$$\zeta_{k,S}(s) = \sum_{(\mathfrak{U},S)=1} \frac{1}{(\mathrm{N}\mathfrak{U})^s}.$$

Here the first sum runs over the nonzero integral ideals $\mathfrak{U}$ of k and the second over those ideals $\mathfrak{U}$ which are relatively prime to the finite primes in S. We have the Euler product

$$\zeta_{k,S}(s) = \prod_{\mathfrak{p} \notin S} \frac{1}{1 - (\mathrm{N}\mathfrak{p})^{-s}}$$

for $\mathrm{Re}(s) > 1$. The function ζ_k can be meromorphically continued to the entire complex plane and satisfies a functional equation (see A.8).

A.2 Facts from representation theory

In this paper, we will assume basic facts about representations of finite groups over fields of characteristic zero (see [18]) .

Let G be a finite group and let F be a field of characteristic zero. The inner product on the space C of central functions $G \to F$ is denoted $\langle \cdot, \cdot \rangle_G$:

$$\langle \chi, \theta \rangle_G = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\theta(\sigma^{-1}).$$

The irreducible characters of G form an orthogonal basis for C with respect to this inner product (and orthonormal when F is algebraically closed).

We now state some basic constructions and isomorphisms of representations.

If V and W are two $F[G]$-modules, we can construct the tensor product $V \otimes_F W$, which has a $F[G]$-module structure with G acting on both factors. The character of the tensor product $V \otimes_F W$ is the product of the characters of V and W.

Now let H be a subgroup of G. Suppose that W is a representation of H over F with character χ. From W we can construct a representation of G called the *induced representation*:

$$\mathrm{Ind}_H^G W = F[G] \otimes_{F[H]} W,$$

where $F[G]$ acts by multiplication on the left factor. The character of $\operatorname{Ind}_H^G W$ is written $\operatorname{Ind}_H^G \chi$ and is given by

$$\operatorname{Ind}_H^G \chi(\sigma) = \frac{1}{h} \sum_{\substack{\tau \in G \\ \tau^{-1}\sigma\tau \in H}} \chi(\tau^{-1}\sigma\tau).$$

The character χ is related to its induced character $\operatorname{Ind}_H^G \chi$ by the following theorem.

Theorem A.2.1 (Frobenius Reciprocity). *Let H be a subgroup of a finite group G. If χ is a character of H and θ is a character of G, then*

$$\langle \operatorname{Ind}_H^G \chi, \theta \rangle_G = \langle \chi, \theta|_H \rangle_H.$$

If H is a normal subgroup of G, then any G/H-module has a natural G module structure induced by the projection $G \to G/H$. If χ is a character of G/H, then the character of G induced by this projection is called the *inflation* and denoted $\operatorname{Infl} \chi$.

For the next Proposition, recall the notation $\operatorname{Hom}(-,-)$ and $\operatorname{Hom}_G(-,-)$ from section 2.

Proposition A.2.2. *Let $H \subset G$ be finite groups. We have the following natural isomorphisms.*

- *Let V and M be $F[G]$-modules of finite dimension over F. Then*

$$V \otimes_F M \cong \operatorname{Hom}(V^*, M).$$

- *Let W be an $F[H]$-module and M an $F[G]$-module. Then*

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G W, M) \cong \operatorname{Hom}_H(W, M).$$

- *Suppose that H is normal in G. Let V be an $F[G/H]$-module and M an $F[G]$-module. Then*

$$\operatorname{Hom}_G(\operatorname{Infl} V, M) \cong \operatorname{Hom}_{G/H}(V, M^H),$$

  where M^H represents the H-invariants of the G-module M.

In the study of Artin L-functions, one can often reduce questions about general characters to the case of 1-dimensional characters by applying Brauer's Theorem:

Theorem A.2.3 (Brauer's Theorem). *Let G be a finite group and χ a character of G. There exist subgroups $H_i \subset G$ and 1-dimensional characters χ_i of H_i such that χ can be written*

$$\chi = \sum_i n_i \operatorname{Ind}_{H_i}^G \chi$$

with $n_i \in \mathbf{Z}$.

A.3 Global class field theory

The reader is referred to [8] and [10] for proofs of the theorems of global class field theory which we state in this section.

Let k be a number field. We define a modulus for k to be a formal product

$$\mathfrak{m} = \prod_v v^{m(v)}$$

where v ranges over the primes of k, with exponents $m(v)$ satisfying:

- $m(v)$ is a non-negative integer and $m(v) = 0$ for all but finitely many v;
- $m(v) = 0$ if v is complex and $m(v) = 0$ or 1 if v is real.

For a modulus $\mathfrak{m}$, we define $I_{k,\mathfrak{m}}$ to be the subgroup of I_k consisting of those fractional ideals relatively prime to the finite primes dividing $\mathfrak{m}$. We define $P_{k,\mathfrak{m}}$ to be the set of principal fractional ideals (α) in I_k such that

- if $\mathfrak{p}$ is a prime ideal with $m(\mathfrak{p}) > 0$, then $v_{\mathfrak{p}}(\alpha - 1) \geq m(\mathfrak{p})$;
- if v is a real prime and $m(v) = 1$, then $\alpha > 0$ in the real embedding corresponding to v.

For an element α satisfying the above two conditions, we write $\alpha \equiv 1 \pmod{^* \mathfrak{m}}$. It is clear that $P_{k,\mathfrak{m}} \subset I_{k,\mathfrak{m}}$.

Let K/k be a finite abelian extension, and let $\mathfrak{m}$ be a modulus of k divisible by all primes of k ramifying in K (including the ramified real primes). We define the *Artin map*

$$\Phi_{K/k,\mathfrak{m}} \colon I_{k,\mathfrak{m}} \to \mathrm{Gal}(K/k)$$

by sending a prime $\mathfrak{p}$ to the Frobenius element $\sigma_{\mathfrak{p}}$ and extending multiplicatively. The image $\Phi_{K/k,\mathfrak{m}}(\mathfrak{U})$ of a fractional ideal $\mathfrak{U} \in I_{k,\mathfrak{m}}$ under the Artin map is written $\sigma_{\mathfrak{U}}$ or $(\mathfrak{U}, K/k)$.

Proposition A.3.1. *The Artin map satisfies the following functorial properties:*

- *Let K/k be abelian, and let $\sigma \colon K \to K'$ be an isomorphism, not necessarily equal to the identity on k. Then*
$$(\sigma\mathfrak{U}, K'/\sigma k) = \sigma(\mathfrak{U}, K/k)\sigma^{-1}.$$
- *Let $K' \supset K \supset k$ be a bigger abelian extension. Then*
$$\mathrm{res}_K(\mathfrak{U}, K'/k) = (\mathfrak{U}, K/k),$$
where $\mathrm{res}_K \colon \mathrm{Gal}(K'/k) \to \mathrm{Gal}(K/k)$ is the canonical map.
- *Let K/k be abelian and let L/k be finite. Let $\mathfrak{U}$ be a fractional ideal of L such that if $\mathfrak{q}$ is a prime ideal of L appearing in the factorization of $\mathfrak{U}$, and $\mathfrak{q}$ lies above the prime $\mathfrak{p}$ of k, then $\mathfrak{p}$ is unramified in K. Then*
$$\mathrm{res}_K(\mathfrak{U}, KL/L) = (\mathrm{N}_{L/k}\mathfrak{U}, K/k).$$
In particular, if $K \supset L \supset k$, then
$$(\mathfrak{U}, K/L) = (\mathrm{N}_{L/K}\mathfrak{U}, K/k).$$

We now state some of the main theorems of class field theory without proof.

Theorem A.3.2 (Cebotarev Density Theorem). *Let K/k be a Galois extension with Galois group G. Let $\sigma \in G$ and suppose that σ has c conjugates in G. The primes $\mathfrak{p}$ which are unramified in K and for which there exists $\mathfrak{P}$ lying above $\mathfrak{p}$ with $\sigma_{\mathfrak{P}} = \sigma$ have a Dirichlet density, and this density is equal to $c/|G|$.*

In this paper we will only use the fact that for a given σ, there are infinitely many unramified $\mathfrak{p}$ such that $\sigma_{\mathfrak{p}} = \sigma$; we will not need the precise statement about Dirichlet density. Note in particular that the Artin map $\Phi_{K/k,\mathfrak{m}}$ is surjective.

Theorem A.3.3. *Let K/k be an abelian extension, and let $\mathfrak{m}$ be a modulus divisible by all the primes of k which ramify in K. If the exponents of the finite primes dividing $\mathfrak{m}$ are sufficiently large, then*

$$P_{k,\mathfrak{m}} \subset \operatorname{Ker} \Phi_{K/k,\mathfrak{m}} \subset I_{k,\mathfrak{m}}. \tag{44}$$

The modulus $\mathfrak{m}$ of k is said to be *admissible* for K/k if it is divisible by the primes of k ramifying in K and if it satisfies the inclusion (44).

Theorem A.3.4. *Let K/k be an abelian extension. There is a modulus $\mathfrak{f} = \mathfrak{f}(K/k)$, called the Artin conductor of the extension, such that:*

- *The primes dividing $\mathfrak{f}$ are precisely those which ramify in K.*
- *A modulus $\mathfrak{m}$ is admissible if and only if $\mathfrak{f}$ divides $\mathfrak{m}$.*

Theorem A.3.5 (Existence Theorem). *Let $\mathfrak{m}$ be a modulus of k, and let H be a subgroup of $I_{k,\mathfrak{m}}$ containing $P_{k,\mathfrak{m}}$. Then there is a unique abelian extension K of k, all of whose ramified primes divide $\mathfrak{m}$, such that the kernel of the Artin map*

$$\Phi_{K/k,\mathfrak{m}} \colon I_{k,\mathfrak{m}} \to \operatorname{Gal}(K/k)$$

is precisely H.

Theorem A.3.6. *Let K and L be abelian extensions of k. Then $K \subset L$ if and only if there is a modulus $\mathfrak{m}$, divisible by all the primes ramifying in either K or L, such that*

$$P_{k,\mathfrak{m}} \subset \operatorname{Ker}(\Phi_{L/k,\mathfrak{m}}) \subset \operatorname{Ker}(\Phi_{K/k,\mathfrak{m}}).$$

Given a number field k and a modulus $\mathfrak{m}$, the existence theorem constructs a unique abelian extension $k_{\mathfrak{m}}$ such that

$$P_{k,\mathfrak{m}} = \operatorname{Ker} \Phi_{k_{\mathfrak{m}}/k,\mathfrak{m}}.$$

The field $k_{\mathfrak{m}}$ is called the *ray class field modulo* $\mathfrak{m}$. The ray class field k_1 corresponding to the modulus $\mathfrak{m} = 1$ is called the *Hilbert class field* of k. The Hilbert class field H is the maximal everywhere unramified extension of k. The Artin map gives an isomorphism between the ideal class group $\mathrm{Cl}(\mathcal{O})$ and the Galois group of H/k:

$$\Phi_{H/k,1} \colon \ \mathrm{Cl}(\mathcal{O}) = I_{k,1}/P_{k,1} \to \operatorname{Gal}(H/k).$$

As an example of the use of these theorems, we prove the following lemma, which will be needed in proving Stark's Conjecture $\mathrm{St}(K/k, S)$ when $|S| = 2$ (Proposition 4.3.11).

Lemma A.3.7. *If K/k is abelian and unramified at every place except for possibly one archimedean place v, then it is unramified at v.*

Proof. We need only consider real v. We may assume that K is the maximal abelian extension of k unramified outside of v. Then K is the ray class field modulo the cycle $\mathfrak{f} = v$. We want to show that $K = H$, the Hilbert class field of k. The Artin map gives an isomorphism

$$\Phi_{K/k,\mathfrak{f}} \colon I_k/P_{k,\mathfrak{f}} \to \operatorname{Gal}(K/k),$$

and it suffices to show that $P_{k,\mathfrak{f}} = P_{k,1}$. It is clear that $P_{k,\mathfrak{f}} \subset P_{k,1}$. Conversely, given $\alpha \in P_{k,1}$, either $\alpha > 0$ or $-\alpha > 0$ in the embedding given by v, so $(\alpha) = (-\alpha) \in P_{k,\mathfrak{f}}$. Thus $P_{k,\mathfrak{f}} = P_{k,1}$, completing the proof. $\square$

We now prove two basic propositions which will be used to find the conductor in certain explicit examples in section 5.

Proposition A.3.8. *Suppose that K/k is a finite abelian extension with conductor $\mathfrak{f}$ and that L/k is finite. Then the conductor of KL/L divides $\mathfrak{f}\mathcal{O}_L$. Here, if $\prod_{\mathfrak{p}} \mathfrak{p}^{f(\mathfrak{p})}$ is the "finite" part of $\mathfrak{f}$, then we define*

$$\mathfrak{f}\mathcal{O}_L = \left(\prod_{\mathfrak{p}} \mathfrak{p}^{f(\mathfrak{p})}\mathcal{O}_L\right)\left(\prod_{v}\prod_{w|v} w\right),$$

where the left product runs over all finite primes $\mathfrak{p}$ of k, and the right product runs over all real primes v of k which divide $\mathfrak{f}$ and split into real primes w of K.

Proof. We may assume without loss of generality that $[KL : L] = [K : k]$, for if not, we can replace k by $K \cap L$, and $\mathfrak{f}(K/k)$ by $\mathfrak{f}(K/K \cap L)$. The Galois groups of the extensions KL/L and K/k are now canonically isomorphic. To prove the proposition, it suffices to show that the modulus $\mathfrak{f}\mathcal{O}_L$ is admissible for KL/L. By Proposition A.3.1, we have a commutative diagram

$$\begin{array}{ccc} I_{L,\mathfrak{f}\mathcal{O}_L} & \xrightarrow{\mathrm{N}_{L/k}} & I_{k,\mathfrak{f}} \\ \Big\downarrow{\scriptstyle \Phi_{KL/L,\mathfrak{f}\mathcal{O}_L}} & & \Big\downarrow{\scriptstyle \Phi_{K/k,\mathfrak{f}}} \\ \mathrm{Gal}(KL/L) & \xrightarrow{\mathrm{res}_K} & \mathrm{Gal}(K/k) \end{array}$$

with the bottom row an isomorphism. If $\alpha \in L$ with $\alpha \equiv 1 \pmod{^* \mathfrak{f}\mathcal{O}_L}$, then $\mathrm{N}_{L/k}\alpha \equiv 1 (\bmod^* \mathfrak{f})$. Therefore

$$\mathrm{N}_{L/k}P_{L,\mathfrak{f}\mathcal{O}_L} \subset P_{k,\mathfrak{f}} \subset \mathrm{Ker}\,\Phi_{K/k,\mathfrak{f}}$$

and so

$$P_{L,\mathfrak{f}\mathcal{O}_L} \subset \mathrm{Ker}\,\Phi_{KL/L,\mathfrak{f}\mathcal{O}_L}$$

as desired. □

Proposition A.3.9. *Let K_1/k and K_2/k be abelian extensions with conductors $\mathfrak{f}_1$ and $\mathfrak{f}_2$, respectively. Then K_1K_2/k has conductor dividing* $\mathrm{lcm}(\mathfrak{f}_1, \mathfrak{f}_2)$.

Proof. Since the conductor of K_1/k is $\mathfrak{f}_1$, every principal prime of k generated by an element $1 \bmod {}^* \mathfrak{f}_1$ splits completely in K_1, and likewise for K_2. Since a prime that splits completely in each of two extensions splits completely in their compositum, the conductor of K_1K_2/k divides $\mathrm{lcm}(\mathfrak{f}_1, \mathfrak{f}_2)$. □

The following proposition is used in stating an equivalence between the Brumer-Stark conjecture and the abelian Stark conjecture (8.1.7). The proof, which we omit, uses the fact that two finite extensions K and K' of k are the same if and only if the set of primes of k which split completely in K and K' are the same, up to a set of Dirichlet density zero. The proof of this fact uses the Cebotarev Density Theorem.

Proposition A.3.10. *Let K/k be an abelian extension. Let T be a set of finite primes of k containing all but finitely many of those which split completely in K. Then the elements of T generate the ideal class group* $\mathrm{Cl}(\mathcal{O}_K) = I_K/P_K$.

A.4 Statements from local class field theory

Throughout this thesis, we have chosen to apply the global theory as opposed to the local theory when possible. However, there are certain definitions and results from the local theory which we will require. In this section we state those results; the reader is referred to [19] for the proofs.

Let K/k be a Galois extension, and let χ be the character of a representation V of $G = \mathrm{Gal}(K/k)$ over $\mathbf{C}$. Let $\mathfrak{p}$ be a finite prime of k and let $\mathfrak{P}$ be a prime of K lying above $\mathfrak{p}$.

Definition A.4.1. For $i \geq 0$, the higher ramification group $G_i = G_i(\mathfrak{P})$ is defined to be the subgroup of elements $\sigma \in G_{\mathfrak{P}}$ which act trivially on the quotient $\mathcal{O}_K/\mathfrak{P}^{i+1}$. The G_i are normal subgroups of $G_{\mathfrak{P}}$, and we have

$$I_{\mathfrak{P}} = G_0 \rhd G_1 \rhd \cdots ,$$

with $G_i = \{1\}$ for large i. We define

$$f(\chi, \mathfrak{p}) = \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|} \operatorname{codim} V^{G_i}.$$

This is independent of the choice of $\mathfrak{P}$ lying above $\mathfrak{p}$. Note that $f(\chi, \mathfrak{p}) = 0$ unless $\mathfrak{p}$ is ramified in K. It is in fact true that $f(\chi, \mathfrak{p}) \in \mathbf{Z}$, but this is not obvious. Define the *Artin conductor* of χ by

$$\mathfrak{f}(\chi) = \prod_{\mathfrak{p}} \mathfrak{p}^{f(\chi, \mathfrak{p})} \in I_k.$$

For our purposes, the Artin conductor will be important because it enters into the functional equation for the function $L(s, \chi)$. The following theorem will allow us to compute the conductor in the specific situations dealt with in this thesis.

Theorem A.4.2. *Let p be the characteristic of $\mathcal{O}_k/\mathfrak{p}$. The quotient G_0/G_1 is isomorphic to a subgroup of $(\mathcal{O}_k/\mathfrak{p})^*$. The quotients G_i/G_{i+1} are direct products of cyclic groups of order p.*

As an immediate corollary we find:

Proposition A.4.3. *If $\mathfrak{P}$ is tamely ramified (that is, $|I_{\mathfrak{P}}|$ is relatively prime to p), then $I_{\mathfrak{P}}$ injects into the cyclic group $(\mathcal{O}_k/\mathfrak{p})^*$ and $f(\chi, \mathfrak{p}) = \operatorname{codim} V^{I_{\mathfrak{P}}}$.*

A.5 Finite sets of primes S

Definition A.5.1. Let S be a finite set of places of k containing the archimedean places. Define the S-integers of k by

$$\begin{aligned} \mathcal{O}_S &= \bigcap_{\mathfrak{p} \notin S} \mathcal{O}_{\mathfrak{p}} = \{x \in k : v_{\mathfrak{p}}(x) \geq 0 \text{ for } \mathfrak{p} \notin S\} \\ &= T^{-1}\mathcal{O} \text{ where } T = \{x \in \mathcal{O} : v_{\mathfrak{p}}(x) = 0 \text{ for } \mathfrak{p} \notin S\}. \end{aligned}$$

The size of the ideal class group $\mathrm{Cl}(\mathcal{O}_S) = I(\mathcal{O}_S)/P(\mathcal{O}_S)$ of the Dedekind ring $\mathcal{O}_S$ will be denoted $h_{k,S}$.

Remark A.5.2. The prime ideals $\mathfrak{p}$ of $\mathcal{O}$ not contained in S are in bijection with the prime ideals $\mathfrak{q}$ of $\mathcal{O}_S$ via $\mathfrak{p} \mapsto T^{-1}\mathfrak{p}$ and $\mathfrak{q} \mapsto \mathfrak{q} \cap \mathcal{O}$. If $\mathfrak{m}$ is the modulus representing the product of the finite primes in S, this bijection gives an isomorphism between the group of fractional ideals of $\mathcal{O}$ relatively prime to $\mathfrak{m}$ and the group of fractional ideals of $\mathcal{O}_S$:

$$\begin{array}{rcl} I_{\mathfrak{m}} & \cong & I(\mathcal{O}_S) \\ \mathfrak{U} & \longmapsto & T^{-1}\mathfrak{U}. \end{array}$$

Under this isomorphism, the subgroup of $I(\mathcal{O}_S)$ consisting of principal fractional ideals corresponds to a subgroup $P'_{\mathfrak{m}} \subset I_{\mathfrak{m}}$. The subgroup $P'_{\mathfrak{m}}$ is the set of all $\mathfrak{U} \in I_{\mathfrak{m}}$ of the form $\mathfrak{U} = \mathfrak{U}' \cdot (x)$, where $\mathfrak{U}'$ is a fractional ideal of $\mathcal{O}$ with $v_{\mathfrak{p}}(\mathfrak{U}') = 0$ for $\mathfrak{p} \notin S$, and (x) is a principal fractional ideal of $\mathcal{O}$.

Proposition A.5.3. *Let K/k be an everywhere unramified abelian extension such that all the elements of S split completely in K. Then $[K : k]$ divides $h_{k,S}$.*

Proof. Letting the notation be as in Remark A.5.2, we see that $\mathrm{Cl}(\mathcal{O}_S)$ is isomorphic to $I_{\mathfrak{m}}/P'_{\mathfrak{m}}$. Since $P_{\mathfrak{m}} \subset P'_{\mathfrak{m}}$, Theorem A.3.5 yields an abelian extension H_S/k such that the Artin map

$$\Phi_{H_S/k,\mathfrak{m}} \colon I_{\mathfrak{m}}/P'_{\mathfrak{m}} \to \mathrm{Gal}(H_S/k)$$

is an isomorphism. In particular, $[H_S : k] = h_{k,S}$, so it suffices to show $K \subset H_S$ as extensions of k. Since K/k is unramified, we may define the Artin map $\Phi_{K/k,\mathfrak{m}} \colon I_{\mathfrak{m}} \to \mathrm{Gal}(K/k)$ and it suffices to show that $P'_{\mathfrak{m}} \subset \ker(\Phi_{K/k,\mathfrak{m}})$. But since K is contained in the Hilbert class field H of k, the Artin map $\Phi_{K/k,1}$ is trivial on all principal fractional ideals of $\mathcal{O}$. Moreover, the primes in S split completely in K and hence $\Phi_{K/k,1}$ is trivial on these primes. Thus $\Phi_{K,1}$ is trivial on $P'_{\mathfrak{m}}$, so obviously $\Phi_{K/k,\mathfrak{m}}$ is trivial on $P'_{\mathfrak{m}}$, as desired. ☐

We conclude this section by defining the regulator R_S which appears in the Dirichlet class number formula.

Definition A.5.4. The unit theorem (3.3.2) implies that the maximal torsion-free quotient of $\mathcal{O}_S^*$ has rank $r = |S| - 1$. Let $\{u_1, \ldots, u_r\}$ be any set of independent units in $(\mathcal{O}_S)^{\mathrm{free}} = \mathcal{O}_S^*/(\mathcal{O}_S^*)_{\mathrm{tors}}$, and pick an arbitrary $v_0 \in S$. Define

$$R_S(u_1, \ldots, u_r) = \left| \det(\log |u_i|_v)_{\substack{1 \le i \le r \\ v \in S - \{v_0\}}} \right|.$$

The product formula shows that $R_S(\{u_i\})$ is independent of the choice of v_0. Let $\langle u_i \rangle$ be the subgroup of $(\mathcal{O}_S^*)^{\mathrm{free}}$ generated by the u_i, and let $\{\epsilon_1, \ldots \epsilon_r\}$ be independent units of $(\mathcal{O}_S^*)^{\mathrm{free}}$ such that $\langle \epsilon_i \rangle \subset \langle u_i \rangle$. One can then show from the structure theorem for finitely generated abelian groups that

$$R_S(u_1, \ldots, u_r) = R_S(\epsilon_1, \ldots, \epsilon_r) \cdot [\langle u_i \rangle : \langle \epsilon_i \rangle].$$

Therefore, we can define the *regulator* R_S as the value $R_S(\{u_i\})$ for any basis $\{u_i\}$ of $(\mathcal{O}_S^*)^{\mathrm{free}}$. The regulator is a positive real number independent of the choice of basis $\{u_i\}$.

A.6 Abelian L-functions

Notation A.6.1. Let K/k be an abelian extension, and let χ be a character of the Galois group $G = \mathrm{Gal}(K/k)$. Let S be a set of primes of k containing the infinite primes and those finite primes $\mathfrak{p}$ such that χ does not act trivially on the inertia group $I_{\mathfrak{p}}$. We can then generalize the Dedekind zeta-function by defining, for $\mathrm{Re}(s) > 1$,

$$L_{K/k,S}(s, \chi) = \sum_{(\mathfrak{U},S)=1} \frac{\chi(\sigma_{\mathfrak{U}})}{(\mathrm{N}\mathfrak{U})^s}.$$

The function $L_{K/k,S}$ can be meromorphically continued to the entire complex plane. We will later see that $L_{K/k,S}$ satisfies a functional equation.

Define the partial zeta function of K/k associated to $\sigma \in G$ as (for $\mathrm{Re}(s) > 1$):

$$\zeta_{K/k,S}(s,\sigma) = \sum_{\substack{(\mathfrak{A},S)=1 \\ \sigma_{\mathfrak{A}}=\sigma}} (\mathrm{N}\mathfrak{A})^{-s}.$$

The partial zeta-functions can be meromorphically continued to the entire complex plane and are readily seen to be related to the L-functions by

$$L_S(s,\chi) = \sum_{\sigma\in G} \chi(\sigma)\zeta_S(s,\sigma) \quad \text{and} \quad \zeta_S(s,\sigma) = \frac{1}{|G|}\sum_{\chi\in\widehat{G}} \overline{\chi}(\sigma) L_S(s,\chi).$$

The essential analytic input into the proof of Stark's Conjectures for characters χ with $r(\chi) = 0$ in section 4.1 is:

Theorem A.6.2 (Siegel). *If K/k is an abelian extension and $\sigma \in \mathrm{Gal}(K/k)$, then $\zeta_S(0,\sigma)$ is a rational number.*

Siegel's proof of this theorem can be found in [17], and an alternate proof using the explicit formulas of Shintani can be found in [16].

Proposition A.6.3. *Let K/k be an abelian extension, and let χ be a non-trivial irreducible character of $\mathrm{Gal}(K/k)$. Then $L_{K/k,S}(s,\chi)$ is analytic in a neighborhood of $s = 1$, and $L_{K/k,S}(1,\chi) \neq 0$.*

This proposition is the essential step in Dirichlet's proof of his theorem on primes in arithmetic progressions.

A.7 Non-abelian Artin L-functions

Artin was able to generalize the definition of abelian L-functions to non-abelian extensions by considering the Euler product representation for abelian L-functions:

$$L_{K/k,S}(s,\chi) = \prod_{\mathfrak{p}\notin S} \frac{1}{1-\chi(\sigma_{\mathfrak{p}})(\mathrm{N}\mathfrak{p})^{-s}}. \tag{45}$$

Recall that S contains the finite primes such that $\chi(I_{\mathfrak{p}}) \neq 1$, so the expression $\chi(\sigma_{\mathfrak{p}})$ is well defined for $\mathfrak{p} \notin S$, even if $\mathfrak{p}$ is ramified in K.

Definition A.7.1. Let K/k be an arbitrary Galois extensions, and let $G = \mathrm{Gal}(K/k)$. Let V be a representation of G over $\mathbf{C}$ with character χ, and let S be any set of primes of k containing the infinite primes. For each place $\mathfrak{p} \notin S$ we choose an arbitrary prime $\mathfrak{P}$ of K lying above $\mathfrak{p}$. The element $\sigma_{\mathfrak{P}} \in G_{\mathfrak{P}}/I_{\mathfrak{P}}$ acts on $V^{I_{\mathfrak{P}}}$. We then define, with equation (45) as motivation,

$$L_{K/k,S}(s,\chi) = \prod_{\mathfrak{p}\notin S} \det\left((1-\sigma_{\mathfrak{P}}\,\mathrm{N}\mathfrak{p}^{-s})|_{V^{I_{\mathfrak{P}}}}\right)^{-1} \tag{46}$$

for $\mathrm{Re}(s) > 1$. Since the elements $\sigma_{\mathfrak{P}}$ for the various $\mathfrak{P}$ lying above a fixed $\mathfrak{p}$ are conjugate, the determinant in equation (46) is independent of the choice of $\mathfrak{P}$.

Here are some formal properties which the L-function satisfies.

Proposition A.7.2. *Let K/k be Galois with group G, and let S be a set of primes of k containing the infinite primes. The Artin L-function satisfies:*

- *if G is abelian, the Artin L-function defined in A.7.1 is equal to the abelian L-function defined in A.6.1;*
- *if χ_1 and χ_2 are two characters of G, then*
$$L_{K/k,S}(s, \chi_1 + \chi_2) = L_{K/k,S}(s, \chi_1) \cdot L_{K/k,S}(s, \chi_2);$$
- *if H is a subgroup of G with fixed field F, and χ is a character of H, then*
$$L_{K/k,S}(s, \operatorname{Ind}_H^G \chi) = L_{K/F,S_F}(s, \chi);$$
- *if H is a normal subgroup of G with fixed field F, and χ is a character of G/H, then*
$$L_{K/k,S}(s, \operatorname{Infl} \chi) = L_{F/k,S}(s, \chi).$$

A.8 The functional equation for Artin L-functions

Let K/k be a Galois extension, and let S_∞ be the set of infinite primes of k. Let V be a representation of G over $\mathbf{C}$ with character χ. In this section, we give the functional equation for the Artin L-function
$$L(s, \chi) = L_{K/k,S_\infty}(s, \chi).$$
To state the functional equation, we need to complete the L-function by adding local factors at the archimedean places of k. We define the functions
$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$$
and
$$\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s).$$
If v is a complex place of k, then the local factor L_v is defined as
$$L_v(s, \chi) = \Gamma_{\mathbf{C}}(s).$$
If v is a real place of k, we choose a place w of K above v and define $n_+ = \dim V^{G_w}$ and $n_- = \operatorname{codim} V^{G_w}$. These constants are independent of the choice of w. We then define the local factor L_v as
$$L_v(s, \chi) = \Gamma_{\mathbf{R}}(s)^{n_+}\Gamma_{\mathbf{R}}(s+1)^{n_-}.$$

Recall the definition of the global Artin conductor
$$\mathfrak{f}(\chi) = \prod_{\mathfrak{p}} \mathfrak{p}^{f(\chi,\mathfrak{p})} \in I_k$$
from A.4.1. We can now state the functional equation for $L(s, \chi)$.

Theorem A.8.1 (Hecke, Tate, Brauer). *Let the notation be as above. Define the completed L-function*

$$\Lambda(s,\chi) = (|D_k|\,\mathrm{N}\mathfrak{f}(\chi))^{s/2} \cdot \left(\prod_{v|\infty} L_v(s,\chi)\right) \cdot L(s,\chi),$$

where D_k is the discriminant of $k/\mathbf{Q}$. Then Λ can be extended to a meromorphic function on the complex plane satisfying the functional equation

$$\Lambda(1-s,\chi) = W(\chi)\Lambda(s,\overline{\chi}),$$

where $W(\chi)$ is a complex number of absolute value 1. If $\chi = 1_G$ is the trivial character, then $W(\chi) = 1$.

The number $W(\chi)$ is called the *root number*. There are explicit formulas for calculating the root number in terms of Gauss sums, which can be found in [11]. We do not restate those formulas here, but we give an example of computing a root number in Proposition 5.2.2.

A.9 Group cohomology

In this section we define Tate cohomology, which will be the main algebraic tool used in the proof of Stark's conjectures for rational characters. Fix a finite group G.

For a G-module A, let A^G (resp. A_G) denote the invariants (resp. co-invariants) of G. The element $\mathrm{N}G = \sum_{g\in G} g \in \mathbf{Z}[G]$ induces a natural homomorphism of abelian groups

$$\widetilde{\mathrm{N}G}\colon A_G \longrightarrow A^G.$$

Given a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G-modules, we can connect the corresponding long exact sequences in homology and cohomology with the homomorphism $\widetilde{\mathrm{N}G}$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_1(G,C) \arrow[r] \arrow[d] & A_G \arrow[r] \arrow[d, "\widetilde{\mathrm{N}G}^A"] & B_G \arrow[r] \arrow[d, "\widetilde{\mathrm{N}G}^B"] & C_G \arrow[r] \arrow[d, "\widetilde{\mathrm{N}G}^C"] & 0 \arrow[d] & \\
& 0 \arrow[r] & A^G \arrow[r] & B^G \arrow[r] & C^G \arrow[r] & H^1(G,A) \arrow[r] & \cdots
\end{tikzcd}$$

From the snake lemma, we can splice the sequences together into one doubly infinite exact sequence

$$\begin{aligned}
\cdots \longrightarrow H_1(G,C) \longrightarrow \operatorname{Ker}\widetilde{\mathrm{N}G}^A \longrightarrow \operatorname{Ker}\widetilde{\mathrm{N}G}^B \longrightarrow \operatorname{Ker}\widetilde{\mathrm{N}G}^C \longrightarrow \\
\operatorname{Coker}\widetilde{\mathrm{N}G}^A \longrightarrow \operatorname{Coker}\widetilde{\mathrm{N}G}^B \longrightarrow \operatorname{Coker}\widetilde{\mathrm{N}G}^C \longrightarrow H^1(G,A) \longrightarrow \cdots.
\end{aligned}$$

We therefore define the Tate cohomology groups $\widehat{H}^r(G,A)$ for $r \in \mathbf{Z}$ as

$$\begin{cases}
\widehat{H}^r(G,A) = H^r(G,A) & \text{for } r \geq 1,\\
\widehat{H}^0(G,A) = \operatorname{Coker}\widetilde{\mathrm{N}G}, & \\
\widehat{H}^{-1}(G,A) = \operatorname{Ker}\widetilde{\mathrm{N}G}, & \\
\widehat{H}^r(G,A) = H_{-1-r}(G,A) & \text{for } r \leq -2.
\end{cases}$$

With these definitions, we correspond to each short exact sequence of G-modules a doubly infinite long exact sequence of groups with the functors $\widehat{H}^r(G,-)$. This makes $\widehat{H}^*(G,-)$ a "doubly infinite" δ-functor from G-modules to abelian groups.

Lemma A.9.1. *Consider an exact sequence of G-modules*

$$0 \longrightarrow P \xrightarrow{\varphi} A \longrightarrow B \longrightarrow Q \longrightarrow 0$$

such that $\widehat{H}^r(G,A) = 0 = \widehat{H}^r(G,B)$ for all $r \in \mathbf{Z}$. There is a natural isomorphism $\widehat{H}^r(G,Q) \cong \widehat{H}^{r+2}(G,P)$ for all $r \in \mathbf{Z}$.

Proof. The given exact sequence can be broken into the two exact sequences

$$0 \longrightarrow P \longrightarrow A \longrightarrow \operatorname{Coker}\varphi \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Coker}\varphi \longrightarrow B \longrightarrow Q \longrightarrow 0.$$

The long exact sequences in cohomology corresponding to these short exact sequences, along with the assumption $\widehat{H}^r(G,A) = 0 = \widehat{H}^r(G,B)$, give natural isomorphisms

$$\widehat{H}^r(G,Q) \cong \widehat{H}^{r+1}(G,\operatorname{Coker}\varphi) \cong \widehat{H}^{r+2}(G,P)$$

for all $r \in \mathbf{Z}$. $\square$

Proposition A.9.2. *Let A be a finite G-module. Then $\widehat{H}^r(G,A)$ has exponent dividing $|G|$ and $|A|$ for all $r \in \mathbf{Z}$.*

Proof. It is clear that $|A|$ annihilates $\widehat{H}^r(G,A)$ for all $r \in \mathbf{Z}$. Furthermore, if H is any normal subgroup of G, then the composition of the co-restriction and restriction maps

$$\widehat{H}^r(G,A) \xrightarrow{\text{res}} \widehat{H}^r(H,A) \xrightarrow{\text{cores}} \widehat{H}^r(G,A)$$

is multiplication by $[G:H]$ (see [1]). Applying this with $H = \{1\}$ gives the desired result. $\square$

A.10 The Herbrand Quotient

In this section, we define the Herbrand quotient and state some of its basic properties. The Herbrand quotient is an important algebraic invariant entering into the proof of Stark's conjecture for rational characters.

Proposition A.10.1. *Suppose we have an exact sequence*

$$0 \longrightarrow A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} A_2 \xrightarrow{f_2} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} 0$$

with A_i and B_i finite groups. Then

$$\prod_{i=1}^{n} |A_i| = \prod_{i=1}^{n} |B_i|.$$

Proof. For each $i = 1, \ldots, n$, we have

$$\frac{|A_i|}{|\operatorname{Ker} f_i|} = |\operatorname{Im} f_i| \text{ and } \frac{|B_i|}{|\operatorname{Ker} g_i|} = |\operatorname{Im} g_i|.$$

Furthermore, the exactness of the sequence gives $\operatorname{Im} f_i = \operatorname{Ker} g_i$ for $i = 1, \ldots, n$, $\operatorname{Im} g_i = \operatorname{Ker} f_{i+1}$ for $i = 1, \ldots, n-1$, and $|\operatorname{Im} g_n| = |\operatorname{Ker} f_1| = 1$. Combining these equations gives the desired result. $\square$

Proposition A.10.2. *Let A be an abelian group and φ a homomorphism of A into some other group. If B is a subgroup of finite index in A, then* $[\operatorname{Ker}\varphi\colon \operatorname{Ker}\varphi \cap B]$ *and* $[\varphi(A) : \varphi(B)]$ *are finite and*

$$[A : B] = [\operatorname{Ker}\varphi : \operatorname{Ker}\varphi \cap B] \cdot [\varphi(A) : \varphi(B)].$$

Proof. Consider the composite homomorphism

$$A \xrightarrow{\varphi} \varphi(A) \longrightarrow \varphi(A)/\varphi(B).$$

It is evidently surjective with kernel $B + \operatorname{Ker}\varphi$. Thus $[A : B + \operatorname{Ker}\varphi] = [\varphi(A) : \varphi(B)]$. But $[B + \operatorname{Ker}\varphi : B] = [\operatorname{Ker}\varphi : \operatorname{Ker}\varphi \cap B]$, so the result follows. □

Definition A.10.3. If φ is any homomorphism of abelian groups with finite kernel and cokernel, we define the Herbrand quotient

$$q(\varphi) = \frac{|\operatorname{Coker}\varphi|}{|\operatorname{Ker}\varphi|}.$$

Proposition A.10.4. *Let $\varphi_1\colon A \to B$ and $\varphi_2\colon B \to C$ be maps of abelian groups with finite kernel and cokernel. Then $\varphi_2 \circ \varphi_1$ satisfies the same condition and $q(\varphi_2 \circ \varphi_1) = q(\varphi_2)q(\varphi_1)$.*

Proof. One easily checks that

$$\begin{aligned} |\operatorname{Ker}\varphi_2 \circ \varphi_1| &\leq |\operatorname{Ker}\varphi_2| \cdot |\operatorname{Ker}\varphi_1| \\ \text{and } |\operatorname{Coker}\varphi_2 \circ \varphi_1| &\leq |\operatorname{Coker}\varphi_2| \cdot |\operatorname{Coker}\varphi_1|. \end{aligned}$$

More specifically, we have

$$\begin{aligned} |\operatorname{Ker}\varphi_2 \circ \varphi_2| &= |\varphi_1^{-1}(\operatorname{Ker}\varphi_2)| = |\operatorname{Ker}\varphi_1| \cdot |\varphi_1(A) \cap \operatorname{Ker}\varphi_2| \\ \text{and } |\operatorname{Coker}\varphi_2 \circ \varphi_1| &= [C : \varphi_2(B)][\varphi_2(B) : \varphi_2 \circ \varphi_1(A)] = [C : \varphi_2(B)]\frac{[B : \varphi_1(A)]}{[\operatorname{Ker}\varphi_2 : \operatorname{Ker}\varphi_2 \cap \varphi_1(A)]} \end{aligned}$$

by Proposition A.10.2. Combining these equations gives the desired result. □

Proposition A.10.5. *Suppose we have two exact sequences connected as in the commutative diagram below:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

If any two of the φ_i's have finite kernel and cokernel, then so does the third and $q(\varphi_2) = q(\varphi_1)q(\varphi_3)$.

Proof. The snake lemma yields the exact sequence

$$0 \to \operatorname{Ker}\varphi_1 \to \operatorname{Ker}\varphi_2 \to \operatorname{Ker}\varphi_2 \to \operatorname{Coker}\varphi_1 \to \operatorname{Coker}\varphi_2 \to \operatorname{Coker}\varphi_3 \to 0.$$

It is clear that if any two of φ_i have finite kernel and cokernel, then so does the third. Proposition A.10.1 now gives the desired result. □

A.11 Basic field theory

We now state some results from basic field theory which are needed in the proof of the equivalence of two different formulations of the non-abelian Stark conjecture.

Proposition A.11.1. *Let F be a countable subfield of an uncountable algebraically closed field C, and let $\alpha\colon F \hookrightarrow C$ be any field map. Then there is an automorphism of C taking the original embedding $F \subset C$ to α.*

Proof. Let $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ be transcendence bases for C over F and $\alpha(F)$, respectively. Since F is countable, there must exist a bijection a between I and J. The maps α and $x_i \mapsto y_{a(i)}$ give an isomorphism $\beta\colon F(\{x_i\}) \to (\alpha(F))(\{y_j\})$. We have reduced to showing that any two algebraic closures of a field are isomorphic. This follows from Zorn's Lemma and field theory. □

Remark A.11.2. Proposition A.11.1 is false without a restriction on the cardinality of F, since $\alpha(F)$ might not equal F.

Proposition A.11.3. *Let F be a subfield of an algebraically closed field C of characteristic zero. Let $G \subset \operatorname{Aut} C$ be the group of automorphisms of C fixing F. The subfield of C fixed by G is F.*

Proof. Given an element $x \in C - F$, we must produce an automorphism $\sigma \in G$ such that $x^\sigma \neq x$. If x is not algebraic over F, then there is an automorphism of $F(x)$ fixing F which sends x to $x+1$. Choosing a transcendence basis for C over $F(x)$ and using the "uniqueness" of algebraic closures, this can be extended to an automorphism of C. If x is algebraic over F, then let K be the normal closure of $F(x)$ over F. By our characteristic zero assumption and basic Galois theory, there is an automorphism $\sigma' \in \operatorname{Gal}(K/F)$ which does not fix x. Arguing as in the previous case, we can lift this to the desired $\sigma \in G$. □

A.12 Galois descent and Schur indices

In this section, we analyze questions of realizability of given characters over certain fields. We consider a field F of characteristic zero with an algebraic closure C. Let G be a finite group.

Suppose L/F is a finite Galois extension with Galois group Γ, and W is an $L[G]$-module. By a *semi-linear Γ-action* on W we mean a (left) action of Γ on W which commutes with the action of G and such that $\gamma(lw) = \gamma(l)\gamma(w)$ for $\gamma \in \Gamma$, $l \in L$, and $w \in W$. Let $\mathcal{W}$ be the category of $L[G]$-modules with a semi-linear Γ action (these form a category in an obvious way) and let $\mathcal{V}$ be the category of $F[G]$-modules. of G over F. For any $W \in \mathcal{W}$, consider the F-subspace invariant under Γ:

$$W^\Gamma = \{w \in W : \gamma(w) = w \text{ for all } \gamma \in \Gamma\}.$$

Clearly W^Γ is an $F[G]$-module. Conversely, given $V \in \mathcal{V}$, we can extend scalars to produce the $L[G]$-module $V \otimes_F L$, with G acting by its action on V on the first factor and L acting by multiplication on the second factor. Furthermore, we can give $V \otimes_F L$ a semi-linear Γ-action on by acting on the second factor.

Proposition A.12.1. *There is an equivalence of categories $\mathcal{W} \longleftrightarrow \mathcal{V}$ given by*

$$\begin{array}{rcl} W & \longmapsto & W^\Gamma \\ V \otimes_F L & \longleftarrow & V. \end{array}$$

Furthermore, this equivalence is compatible with formation of the character of a representation.

Proof. Let $V \in \mathcal{V}$ and consider the map $\varphi\colon V \to (V \otimes_F L)^\Gamma$ given by $v \mapsto v \otimes 1$. This is clearly an injective homomorphism of $F[G]$-modules. To show surjectivity, let $\{v_1, \ldots, v_m\}$ be a basis for V over F. Any element $w \in V \otimes_F L$ can be written uniquely as $\sum_{i=1}^m v_i \otimes l_i$ with $l_i \in L$, and $w \in (V \otimes_F L)^\Gamma$ implies that $\sum v_i \otimes l_i = \sum v_i \otimes \gamma(l_i)$ for all $\gamma \in \Gamma$. Hence $l_i \in L^\Gamma = F$, so that $w = (\sum l_i v_i) \otimes 1$ is in the image of φ. Thus $V \cong (V \otimes_F L)^\Gamma$ as desired.

Conversely, consider $W \in \mathcal{W}$, and define a linear map $\phi\colon W^\Gamma \otimes_F L \to W$ by $w \otimes l \mapsto lw$. This is clearly a homomorphism of semi-linear $L[G]$-modules. First we check the injectivity of this map. Let $\Gamma = \{\sigma_1, \ldots, \sigma_n\}$ and let $\{\sigma_1(x), \ldots, \sigma_n(x)\}$ be a normal basis for L over F, for some $x \in L$ (see [9]). Any $\nu \in W^\Gamma \otimes_F L$ can be written in a unique way as $\nu = \sum_{j=1}^n \nu_j \otimes \sigma_j(x)$ with $\nu_j \in W^\Gamma$. If $\phi(\nu) = 0$, then $\sum \sigma_j(x)\nu_j = 0$ in W. Applying σ_i to this equation for any $\sigma_i \in \Gamma$ we obtain the matrix equation $A\overrightarrow{\nu} = \overrightarrow{0}$, where A is the $n \times n$ matrix whose (i,j) entry is $\sigma_i\sigma_j(x)$ and $\overrightarrow{\nu} = (\nu_1, \ldots, \nu_n)^t$ is a column vector in W^n. The proof of the normal basis theorem shows that A is invertible, so in fact $\overrightarrow{\nu} = 0$, implying that $\nu = 0$.

For the surjectivity of ϕ, consider an arbitrary $w \in W$ and create the column vector $\overrightarrow{w} = (\sigma_1(w), \ldots, \sigma_n(w))^t \in W^n$. Define $\overrightarrow{\nu} = (\nu_1, \ldots, \nu_n)^t \in W^n$ by the matrix equation $A\overrightarrow{\nu} = \overrightarrow{w}$. This matrix equation is equivalent to

$$\sum_{j=1}^n \sigma\sigma_j(x)\nu_j = \sigma(w) \tag{47}$$

for all $\sigma \in \Gamma$. In particular, $\sum \sigma_j(x)\nu_j = w$.

Since A is invertible, the equation (47) admits a unique solution for each $\sigma \in \Gamma$. Applying any $\gamma \in \Gamma$ to (47) and relabeling $\tau = \gamma\sigma$ yields

$$\sum_{j=1}^n \tau\sigma_j(x)\gamma(\nu_j) = \tau(w)$$

for all $\tau \in \Gamma$. In other words, the vector $\gamma(\overrightarrow{\nu}) = (\gamma(\nu_i), \ldots, \gamma(\nu_n))^t$ satisfies the equation defining $\overrightarrow{\nu}$, so by uniqueness each ν_i lies in W^Γ and $\phi\left(\sum \nu_j \otimes \sigma_j(x)\right) = w$, demonstrating the surjectivity of ϕ. Therefore ϕ is an isomorphism between the semi-linear $L[G]$-modules $W^\Gamma \otimes_F L$ and W.

The fact that corresponding representations have equal characters follows from the fact that extending scalars does not change the trace of an endomorphism. $\square$

Corollary A.12.2. *Let W be a representation of G over L with character θ. There exists an $F[G]$-module, unique up to non-canonical isomorphism, with character $\psi = \operatorname{Tr}_{L/F}\theta$*

Proof. For $\sigma \in \Gamma$, write W^σ for the representation $W \otimes_{L,\sigma^{-1}} L$ of G over L Then it is clear that W^σ has character $\chi^{\sigma^{-1}}$, so ψ is the character of the representation

$$W' = \bigoplus_{\sigma \in \Gamma} W^\sigma.$$

By Proposition A.12.1, it suffices to demonstrate that W' has a semi-linear Γ-action. We denote by $[v_\sigma \otimes 1]_\sigma$ the element of W' whose coordinate in the W^σ-entry is $v_\sigma \otimes 1$. For $\gamma \in \Gamma$, we define $\gamma[v_\sigma \otimes 1]_\sigma = [v_{\sigma\gamma} \otimes 1]_\sigma$. Note that this gives a *left* action of Γ on W'. We check

$$\begin{aligned}
\gamma(l[v_\sigma \otimes 1]_\sigma) &= \gamma[v_\sigma \otimes l]_\sigma = \gamma[(\sigma(l)v_\sigma) \otimes 1]_\sigma \\
&= [(\sigma\gamma(l)v_{\sigma\gamma}) \otimes 1]_\sigma = [v_{\sigma\gamma} \otimes \gamma(l)]_\sigma \\
&= \gamma(l)[v_{\sigma\gamma} \otimes 1]_\sigma = \gamma(l)\gamma[v_\sigma \otimes 1]_\sigma.
\end{aligned}$$

Furthermore, the action of γ clearly commutes with that of G, so we have a semi-linear action as desired. $\square$

We now employ the theory of Schur indices (see [18]) which will enable us to demonstrate the realizability of certain representations over $\mathbf{Q}$. This will be used in reducing the proof of Stark's non-abelian conjecture for rational characters to characters of the form $\mathrm{Tr}_{L/\mathbf{Q}}\,\chi$, where χ is a character of an $L[G]$-module (section 9.3).

Let the irreducible representations of G over F be V_i with characters θ_i. Then $D_i = \mathrm{End}_{F[G]} V_i$ is a division ring, and its degree over its center E_i is a square, m_i^2. The integer m_i is called the *Schur index* of θ_i over F.

Lemma A.12.3. [18, Chapter 12.2, Proposition 35] The characters θ_i/m_i form a $\mathbf{Z}$-basis for the space of virtual characters of G over C with values in F.

Theorem A.12.4. *Let θ be the character of an irreducible representation of G over C. Then there is an irreducible representation of G over $F(\theta)$ with character $\theta' = m\theta$, where m is the Schur index of θ' over $F(\theta)$. Furthermore, $\varphi = \mathrm{Tr}_{F(\theta)/F}\,\theta'$ is the character of an irreducible representation W of G over F. Finally, $D = \mathrm{End}_{F[G]} W$ is a division algebra with center $E \cong F(\theta)$ and $[D : E] = m^2$.*

Proof. Let the irreducible representations of G over $F(\theta)$ be V_i, with character θ_i. Let D_i, E_i, and m_i be defined as above for V_i over $F(\theta)$. By Lemma A.12.3, we see that θ can be written as $\theta = \sum d_i\theta_i/m_i$, where the d_i are integers. By the irreducibility of θ over C, we obtain

$$1 = \langle\theta, \theta\rangle = \sum \frac{d_i^2}{m_i^2}\langle\theta_i, \theta_i\rangle.$$

Now

$$\langle\theta_i, \theta_i\rangle = [D_i : F(\theta)] = [D_i : E_i][E_i : F(\theta)] = m_i^2[E_i : F(\theta)].$$

Hence we obtain

$$1 = \sum d_i^2[E_i : F(\theta)].$$

Therefore, all but one of the d_i's are zero, and for exactly one i_0 we have $d_{i_0}^2 = 1$ and $E_{i_0} = F(\theta)$. We then have, for $m = m_{i_0}$, the equation $m\theta = d_{i_0}\theta_{i_0}$. Evaluating at $1 \in G$ shows $d_{i_0} > 0$, do $d_{i_0} = 1$. This proves the first part of the theorem.

Since $F(\theta_{i_0}) = F(\theta)$, we deduce in particular that θ_{i_0} has Schur index 1 over $F(\theta)$. Fix an $F(\theta)[G]$-module V with character θ_{i_0}, so V is irreducible over $F(\theta)$ and

$$Z(\mathrm{End}_{F(\theta)[G]}(V)) = F(\theta), \tag{48}$$

where $Z(-)$ denotes the center of a ring.

If $\Gamma = \mathrm{Gal}(F(\theta)/F)$, we can consider the traces $\psi = \mathrm{Tr}_{F(\theta)/F}\,\theta$ and $\varphi = \mathrm{Tr}_{F(\theta)/F}\,\theta_i$, so

$$m\psi = \varphi = \sum_{\gamma\in\Gamma}\theta_{i_0}^{\gamma}.$$

This character is realizable over F by Corollary A.12.2 (note $F(\theta_{i_0}) = F(\theta)$); that is, φ is the character of a $F[G]$-module W. We claim that W is actually an irreducible representation over F. For suppose $W = W_1 \oplus W_2$ is a decomposition of W into a direct sum of $F[G]$-submodules. We have

$$(W_1 \otimes_F F(\theta)) \oplus (W_2 \otimes_F F(\theta)) = W \otimes_F F(\theta) \cong \bigoplus_{\sigma\in\Gamma} V^{\sigma}.$$

One of the factors in the direct sum on the left, say the first, contains a subspace isomorphic to V. But since $W_1 \otimes_F F(\theta)$ is stable under the action of Γ, we see that this factor contains a copy of each V^σ. Furthermore, no $\sigma \neq 1 \in \Gamma$ can satisfy $\theta_{i_0}^\sigma = \theta_{i_0}$, or else σ fixes $F(\theta_{i_0}) = F(\theta)$. Thus, for $\sigma \neq \gamma$, $V^\sigma \ncong V^\gamma$. We conclude that $\bigoplus_{\sigma\in\Gamma} V^\sigma \cong W_1 \otimes_F F(\theta)$, so $W_2 \otimes_F F(\theta) = 0$ and hence $W_2 = 0$. Thus W is indeed an irreducible representation over F.

Let D be the division algebra $\operatorname{End}_{F[G]} W$. It remains to show that its center E is F-isomorphic to $F(\theta)$ and that $[D : E] = m^2$. To see this, we will construct an isomorphism of right $F(\theta)$-algebras

$$E \otimes_F F(\theta) \cong F(\theta) \otimes_F F(\theta). \tag{49}$$

Note that this implies that E and $F(\theta)$ have the same degree over F. Furthermore, such an isomorphism defines a map over F $E \to F(\theta)$ by the composition of maps shown below:

$$E \xrightarrow{e\mapsto e\otimes 1} E \otimes_F F(\theta) \xrightarrow{\sim} F(\theta) \otimes_F F(\theta) \xrightarrow{x\otimes y\mapsto xy} F(\theta)$$

Since this is a map of fields of the same degree over F, it is an isomorphism.

To construct (49), note that

$$\begin{aligned} E \otimes_F F(\theta) &= Z\left(D \otimes_F F(\theta)\right) \\ &= Z\left(\operatorname{End}_{F(\theta)[G]}(W \otimes_F F(\theta))\right) \\ &\cong Z\left(\operatorname{End}_{F(\theta)[G]}\left(\bigoplus_{\sigma\in\Gamma} V^\sigma\right)\right). \end{aligned}$$

We have seen that the V^σ are distinct irreducible representations of G over $F(\theta)$, so by (48) we find

$$E \otimes_F F(\theta) = \bigoplus_{\sigma\in\Gamma} Z(\operatorname{End}_{F(\theta)[G]} V^\sigma) \cong \bigoplus_{\sigma\in\Gamma} F(\theta)^\sigma, \tag{50}$$

where $F(\theta)^\sigma$ denotes the extension field $\sigma^{-1}\colon F(\theta) \to F(\theta)$ of $F(\theta)$. By Galois theory, the right side of (50) is $F(\theta)$-isomorphic to $F(\theta) \otimes_F F(\theta)$ (using the right structure). Finally,

$$[D : F] = \langle \varphi, \varphi \rangle = m^2 \langle \psi, \psi \rangle = m^2 \sum_{\sigma,\gamma\in\Gamma} \langle \theta^\sigma, \theta^\gamma \rangle.$$

For each σ, $\langle \theta^\sigma, \theta^\sigma \rangle = \langle \theta, \theta \rangle = 1$. By the same argument as applied to θ_{i_0} before, we have $\theta^\sigma \neq \theta^\gamma$ for $\sigma \neq \gamma$, so $\langle \theta^\sigma, \theta^\gamma \rangle = 0$ by irreducibility. Hence $[D : F] = m^2[F(\theta) : F]$, giving $[D : E] = m^2$. $\square$

B Specifics of the numerical confirmation

B.1 Basic propositions about the Γ-function

Proposition B.1.1. *$\Gamma(z)$ has no zeroes and only has poles at non-positive integers. Furthermore, if j is an integer $j \geq 0$, then the Laurent series at $-j$ is given by*

$$\Gamma(z) = \frac{(-1)^j}{j!}\left(\frac{1}{z+j} + (H_j - \gamma) + \cdots\right)$$

where $H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j}$ and $\gamma = 0.577\ldots$ is Euler's constant.

Proposition B.1.2. *Let j be even and $j \geq 0$. Then*

$$\Gamma\left(\frac{1-j}{2}\right) = \frac{\sqrt{\pi}(-1)^{j/2}(j/2)!2^j}{j!}.$$

B.2 The integrals $F(t)$ and $G(t)$

Proposition B.2.1. *For any integer $J > 0$, we have*

$$\begin{aligned} F(t) &= \sqrt{\pi}\left(-\frac{3\gamma}{2} - \log 2 + \log t\right) + \sum_{j=2\ even}^{J} f_j + \sum_{j=1\ odd}^{J} g_j h_j \\ &\quad + \frac{1}{2\pi i}\int_{-J-\frac{1}{2}-i\infty}^{-J-\frac{1}{2}+i\infty} t^z \Gamma(z)\Gamma\left(\frac{1+z}{2}\right)\frac{dz}{z}, \end{aligned}$$

with

$$\begin{aligned} f_j &= \frac{\sqrt{\pi}(-1)^{j/2+1}(j/2-1)!2^{j-1}}{t^j j!^2}, \\ g_j &= \frac{(-1)^{(j-1)/2}}{\left(\frac{j-1}{2}\right)!t^j j! j}, \\ h_j &= \left(2\log t + \frac{2}{j} + 2H_j + H_{(j-1)/2} - 3\gamma\right). \end{aligned}$$

where $H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j}$ and $\gamma = 0.577\ldots$ is Euler's constant.

Proof. At $z = 0$, we have the Laurent expansions:

$$\begin{aligned} \frac{\Gamma(z)}{z} &= \frac{1}{z^2} - \frac{\gamma}{z} + \cdots \\ \Gamma\left(\frac{1+z}{2}\right) &= \sqrt{\pi} - \frac{\sqrt{\pi}}{2}(\gamma + 2\log 2) + \cdots \\ t^z &= 1 + (\log t)z + \cdots, \end{aligned}$$

so the residue at zero is

$$\sqrt{\pi}\left(-\frac{3\gamma}{2} - \log 2 + \log t\right).$$

Now consider the residue at $z = -j$ for $j > 0$ even. In this case only the $\Gamma(z)$ factor has a pole, and it is a simple pole. Hence the residue is

$$t^{-j}\frac{1}{j!}\Gamma\left(\frac{1-j}{2}\right)\frac{1}{-j} = -\frac{1}{t^j j! j}\cdot\frac{\sqrt{\pi}(-1)^{j/2}(j/2)!2^j}{j!} = \frac{\sqrt{\pi}(-1)^{j/2+1}(j/2-1)!2^{j-1}}{t^j j!^2}$$

Finally, consider $z = -j$ for $j > 0$ odd. We have the Laurent expansions:

$$\begin{aligned}
\Gamma(z) &= -\frac{1}{j!}\left(\frac{1}{z+j} + (H_j - \gamma)\cdots\right)\\
\Gamma\left(\frac{1+z}{2}\right) &= \frac{(-1)^{(j-1)/2}}{\left(\frac{j-1}{2}\right)!}\left(\frac{2}{z+j} + \left(H_{(j-1)/2} - \gamma\right) + \cdots\right)\\
t^z &= t^{-j}(1 + (\log t)(z+j) + \cdots)\\
\frac{1}{z} &= -\frac{1}{j}\left(1 + \frac{1}{j}(z+j) + \cdots\right).
\end{aligned}$$

Hence we have the residue

$$\frac{(-1)^{(j-1)/2}}{\left(\frac{j-1}{2}\right)! t^j j! j}\cdot\left(2\log t + \frac{2}{j} + \left(H_{(j-1)/2} - \gamma\right) + 2(H_j - \gamma)\right).$$

This proves the proposition. □

Proposition B.2.2.

$$G(t) = \left(t - \sqrt{\pi}\right) + \sum_{j=2 \text{ even}}^{J} f'_j + \sum_{j=1 \text{ odd}}^{J} g'_j h'_j + \frac{1}{2\pi i}\int_{-J-\frac{1}{2}-i\infty}^{-J-\frac{1}{2}+i\infty} t^z\Gamma(z)\Gamma\left(\frac{1+z}{2}\right)\frac{dz}{z-1},$$

with

$$\begin{aligned}
f'_j &= \frac{\sqrt{\pi}(-1)^{j/2+1}(j/2-1)!2^{j-1}}{t^j(j+1)!(j-1)!},\\
g'_j &= \frac{(-1)^{(j-1)/2}}{\left(\frac{j-1}{2}\right)! t^j (j+1)!},\\
h'_j &= \left(2\log t + \frac{2}{j+1} + 2H_j + H_{(j-1)/2} - 3\gamma\right).
\end{aligned}$$

Proof. The residue at $z = 1$ is $t\Gamma(1)^2 = t$, and the residue at $z = 0$ is $-\Gamma(1/2) = -\sqrt{\pi}$. At $z = -j$ for $j > 0$ even, only the $\Gamma(s)$ term has a pole, and this pole is simple. Thus the residue is

$$t^{-j}\frac{1}{j!}\Gamma\left(\frac{1-j}{2}\right)\frac{1}{-j-1} = -\frac{1}{t^j(j+1)!}\cdot\frac{\sqrt{\pi}(-1)^{j/2}(j/2)!2^j}{j!} = \frac{\sqrt{\pi}(-1)^{j/2+1}(j/2-1)!2^{j-1}}{t^j(j+1)!(j-1)!}.$$

At $z = -j$ for $j > 0$ odd, we have the Laurent expansions

$$\begin{aligned}
\Gamma(z) &= -\frac{1}{j!}\left(\frac{1}{z+j} + (H_j - \gamma)\cdots\right)\\
\Gamma\left(\frac{1+z}{2}\right) &= \frac{(-1)^{(j-1)/2}}{\left(\frac{j-1}{2}\right)!}\left(\frac{2}{z+j} + \left(H_{(j-1)/2} - \gamma\right) + \cdots\right)\\
t^z &= t^{-j}(1 + (\log t)(z+j) + \cdots)\\
\frac{1}{z-1} &= -\frac{1}{j+1}\left(1 + \frac{1}{j+1}(z+j) + \cdots\right).
\end{aligned}$$

Hence we have the residue

$$\frac{(-1)^{(j-1)/2}}{\left(\frac{j-1}{2}\right)!t^j(j+1)!} \cdot \left(2\log t + \frac{2}{j+1} + \left(H_{(j-1)/2} - \gamma\right) + 2(H_j - \gamma)\right).$$

□

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