On the Equality of Three Formulas for Brumer–Stark Units

Samit Dasgupta
Matthew H. Honnor

October 24, 2023

Abstract

We prove the equality of three conjectural formulas for the Brumer–Stark units. The first formula has essentially been proven, so the present paper also verifies the validity of the other two formulas.

Contents

1 Introduction

In this paper we prove the equality of three conjectural formulas for Brumer–Stark units made by the first author in [5], [8], and [9] (the last two of these, in collaboration with Spieß).

One significance of this result is that the first formula has essentially been proven in [6], so the present paper also verifies the validity of the other two formulas. Additionally, the third formula, made in [9], relates to a conjecture for the principal minors of the Gross–Regulator matrix. The validation of the third formula here gives a proof of this conjecture for the diagonal entries. Our work generalizes a partial result in this direction established in [12]. See [6] for a discussion of the application of the these formulas toward explicit class field theory.

We now describe our results more precisely. Let F denote a totally real field, and let H denote a finite abelian extension of F. Write $G = \mathrm{Gal}(H/F)$. Let R_∞ denote the set of archimedean places of F. Let R be a finite set of places of F containing R_∞ and the places that are ramified in H. Fix a prime ideal $\mathfrak{p} \notin R$ that splits completely in H and let $S = R \cup \{\mathfrak{p}\}$. Finally, we consider an auxiliary finite set T of primes of F, disjoint from $R \cup \{\mathfrak{p}\}$ and satisfying a standard minor condition (see §2.2). The following conjecture was first stated by Tate and called the Brumer–Stark conjecture, [16, Conjecture 5.4].

Conjecture 1.1. *Let $\mathfrak{P}$ be a prime in H above $\mathfrak{p}$. There exists an element*

$$u_T \in \mathcal{U}_\mathfrak{p} = \{u \in H^* : |\, u\,|_v = 1 \text{ if } v \text{ does not divide } \mathfrak{p}\}$$

such that $u_T \equiv 1 \pmod{T}$, and for all $\sigma \in G$, we have $\mathrm{ord}_\mathfrak{P}(u_T^\sigma) = \zeta_{R,T}(H/F,\sigma,0)$.

Here v ranges over all finite and archimedean places of H; in particular, each complex conjugation in H acts as an inversion on $\mathcal{U}_\mathfrak{p}$. The definition of the partial zeta function $\zeta_{R,T}(H/F,\sigma,0)$ is recalled in §2.2. The conjectural element $u_T \in \mathcal{U}_\mathfrak{p}$ satisfying Conjecture 1.1 is called the **Brumer-Stark unit** for the data $(S,T,H,\mathfrak{P})$.

Conjecture 1.1 has been recently proved away from 2 in joint work of the first author and Kakde [7]. It is convenient for us to package together u_T and its conjugates over F into an element of $H^* \otimes \mathbb{Z}[G]$ that we call the **Brumer–Stark element**:

$$u_\mathfrak{p} = \sum_{\sigma \in G} u_T^\sigma \otimes [\sigma^{-1}] \in H^* \otimes \mathbb{Z}[G].$$

There have been three formulas conjectured for the image of the Brumer–Stark element $u_\mathfrak{p}$ in $F_\mathfrak{p}^* \otimes \mathbb{Z}[G]$. In [5] the first author conjectured a p-adic analytic formula for $u_\mathfrak{p}$ following the methods of Shintani and Cassou-Noguès. We denote this formula by u_1 and state it precisely in §3. The other two formulas, which we denote u_2 and u_3, were defined in joint work of the first author with Spieß in [8] and [9], respectively. Both of these formulas are cohomological in nature and are defined using the Eisenstein cocycle. They are stated precisely in §5 and §6. The following combines the conjectures of the first author (for $i = 1$) and the first author with Speiß (for $i = 2, 3$).

Conjecture 1.2. *For $i = 1, 2, 3$ we have $u_i = u_{\mathfrak{p}}$.*

The main result of this paper is the following.

Theorem 1.3. *The three conjectural formulas for the Brumer–Stark element $u_{\mathfrak{p}}$ are equal, i.e. $u_1 = u_2 = u_3$ in $F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$.*

Recent work of the first author with Kakde has proved that $u_1 = u_{\mathfrak{p}}$ up to a root of unity under some mild assumptions. Write $\mu(F_{\mathfrak{p}}^*)$ for the group of roots of unity in $F_{\mathfrak{p}}^*$.

Theorem 1.4 (Theorem 1.6, [6])**.** *Suppose that the rational prime p below $\mathfrak{p}$ is odd and unramified in F. Suppose further that there exists $\mathfrak{q} \in S$ that is unramified in H whose associated Frobenius $\sigma_{\mathfrak{q}}$ is the complex conjugation in G. Then Conjecture 1.2 for u_1 holds up to multiplication by a root of unity in $F_{\mathfrak{p}}^*$:*

$$u_1 = u_{\mathfrak{p}} \ \text{in} \ (F_{\mathfrak{p}}^*/\mu(F_{\mathfrak{p}}^*)) \otimes \mathbb{Z}[G].$$

Remark 1.5. Theorem 1.3 implies that $u_2 = u_3 = u_{\mathfrak{p}}$ in $(F_{\mathfrak{p}}^*/\mu(F_{\mathfrak{p}}^*)) \otimes \mathbb{Z}[G]$ under the assumptions of Theorem 1.4.

In §7 we prove that $u_2 = u_3$ via a direct calculation that was foretold in [9]. The proof that $u_1 = u_2$, which takes up §8, is more interesting and involves a new idea not present in prior work in this direction. It can be broken into two parts. Suppose that H/F a CM abelian extension H/F of conductor $\mathfrak{f}$ such that $\mathfrak{p}$ splits completely in H. We note that if $\mathfrak{q} \mid \mathfrak{f}$ then we must have $\mathfrak{q} \in R$. Denote by $E_+(\mathfrak{f}) \subset \mathcal{O}_F^*$ the subgroup of totally positive units congruent to 1 modulo $\mathfrak{f}$. We then prove by a direct calculation that

$$u_1(\sigma) \equiv u_2(\sigma) \pmod{E_+(\mathfrak{f})}, \tag{1}$$

where $u_i(\sigma)$ denotes the σ component of u_i. We remark here that showing the above equation first requires the proof that $u_2 = u_3$.

Next, Let $\mathfrak{f}'$ be an auxiliary ideal of $\mathcal{O}_F$ that is divisible only by primes dividing $\mathfrak{f}$. Let $H' \supset H$ H' be another finite abelian CM extension of F in which $\mathfrak{p}$ splits completely, such that the conductor of H'/F divides $\mathfrak{f}\mathfrak{f}'$. In particular, the extension H'/F is unramified outside R. For each $\sigma \in G$, we then show the norm compatibility relation for $i = 1, 2$,

$$u_i(\sigma, H) = \prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} u_i(\tau, H'). \tag{2}$$

Applying (1) with H replaced by H' and combining with (2), we obtain

$$u_1(\sigma, H) \equiv u_2(\sigma, H) \pmod{E_+(\mathfrak{f}\mathfrak{f}')}. \tag{3}$$

If $R \neq R_\infty$, then taking larger and larger conductors $\mathfrak{f}\mathfrak{f}'$ and passing to a limit, we obtain the desired result

$$u_1(\sigma, H) = u_2(\sigma, H).$$

In the case $R = R_\infty$ we are required to do a little more work. The issue in this case is that $\mathfrak{f}\mathfrak{f}' = 1$ for all possible choices and thus working with bigger extensions does not yield more information.

In this case, by adding auxillary primes into R, we are able to show that there exists $\varepsilon \in E_+$ such that for each $\sigma \in G$ we have

$$u_1(\sigma, H) = \varepsilon u_2(\sigma, H).$$

We then extend the definitions for u_1 and u_2 to work with the trivial extension F/F. We note that this was already done for u_2 in [8]. In fact, u_2 is defined for any finite abelian extension H/F. Furthermore, [8, Proposition 6.3] it is proved that $u_2(H/F) = 1$ if H has at least two real places. In particular, $u_2(F/F) = 1$. We prove that also $u_1(F/F) = 1$. By the norm compatibility property satisfied by u_1 and u_2 we have

$$1 = u_1(F) = \prod_{\sigma \in G} u_1(\sigma, H) = \varepsilon^{|G|} \prod_{\sigma \in G} u_2(\sigma, H) = \varepsilon^{|G|}.$$

Thus $\varepsilon = 1$ and so $u_1 = u_2$.

Acknowledgements. The first author was supported by NSF grant DMS 1901939 for the duration of this project. The second author is grateful for the support of the Heilbronn Institute for Mathematical Research and Imperial College London. We thank Mahesh Kakde and Michael Spieß for helpful discussions.

2 Preliminaries for the multiplicative integral formula

2.1 Notation

Recall that we have let F be a totally real field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O} = \mathcal{O}_F$. Let $E_F = \mathcal{O}_F^*$ denote the group of global units. More generally, for a finite set S of nonarchimedean places of F we denote by $E_S = E_{F,S}$ the group of S-units of F. We define

$$\overline{S} = \{\mathfrak{q} : \mathfrak{q} \mid q \text{ where, for some } \mathfrak{r} \in S,\ \mathfrak{r} \mid q\}. \tag{4}$$

We also let H/F be a totally complex extension containing a CM-subfield. Let $\mathfrak{f}$ denote the conductor of the extension H/F. We write $E_+(\mathfrak{f})$ for the totally positive units of F that are congruent to 1 $\pmod{\mathfrak{f}}$. Write $G_\mathfrak{f}$ for the narrow ray class group of conductor $\mathfrak{f}$. Let e be the order of $\mathfrak{p}$ in $G_\mathfrak{f}$ and suppose that $\mathfrak{p}^e = (\pi)$ with $\pi \equiv 1 \pmod{\mathfrak{f}}$ and π totally positive. We write $\mathbb{O} = \mathcal{O}_\mathfrak{p} - \pi\mathcal{O}_\mathfrak{p} \subset F_\mathfrak{p}^*$.

Define $\mathbb{A} = \mathbb{A}_F$ as the adele ring of F. For a $\mathbb{Q}$-vector space W fix the notation $W_{\widehat{\mathbb{Z}}} = W \otimes_\mathbb{Z} \widehat{\mathbb{Z}} = W \otimes_\mathbb{Q} \mathbb{A}_\mathbb{Q}$. For an abelian group A and prime number L, we put $A_\ell = A \otimes_\mathbb{Z} \mathbb{Q}_\ell$.

For a place v of F we put $U_v = \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ if $v \mid \infty$ and $U_v = \mathcal{O}_v^*$ if v is finite. For a set S of places of F we let $\mathbb{A}^S$ denote the adele ring away from S. We also define $U^S = \prod_{v \notin S} U_v$, and $U_S = \prod_{v \in S} U_v$. We shall also use the notation $F^S = (\mathbb{A}_F^S \times U_S) \cap F^*$.

Finally we note that if we have a function $f : X \to Z$ and $X \subseteq Y$ then we can extend f to a function $f_! : Y \to Z$ by defining

$$f_!(y) = \begin{cases} f(y) & \text{if } y \in X \\ 0 & \text{if } y \in Y - X, \end{cases} \tag{5}$$

we call this the extension of f to Y by 0.

2.2 Partial zeta functions

For $\sigma \in G = \mathrm{Gal}(H/F)$, we define the **partial zeta function**

$$\zeta_R(H/F, \sigma, s) = \sum_{\substack{(\mathfrak{a}, R)=1 \\ \sigma_{\mathfrak{a}} = \sigma}} \mathrm{N}\mathfrak{a}^{-s}. \tag{6}$$

Here the sum ranges over all integral ideals $\mathfrak{a} \subset \mathcal{O}$ that are relatively prime to the elements of R and whose associated Frobenius element $\sigma_{\mathfrak{a}} \in G$ is equal to σ. The series (6) converges for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$, regular outside $s = 1$. When the field extension H/F is clear from context, we drop it from the notation and simply write $\zeta_R(\sigma, s)$. Since $\mathfrak{p}$ splits completely in H, the zeta functions associated to the sets of primes R and $S = R \cup \{\mathfrak{p}\}$ are related by the formula

$$\zeta_S(\sigma, s) = (1 - \mathrm{N}\mathfrak{p}^{-s})\zeta_R(\sigma, s).$$

Recall that we have fixed an auxiliary finite set of primes of F, denoted T, that is disjoint from S. The partial zeta function associated to the sets R and T is defined by the group ring equation

$$\sum_{\sigma \in G} \zeta_{R,T}(\sigma, s)[\sigma^{-1}] = \prod_{\eta \in T} (1 - [\sigma_\eta^{-1}]\mathrm{N}\eta^{1-s}) \sum_{\sigma \in G} \zeta_R(\sigma, s)[\sigma^{-1}]. \tag{7}$$

We assume that the set T contains at least two primes of different residue characteristic or at least one prime η with absolute ramification degree at most $\ell - 2$ where η lies above ℓ. With this in place, the values $\zeta_{R,T}(K/F, \sigma, 0)$ are rational integers for any finite abelian extension K/F unramified outside R and any $\sigma \in \mathrm{Gal}(K/F)$. This was shown by Deligne-Ribet [10] and Cassou-Nogués [2].

Our assumption on T implies that there are no nontrivial roots of unity in H that are congruent to 1 modulo T. Thus the $\mathfrak{p}$-unit u_T in Conjecture 1.1, if it exists, is unique. Note also that our u_T is actually the inverse of the u in [11, Conjecture 7.4]. Throughout this paper, for ease of notation, we fix $T = \{\lambda\}$ for an appropriate choice of λ.

2.3 Shintani zeta functions

Shintani zeta functions are a crucial ingredient in each of the constructions we study. We establish the necessary notation here, following Shintani [13].

For each $v \in R_\infty$ we write $\sigma_v : F \to \mathbb{R}$ and fix the order of these embeddings. We can then embed F into $\mathbb{R}^n$ by $x \mapsto (\sigma_v(x))_{v \in R_\infty}$. Note that F^* acts on $\mathbb{R}^n$ with $x \in F^*$ acting by multiplication by $\sigma_v(x)$ on the v-component of any vector in $\mathbb{R}^n$. For linearly independent $v_1, \ldots, v_r \in \mathbb{R}^n_+$, define the simplicial cone

$$C(v_1, \ldots, v_r) = \left\{ \sum_{i=1}^r c_i v_i \in \mathbb{R}^n_+ : c_i > 0 \right\}.$$

Definition 2.1. *A* ***Shintani cone*** *is a simplicial cone $C(v_1, \dots, v_r)$ generated by elements $v_i \in F \cap \mathbb{R}^n_+$. A* ***Shintani set*** *is a subset of $\mathbb{R}^n_+$ that can be written as a finite disjoint union of Shintani cones.*

We now recall the definition of **Shintani zeta functions**. Write $\mathfrak{f}$ for the conductor of the extension H/F. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and $\overline{T}$, and let D be a Shintani set. For each compact open $U \subseteq \mathcal{O}_\mathfrak{p}$, define, for $\mathrm{Re}(s) > 1$,

$$\zeta_R(\mathfrak{b}, D, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in F \cap D,\ \alpha \in U \\ (\alpha, R) = 1,\ \alpha \in \mathfrak{b}^{-1} \\ \alpha \equiv 1 \pmod{\mathfrak{f}}}} \mathrm{N}\alpha^{-s}.$$

We define $\zeta_{R,T}(\mathfrak{b}, D, U, s)$ in analogy with (7) i.e., by the group ring equation

$$\sum_{\sigma \in G} \zeta_{R,T}(\mathfrak{b}, D, U, s)[\sigma^{-1}] = \prod_{\eta \in T} (1 - [\sigma_\eta^{-1}]\mathrm{N}\eta^{1-s}) \sum_{\sigma \in G} \zeta_R(\mathfrak{b}, D, U, s)[\sigma^{-1}]. \tag{8}$$

It follows from Shintani's work in [13] that the function $\zeta_{R,T}(\mathfrak{b}, D, U, s)$ has a meromorphic continuation to $\mathbb{C}$. We now want to define conditions on the set of primes T and the Shintani set D to allow our Shintani zeta functions to be integral at 0.

Definition 2.2. *A prime ideal η of F is called* ***good*** *for a Shintani cone C if*

- *$\mathrm{N}\eta$ is a rational prime ℓ; and*
- *the cone C may be written $C = C(v_1, \dots, v_r)$ with $v_i \in \mathcal{O}$ and $v_i \notin \eta$.*

We also say that η is ***good*** *for a Shintani set D if D can be written as a finite disjoint union of Shintani cones for which η is good.*

Definition 2.3. *The set T is* ***good*** *for a Shintani set D if D can be written as a finite disjoint union of Shintani cones $D = \bigsqcup C_i$ so that for each cone C_i, there are at least two primes in T that are good for C_i (necessarily of different residue characteristic by our earlier assumption) or one prime $\eta \in T$ that is good for C_i such that $\mathrm{N}\eta \geq n + 2$.*

Remark 2.4. Given any Shintani set D, it is possible to choose a set of primes T such that T is good for D. In fact, all but a finite number of prime ideals with prime norm are good for a given Shintani set.

We can now note the required property to allow our Shintani zeta functions to be integral at zero. The proposition below is proved in [5, p.15].

Proposition 2.5. *If the set of primes T is good for a Shintani set D, then*

$$\zeta_{R,T}(\mathfrak{b}, D, U, 0) \in \mathbb{Z}.$$

We define a $\mathbb{Z}$-valued measure $\nu(\mathfrak{b}, D)$ on $\mathcal{O}_\mathfrak{p}$ by

$$\nu(\mathfrak{b}, D, U) := \zeta_{R,T}(\mathfrak{b}, D, U, 0), \tag{9}$$

for $U \subseteq \mathcal{O}_\mathfrak{p}$ compact open.

We are mostly interested in a particular type of Shintani set, one which is a fundamental domain for the action of $E_+(\mathfrak{f})$.

Definition 2.6. *We call a Shintani set D a* ***Shintani domain*** *if D is a fundamental domain for the action of $E_+(\mathfrak{f})$ on $\mathbb{R}^n_+$. That is,*

$$\mathbb{R}^n_+ = \bigsqcup_{\epsilon \in E_+(\mathfrak{f})} \epsilon D \quad \textit{(disjoint union)}.$$

The existence of such domains follows the work of Shintani, in particular from [13, Proposition 4]. We note here some simple equalities that follow from the definitions. More details are given in §3.3 of [5]. Recall we have written $G_{\mathfrak{f}}$ for the narrow ray class group of conductor $\mathfrak{f}$. Let e be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$, and write $\mathfrak{p}^e = (\pi)$ with $\pi \equiv 1 \pmod{\mathfrak{f}}$ and π totally positive. We denote by $H_{\mathfrak{f}}$ the narrow ray class field of F of conductor $\mathfrak{f}$. Let D be a Shintani domain and write $\mathbb{O} = \mathcal{O}_{\mathfrak{p}} - \pi\mathcal{O}_{\mathfrak{p}}$. Then,

$$\nu(\mathfrak{b}, D, \mathbb{O}) = \zeta_{S,T}(H/F, \mathfrak{b}, 0) = 0, \quad \text{and} \quad \nu(\mathfrak{b}, D, \mathcal{O}_{\mathfrak{p}}) = \zeta_{R,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0).$$

We now give two technical definitions that are necessary in the definition of u_1.

Definition 2.7. *Let $V \subset E_+(\mathfrak{f})$ be a finite index subgroup (which is necessarily free of rank $n-1$). We call a Shintani set D a* ***Colmez domain*** *for V if D is a fundamental domain for the action of V on $\mathbb{R}^n_+$. That is,*

$$\mathbb{R}^n_+ = \bigsqcup_{\epsilon \in V} \epsilon D \quad \textit{(disjoint union)}.$$

We note that in the definition of a Colmez domain we allow ourselves to work with $V = E_+(\mathfrak{f})$, thus the definition includes Shintani domains.

Proposition 2.8. *Let $V \subset E_+(\mathfrak{f})$ be a finite index subgroup. Let D and D' be Colmez domains for V. We may write D and D' as finite disjoint unions of the same number of simplicial cones*

$$D = \bigcup_{i=1}^{d} C_i, \quad D' = \bigcup_{i=1}^{d} C_i', \tag{10}$$

with $C_i' = \epsilon_i C_i$ for some $\epsilon_i \in V$, $i = 1, \dots, d$.

Proof. [5, Proposition 3.15] proves this result when $V = E_+(\mathfrak{f})$. The proof of this proposition is analogous. □

A decomposition as in (10) is called a **simultaneous decomposition** of the Colmez domains (D, D').

Definition 2.9. *Let (D, D') be a pair of Colmez domains. A set T is* ***good*** *for the pair (D, D') if there is a simultaneous decomposition as in (10) such that for each cone C_i, there are at least two primes in T that are good for C_i, or there is one prime $\eta \in T$ that is good for C_i such that $\mathrm{N}\eta \geq n+2$.*

Definition 2.10. *Let D be a Colmez domain. If $\beta \in F^*$ is totally positive, then T is β-**good** for D if T is good for the pair $(D, \beta^{-1}D)$.*

Lemma 2.11 (Lemma 3.20, [5]). *Let D be a Shintani set and U a compact open subset of $\mathcal{O}_{\mathfrak{p}}$. Let $\mathfrak{b}$ be a fractional ideal of F, and let $\beta \in F^*$ be totally positive so that $\beta \equiv 1 \pmod{\mathfrak{f}}$ and $\mathrm{ord}_{\mathfrak{p}}(\beta) \geq 0$. Suppose that $\mathfrak{b}$ and β are relatively prime to R and that $\mathfrak{b}$ is also relatively prime to $\overline{T}$. Let $\mathfrak{q} = (\beta)\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(\beta)}$. Then*

$$\zeta_{R,T}(\mathfrak{b}\mathfrak{q}, D, U, 0) = \zeta_{R,T}(\mathfrak{b}, \beta D, \beta U, 0).$$

We end this section with a lemma of Colmez that allows us to give an explicit Colmez domain. Let α be, up to a sign, one of the standard basis vectors of $\mathbb{R}^n$. Note that its ray $(\alpha\mathbb{R}_+)$ is preserved by the action of $\mathbb{R}^n_+$. We define $\overline{C}_\alpha(v_1,\ldots,v_r)$ to be the union of the cone $C(v_1,\ldots,v_r)$ with the boundary cones that are brought into the interior of the cone by a small perturbation by α, i.e., the set whose characteristic function is given by

$$\mathbb{1}_{\overline{C}_\alpha(v_1,\ldots,v_r)}(x) = \lim_{h\to 0^+} \mathbb{1}_{C(v_1,\ldots,v_r)}(x+h\alpha). \tag{11}$$

We use the usual bar notation for homogeneous chains

$$[x_1 \mid \ldots \mid x_{n-1}] = (1, x_1, x_1x_2, \ldots, x_1\ldots x_{n-1}).$$

Let $x_1,\ldots,x_{n-1} \in F$. We define the sign map $\delta : F^n \to \{-1,0,1\}$ by the rule

$$\delta(x_1,\ldots,x_n) = \operatorname{sign}(\det(\omega(x_1,\ldots,x_n))), \tag{12}$$

where $\omega(x_1,\ldots,x_n)$ denotes the $n\times n$ matrix whose columns are the images of the x_i in $\mathbb{R}^n$. We adopt the convention $\operatorname{sign}(0) = 0$.

Lemma 2.12 (Lemma 2.2, [4])**.** *Let α be, up to a sign, one of the standard basis vectors of $\mathbb{R}^n$. Let $\varepsilon_1,\ldots,\varepsilon_{n-1} \in E_+(\mathfrak{f})$ such that $V = \langle\varepsilon_1,\ldots,\varepsilon_{n-1}\rangle \subset E_+(\mathfrak{f})$ has finite index. Suppose that for all $\tau\in S_{n-1}$ we have*

$$\delta([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]) = \operatorname{sign}(\tau).$$

Then the Shintani set

$$D = \bigcup_{\tau\in S_{n-1}} \overline{C}_\alpha([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]),$$

is a Colmez domain for V.

The existence of Colmez domains follows from the work of Shintani in [13]. In Lemma 4.5 we show the existence of units $\varepsilon_1,\ldots,\varepsilon_{n-1} \in E_+(\mathfrak{f})$ that satisfy the conditions of Lemma 2.12.

3 The multiplicative integral formula (u_1)

Definition 3.1. *Let I be an abelian topological group that may be written as an inverse limit of discrete groups*

$$I = \varprojlim I_\alpha.$$

*Denote the group operation on I multiplicatively. For each $i \in I_\alpha$, denote by U_i the open subset of I consisting of the elements that map to i in I_α. Suppose that G is a compact open subset of a quotient of $\mathbb{A}_F^*$. Let $f : G \to I$ be a continuous map, and let μ be a $\mathbb{Z}$-valued measure on G. We define the **multiplicative integral**, written with a cross through the integration sign, by*

$$⨰_G f(x)d\mu(x) = \varprojlim \prod_{i\in I_\alpha} i^{\mu(f^{-1}(U_i))} \in I.$$

Let λ be a prime of F such that $\mathrm{N}\lambda = \ell$ for a prime number $\ell\in\mathbb{Z}$ and $\ell \geq n+2$. We assume that no primes in S have residue characteristic equal to ℓ.

Definition 3.2. *Let $\mathscr{D}$ be a Shintani domain, and assume that λ is π-good for $\mathscr{D}$. Define the* ***error term***

$$\epsilon(\mathfrak{b},\mathscr{D},\pi) := \prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b},\epsilon\mathscr{D}\cap\pi^{-1}\mathscr{D},\mathbb{O}_\mathfrak{p})} \in E_+(\mathfrak{f}). \tag{13}$$

By [5, Lemma 3.14], only finitely many of the exponents in (13) are nonzero. [5, Proposition 3.12] and the assumption that λ is π-good for $\mathscr{D}$ implies that the exponents are integers. We recall from (9) that the measure is defined as

$$\nu(\mathfrak{b},\epsilon\mathscr{D}\cap\pi^{-1}\mathscr{D},\mathbb{O}_\mathfrak{p}) = \zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathscr{D}\cap\pi^{-1}\mathscr{D},\mathbb{O}_\mathfrak{p},0).$$

We are now ready to write down the conjectural formula from [5]. We note that for any Shintani domain $\mathscr{D}$ we can always choose a prime λ that is π-good for $\mathscr{D}$. In fact, all but a finite number of primes will satisfy this property. Henceforth, we can assume that λ satisfies the property written above and is π-good for $\mathscr{D}$. We now give the main definition of this section.

Definition 3.3. *Let $\mathscr{D}$ be a Shintani domain, and assume that λ is π-good for $\mathscr{D}$. Define*

$$u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D}) := \epsilon(\mathfrak{b},\mathscr{D},\pi)\pi^{\zeta_{R,\lambda}(H_\mathfrak{f}/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \, d\nu(\mathfrak{b},\mathscr{D},x) \in F_\mathfrak{p}^*.$$

As our notation suggests, we have the following proposition.

Proposition 3.4 (Proposition 3.19, [5])**.** *The element $u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D})$ does not depend on the choice of generator π of $\mathfrak{p}^e$.*

The following is conjectured.

Conjecture 3.5 (Conjecture 3.21, [5])**.** *Let e be the order of $\mathfrak{p}$ in $G_\mathfrak{f}$, and suppose that $\mathfrak{p}^e = (\pi)$ with π totally positive and $\pi \equiv 1 \pmod{\mathfrak{f}}$. Let $\mathscr{D}$ be a Shintani domain, and let λ be π-good for $\mathscr{D}$. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and $\overline{\lambda}$. We have the following.*

1. *The element $u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D}) \in F_\mathfrak{p}^*$ depends only on the class of $\mathfrak{b} \in G_\mathfrak{f}/\langle\mathfrak{p}\rangle$ and no other choices, including the choice of $\mathscr{D}$, and hence may be denoted $u_{\mathfrak{p},\lambda}(\sigma_\mathfrak{b})$, where $\sigma_\mathfrak{b} \in \mathrm{Gal}(H/F)$.*
2. *The element $u_{\mathfrak{p},\lambda}(\sigma_\mathfrak{b})$ lies in $\mathscr{U}_\mathfrak{p}$, and $u_{\mathfrak{p},\lambda}(\sigma_\mathfrak{b}) \equiv 1 \pmod{\lambda}$.*
3. *Shimura reciprocity law: For any fractional ideal $\mathfrak{a}$ of F prime to S and to $\overline{\lambda}$, we have*

$$u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{ab}}) = u_{\mathfrak{p},\lambda}(\sigma_\mathfrak{b})^{\sigma_\mathfrak{a}}.$$

As we noted in the introduction, this conjecture has been proved up to a root of unity (Theorem 1.4).

We want to state the formula over $F_\mathfrak{p}^* \otimes \mathbb{Z}[G]$ to match with the cohomological constructions.

Definition 3.6. *We define*

$$u_1 = \sum_{\mathfrak{b}\in G_\mathfrak{f}/\langle\mathfrak{p}\rangle} u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D}) \otimes [\sigma_\mathfrak{b}^{-1}] \in F_\mathfrak{p}^* \otimes \mathbb{Z}[G].$$

3.1 Transferring to a subgroup

In this section we recall the results [12], which allow us to transfer to a subgroup. Let V be a finite index subgroup of $E_+(\mathfrak{f})$. Recall that π is totally positive, congruent to 1 modulo $\mathfrak{f}$ and satisfies $(\pi) = \mathfrak{p}^e$ where e is the order of $\mathfrak{p}$ in $G_\mathfrak{f}$. Let $\mathscr{D}'_V$ be a Shintani set which is a fundamental domain for the action of V on $\mathbb{R}^n_+$ and assume that λ is π-good for $\mathscr{D}'_V$. As before, we shall refer to such Shintani sets as Colmez domains. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and $\overline{\lambda}$.

We define

$$u_1(V,\sigma_\mathfrak{b}) = u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D}'_V) \coloneqq \prod_{\epsilon\in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathscr{D}'_V\cap\pi^{-1}\mathscr{D}'_V,\mathcal{O}_\mathfrak{p},0)}\pi^{\zeta_{R,\lambda}(\mathfrak{b},\mathscr{D}'_V,\mathcal{O}_\mathfrak{p},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x\, d\nu(\mathfrak{b},\mathscr{D}'_V,x),$$

and write $u_1(V) = \sum_{\sigma\in G} u_1(V,\sigma)\otimes[\sigma^{-1}]$.

Proposition 3.7 (Proposition 6.11, [12]). *Let $\mathscr{K}$ and $\mathscr{K}'$ be two Colmez domains for V and λ a prime of F such that λ is π-good for $\mathscr{K}$ and $\mathscr{K}'$. If λ is also good for $(\mathscr{K},\mathscr{K}')$, then $u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{K}) = u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{K}')$.*

Let $V\subset E_+(\mathfrak{f})$ be a finite index subgroup. The following proposition shows the relation between $u_1(\sigma)$ and $u_1(V,\sigma)$.

Proposition 3.8 (Proposition 6.12, [12]). *Let $\mathscr{D}$ be a Shintani domain for $E_+(\mathfrak{f})$. Let V be a finite index subgroup of $E_+(\mathfrak{f})$. Write $g_1,\dots,g_{n-1}$ for a $\mathbb{Z}$-basis of $E_+(\mathfrak{f})$ such that $g_1^{b_1},\dots,g_{n-1}^{b_{n-1}}$ is a $\mathbb{Z}$-basis for V. Define*

$$\mathscr{D}_V \coloneqq \bigcup_{j_1=0}^{b_1-1}\cdots\bigcup_{j_{n-1}=0}^{b_{n-1}-1} g_1^{j_1}\dots g_{n-1}^{j_{n-1}}\mathscr{D}.$$

Then, if $b_1,\dots,b_{n-1}>M$, where $M = M(\pi,g_1,\dots,g_{n-1})$ is some constant that depends on $g_1,\dots,$ g_{n-1} and π (up to multiplication by an element of $E_+(\mathfrak{f})$), we have

$$u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D}_V) = u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D})^{[E_+(\mathfrak{f}):V]}.$$

4 Preliminaries for the cohomological formulas

4.1 Continuous maps

For topological spaces X and Y let $C(X,Y)$ denote the set of continuous maps $X\to Y$. If R is a topological ring we let $C_c(X,R)$ denote the subset of $C(X,R)$ of continuous maps with compact support. If we consider Y (resp. R) with the discrete topology then we shall also write $C^0(X,Y)$ (resp. $C^0_c(X,R)$) instead of $C(X,Y)$ (resp. $C_c(X,Y)$).

Assume now that X is a totally disconnected topological Hausdorff space and A a locally profinite group. We define subgroups $C^\diamond(X,A)\subseteq C(X,A)$ and $C^\diamond_c(X,A)\subseteq C_c(X,A)$ by

$$C^\diamond(X,A) = C^0(X,A) + \sum_K C(X,K),$$

$$C^\diamond_c(X,A) = C^0_c(X,A) + \sum_K C_c(X,K),$$

where the sums are taken over all compact open subgroups K of A. So $C_c^\diamond(X,A)$ is the subgroup of $C_c(X,A)$ generated by locally constant maps with compact support $X \to A$ and by continuous maps with compact support $X \to K \subseteq A$ for some compact open subgroup $K \subseteq A$. Similarly $C^\diamond(X,A)$ is the subgroup of $C(X,A)$ generated by locally constant maps $X \to A$ and by continuous maps $X \to K \subseteq A$ for some compact open K.

The following notation is used in the formulation of u_2. Given two arbitary finite, disjoint sets Σ_1, Σ_2 of places of F and a locally profinite group A we put

$$\mathscr{C}_?(\Sigma_1, A)^{\Sigma_2} = C_?((\mathbb{A}_F^{\Sigma_2})^*/U^{\Sigma_1\cup\Sigma_2}, A).$$

where $? \in \{\diamond, c, 0\}$. Here, for a set of places S, U^S denotes the subgroup of $\mathbb{A}_F^*$ of ideles $(x_v)_v$ with local components $x_v = 1$ if $v \in S$, $x_v > 0$ if $v \mid \infty$ and x_v is a local unit if $v \notin S \cup R_\infty$.

We also introduce a generalisation of the above notation. For S_1, S_2 disjoint sets of places of F let

$$\mathscr{C}_?(S_1, S_2, A) = C_?(\prod_{\mathfrak{p}\in S_1} F_\mathfrak{p} \times (\mathbb{A}_F^{S_1})^*/U^{S_1\cup S_2}, A).$$

If S_3 is an additional disjoint set of places we also define

$$\mathscr{C}_?(S_1, S_2, A)^{S_3} = C_?(\prod_{\mathfrak{p}\in S_1} F_\mathfrak{p} \times (\mathbb{A}_F^{S_1\cup S_3})^*/U^{S_1\cup S_2\cup S_3}, A).$$

4.2 Measures

We now wish to attach to a homomorphism $\mu : C_c(X,\mathbb{Z}) \to \mathbb{Z}[G]$ an $A \otimes \mathbb{Z}[G]$-valued measure on X for any abelian group A and finite abelian group G. We write the group operation of A multiplicatively. Firstly, by tensoring μ with the identity we obtain a homomorphism

$$\mu_A : C_c(X,\mathbb{Z}) \otimes (A \otimes \mathbb{Z}[G]) \cong C_c^0(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G]. \tag{14}$$

To write this map explicitly we first note that the isomorphism in (14) is given by

$$f \otimes \alpha \mapsto \alpha \cdot f, \text{ with inverse } g \mapsto \sum_{\alpha\in A\otimes\mathbb{Z}[G]} (\alpha \otimes g_\alpha),$$

where $g_\alpha(x) = 1$ if $g(x) = \alpha$ and 0 otherwise. Here we have $f \in C_c(X,\mathbb{Z})$, $\alpha \in A \otimes \mathbb{Z}[G]$ and $g \in C_c^0(X, A \otimes \mathbb{Z}[G])$. Thus the homomorphism μ_A is given by

$$\mu_A(g) = \sum_{\alpha\in A\otimes\mathbb{Z}[G]} \left(\sum_{\sigma\in G} \sum_{\tau\in G} \alpha_\tau^{\mu_\sigma(g_\alpha)} \otimes \sigma\tau \right).$$

Where $\alpha = \sum_{\tau\in G} \alpha_\tau \otimes \tau$, $\mu(g_\alpha) = \sum_{\sigma\in G} \mu_\sigma(g_\alpha)[\sigma]$ and g_α is as defined before. If A is profinite we can consider the homomorphism

$$\mu_A \coloneqq \varprojlim_K \mu_{A/K} : \varprojlim_K C_c(X, A/K \otimes \mathbb{Z}[G]) \to \varprojlim_K A/K \otimes \mathbb{Z}[G] = A \otimes \mathbb{Z}[G]$$

where K ranges over the open subgroups of A. Since $C_c(X, A \otimes \mathbb{Z}[G]) \subseteq \varprojlim_K C_c(X, A/K \otimes \mathbb{Z}[G])$, we see that μ_A extends canonically to a homomorphism $C_c(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G]$ (which we denote by μ_A as well). For a general A (not necessarily profinite) we have seen that μ induces a homomorphism $C_c(X, K \otimes \mathbb{Z}[G]) \to K \otimes \mathbb{Z}[G]$ for every compact open subgroup $K \subset A$. Combining these maps we see that μ induces a canonical homomorphism $\mu_A : C_c^{\diamond}(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G]$. Define the set of $A \otimes \mathbb{Z}[G]$-valued measures on X to be

$$\mathrm{Meas}(X, A \otimes \mathbb{Z}[G]) = \mathrm{Hom}(C_c^{\diamond}(X, A \otimes \mathbb{Z}[G]), A \otimes \mathbb{Z}[G]).$$

The map $\mu \mapsto \mu_A$ defines a homomorphism $\mathrm{Hom}(C_c(X, \mathbb{Z}[G]), \mathbb{Z}) \to \mathrm{Meas}(X, A \otimes \mathbb{Z}[G])$.

In practice, we apply certain specialisations of the general construction above. In the definition of u_2 we construct $\mu \in \mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z})$ rather than in $\mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z}[G])$. We include $\mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z})$ into $\mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z}[G])$ by the map

$$\iota_1 : \mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z}[G]), \quad \iota_1(\mu)(f) = \mu(f)[\mathrm{id}],$$

for $f \in C_c(X, \mathbb{Z})$.

In the definition of u_3 we have a measure on A rather than on $A \otimes \mathbb{Z}[G]$. We include $C_c^{\diamond}(X, A)$ into $C_c^{\diamond}(X, A \otimes \mathbb{Z}[G])$ via the map

$$\iota_2 : C_c^{\diamond}(X, A) \to C_c^{\diamond}(X, A \otimes \mathbb{Z}[G]), \quad \iota_2(f)(x) = f(x) \otimes \mathrm{id}_G,$$

for $x \in X$.

4.3 Eisenstein cocycles

We now define the Eisenstein cocycle. The cohomological constructions for u_2 and u_3 require different variations.

Write S_p for the primes of F above p that split completely in H. Let $\mathcal{O}_{F,S_p}$ denote the ring of S_p-integers of F. For any fractional ideal $\mathfrak{b} \subset F$ relatively prime to S, we let $\mathfrak{b}_{S_p} = \mathfrak{b} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,S_p}$ denote the $\mathcal{O}_{F,S_p}$-module generated by $\mathfrak{b}$. Let

$$U \subset F_{S_p} = \prod_{\mathfrak{q} \in S_p} F_{\mathfrak{q}}$$

be a compact open subset. Let D be a Shintani set. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define the Shintani L-function

$$\mathfrak{L}_R(D, \mathfrak{b}, U, s) = (\mathrm{N}\mathfrak{b})^{-s} \sum_{\substack{\xi \in D \cap \mathfrak{b}_{S_p}^{-1},\ \xi \in U \\ (\xi, R) = 1}} \frac{\mathrm{rec}_{H/F}((\xi))^{-1}}{\mathrm{N}\xi^s} \in \mathbb{C}[G]. \tag{15}$$

Here $\mathrm{rec}_{H/F}$ denotes the Artin reciprocity map for the extension H/F. It follows from work of Shintani that the L-function in (15) has a meromorphic continuation to $\mathbb{C}$. Furthermore, for $D, \mathfrak{b}$ and s fixed, the values $\mathfrak{L}_R(D, \mathfrak{b}, U, s)$ form a distribution on F_{S_p} in the sense that for disjoint compact open sets $U_1, U_2 \subset F_{S_p}$, we have

$$\mathfrak{L}_R(D, \mathfrak{b}, U_1 \cup U_2, s) = \mathfrak{L}_R(D, \mathfrak{b}, U_1, s) + \mathfrak{L}_R(D, \mathfrak{b}, U_2, s).$$

Let λ be a prime of F such that $\mathrm{N}\lambda = \ell$ for a prime number $\ell \in \mathbb{Z}$ and $\ell \geq n + 2$. We assume that no primes in S have residue characteristic equal to ℓ. We then define the smoothed Shintani L-function

$$\mathfrak{L}_{R,\lambda}(D, \mathfrak{b}, U, s) \coloneqq \mathfrak{L}_R(D, \mathfrak{b}\lambda^{-1}, U, s) - \mathrm{rec}_{H/F}(\lambda)^{-1} \ell^{1-s} \mathfrak{L}_R(D, \mathfrak{b}, U, s).$$

Proposition 4.1. *Suppose λ is good for the Shintani set D. Then for a compact open subset $U \subset F_{S_p}$*

$$\mathfrak{L}_{R,\lambda}(D, \mathfrak{b}, U, 0) \in \mathbb{Z}[G].$$

Proof. The result follows from [9, Page 7]. □

Let F_+^* denote the group of totally positive elements of F. Let $E_{S_p,+}$ denote the group of totally positive units in $\mathcal{O}_{S_p}$ which we view as a subgroup of F_+^*. Let $x_1, \ldots, x_{n-1} \in F_+^*$. Recall the definition of $\overline{C}_{e_1}(x_1, \ldots, x_n)$ from (11) and the definition of $\delta(x_1, \ldots, x_n)$ from (12). The following proposition follows directly from [3, Theorem 1.6].

Proposition 4.2. *Let $x_1, \ldots, x_n \in E_{S_p,+}$. For a compact open subset $U \subset F_{S_p}$ let*

$$\mu_{\mathfrak{b},\lambda}(x_1, \ldots, x_n)(U) \coloneqq \delta(x_1, \ldots, x_n)\mathfrak{L}_{R,\lambda}(\overline{C}_{e_1}(x_1, \ldots, x_n), \mathfrak{b}, U, 0).$$

Then $\mu_{\mathfrak{b},\lambda}$ is an $E_{S_p,+}$-invariant homogeneous $(n-1)$-cocycle yielding a class

$$\kappa_{\mathfrak{b},\lambda} = [\mu_{\mathfrak{b},\lambda}] \in H^{n-1}(E_{S_p,+}, \mathrm{Hom}(C_c(F_{S_p}, \mathbb{Z}), \mathbb{Z}[G])).$$

Remark 4.3. The function $\mu_{\mathfrak{b},\lambda}(x_1, \ldots, x_n)$ is viewed as an element of $\mathrm{Hom}(C_c(F_{S_p}, \mathbb{Z}), \mathbb{Z}[G])$ via the following canonical integration pairing

$$(f, \mu) \mapsto \int_{F_R} f(t)d\mu(t) = \lim_{\|\mathcal{V}\| \to 0} \sum_{V \in \mathcal{V}} f(t_V)\mu(V)$$

where the limit is over increasingly finer covers $\mathcal{V}$ of the support of f by compact open subgroups $V \subseteq F_{S_p}$ and $t_V \in V$ is any element of V.

We define the Eisenstein cocycle associated to λ by

$$\kappa_\lambda = \sum_{i=1}^{h} \mathrm{rec}_{H/F}(\mathfrak{b}_i)^{-1}\kappa_{\mathfrak{b}_i,\lambda} \in H^{n-1}(E_{S_p,+}, \mathrm{Hom}(C_c(F_{S_p}, \mathbb{Z}), \mathbb{Z}[G])).$$

Here $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_h\}$ is a set of integral ideals representing the narrow class group of $\mathcal{O}_{F,S_p}$ (i.e., the group of fractional ideals of $\mathcal{O}_{F,S_p}$ modulo the group of fractional principal ideals generated by totally positive elements of F).

For more details on this construction, see §2 of [9]. Note that we use $\mathrm{rec}_{H/F}^{-1}$ rather then $\mathrm{rec}_{H/F}$ to make our formulation of u_3 consistent with $u_{\mathfrak{p}}$, u_1 and u_2. We expand further on this at the start of §6.

We now give a variation on the Eisenstein cocycle. Let $E_+(\mathfrak{f})_{\mathfrak{p}}$ denote the group of $\mathfrak{p}$-units of F that are congruent to 1 (mod $\mathfrak{f}$). The abelian group $E_+(\mathfrak{f})_{\mathfrak{p}}$ is free of rank n. For $x_1, \ldots, x_n \in E_+(\mathfrak{f})_{\mathfrak{p}}$, a fractional ideal $\mathfrak{b}$ coprime to S and ℓ, and compact open $U \subset F_{\mathfrak{p}}$, we put

$$\nu_{\mathfrak{b},\lambda}^{\mathfrak{p}}(x_1, \ldots, x_n)(U) = \delta(x_1, \ldots, x_n)\zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}(x_1, \ldots, x_n), U, 0).$$

Here, the Shintani zeta function is defined in (8), δ is defined in (12) and $\overline{C}_{e_1}(x_1, \ldots, x_n)$ is defined in (11). Then $\nu_{\mathfrak{b},\lambda}^{\mathfrak{p}}$ is a homogeneous $(n-1)$ on $E_+(\mathfrak{f})_{\mathfrak{p}}$ with values in the space of $\mathbb{Z}$-distribution on $F_{\mathfrak{p}}$. This follows from [3, Theorem 2.6]. We obtain a class

$$\omega_{\mathfrak{f},\mathfrak{b},\lambda}^{\mathfrak{p}} \coloneqq [\nu_{\mathfrak{b},\lambda}^{\mathfrak{p}}] \in H^{n-1}(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathrm{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z})).$$

Here $\nu^{\mathfrak{p}}_{\mathfrak{b},\lambda}$ is viewed as an element of $\mathrm{Hom}(C_c(F_{\mathfrak{p}},\mathbb{Z}),\mathbb{Z})$ via the integration pairing from Remark 4.3. We also define

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\lambda} = \sum_{[\mathfrak{b}]\in G_{\mathfrak{f}}/\langle\mathfrak{p}\rangle} \mathrm{rec}_{H/F}(\mathfrak{b})^{-1}\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \in H^{n-1}(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathrm{Hom}(C_c(F_{\mathfrak{p}},\mathbb{Z}),\mathbb{Z}[G])),$$

where the sum ranges over a system of representatives of $G_{\mathfrak{f}}/\langle\mathfrak{p}\rangle$. This construction is adapted from the construction of $\omega^{\mathfrak{p}}_{\mathfrak{f},\lambda}$ in §3.3 of [9].

We write W for F considered as a $\mathbb{Q}$-vector space, and $W_\infty = W\otimes_{\mathbb{Q}}\mathbb{R}$. As before, let λ be a prime of F such that $\mathrm{N}\lambda = \ell$ for a prime number $\ell\in\mathbb{Z}$ and $\ell\geq n+2$. We assume that no primes in S have residue characteristic equal to ℓ. Let $W_\ell = W\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$.

Define $\phi_\lambda\in C_c(W_\ell,\mathbb{Z})$ by $\phi_\lambda = \mathbb{1}_{\mathcal{O}_F\otimes\mathbb{Z}_\ell} - \ell\mathbb{1}_{\lambda\otimes\mathbb{Z}_\ell}$, i.e.

$$\phi_\lambda(v) = \begin{cases} 1 & \text{if } v\in(\mathcal{O}_F\otimes\mathbb{Z}_\ell)-(\lambda\otimes\mathbb{Z}_\ell),\\ 1-\ell & \text{if } v\in\lambda\otimes\mathbb{Z}_\ell,\\ 0 & \text{if } v\in V_\ell-(\mathcal{O}_F\otimes\mathbb{Z}_\ell).\end{cases} \tag{16}$$

By fixing an ordering of the infinite places, $v\in R_\infty$, we fix an identification $W_\infty\cong\mathbb{R}^n$. Let $v\in R_\infty$ be the infinite place corresponding to the standard basis element $e_1\in\mathbb{R}^n$.

We define $F^{\ell,v}$ as in §2.1. If D is a Shintani set and $\Phi\in C_c(W_{\widehat{\mathbb{Z}}},\mathbb{Z})$ then, following [8], we define the Dirichlet series

$$L(D,\Phi;s) = \sum_{v\in W\cap D}\Phi(v)N(v)^{-s}. \tag{17}$$

It is known to converge for $\mathrm{Re}(s)>1$ and extend to the whole complex plane except for possibly a simple pole at $s=0$. Moreover, if D and Φ are as given in the following proposition then $L(D,\Phi;s)$ is holomorphic. We remark that the set S does not appear in the definition of this Dirichlet series. In the following proposition we will decorate the L-function with λ since the choice of Φ incorporates λ into it.

For a subgroup $H\subseteq F^{\ell,v}$ and an H-module M, define $M(\delta) = M\otimes\mathbb{Z}(\delta)$. Thus $M(\delta)$ is the group M with H-action given by $x\cdot m = \delta(x)xm$ for $x\in H$ and $m\in M$.

Proposition 4.4. *Let $\omega_1,\ldots,\omega_n\in F^{\ell,v}$. For a map $\phi\in C_c(W_{\widehat{\mathbb{Z}}^\lambda},\mathbb{Z})$, let*

$$\mathrm{Eis}^0_{F,\lambda}(\omega_1,\ldots,\omega_n)(\phi) = \delta(\omega_1,\ldots,\omega_n)L_\lambda(\overline{C}_{e_1}(\omega_1,\ldots,\omega_n),\Phi;0),$$

where $\Phi=\phi\otimes\phi_\lambda$. Then $\mathrm{Eis}^0_{F,\lambda}$ is an $F^{\ell,v}$-homogeneous $(n-1)$-cocycle yielding a class

$$\mathrm{Eis}^0_{F,\lambda}\in H^{n-1}(F^{\ell,v},\mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\lambda},\mathbb{Z}),\mathbb{Z})(\delta)).$$

Proof. This proposition follows the combination of [8, Definition 4.5] and [8, Lemma 5.1]. □

4.4 Colmez subgroups

In the definitions for the Eisenstein cocycle and its variants the sign map δ appears. For the explicit calculations we want to perform later it is convenient if we can work with a finite index subgroup $V\subseteq E_+$ such that $V=\langle g_1,\ldots,g_{n-1}\rangle$ and that we are able to choose π such that, after writing $g_n=\pi$,

- for $\tau \in S_n$ we have $\delta([g_{\tau(1)} \mid \dots \mid g_{\tau(n-1)}]) = \operatorname{sign}(\tau)$.

We refer to such subgroups as Colmez subgroups. We define

$$\operatorname{Log}: \mathbb{R}_+^n \to \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (\log(x_1), \dots, \log(x_n)).$$

Let $\mathscr{H} \subset \mathbb{R}^n$ be the hyperplane defined by $\operatorname{Tr}(z) = 0$. Then, $\operatorname{Log}(E_+)$ is a lattice in $\mathscr{H}$. If $z = (z_1, \dots, z_n) \in \mathbb{R}_+^n$ and $\operatorname{Log}(z) \in \mathbb{R}^n$ is not an element of $\mathscr{H}$, then we define the projection

$$z_{\mathscr{H}} = (z_1 \dots z_n)^{-\frac{1}{n}} \cdot z.$$

We have $\operatorname{Log}(z_{\mathscr{H}}) \in \mathscr{H}$. Note that z and $z_{\mathscr{H}}$ lie on the same ray in $\mathbb{R}_+^n$. For any $M > 0$ and $i = 0, 1, \dots, n-1$, write $l_i(M)$ for the element of $\mathscr{H}$ which has value M in the $(i+1)$ place and $-M/(n-1)$ in the other places. We endow $\mathbb{R}^n$ with the sup-norm. We denote by $B(x, r)$ the ball centred at x of radius r.

The following lemma, which builds on [4, Lemma 2.1], allows us to find a collection of possible subsets $V = \langle g_1, \dots, g_{n-1} \rangle$ such that we get a nice sign property that allows us to more easily explicitly calculate the Eisenstein cocycle.

Lemma 4.5. *There exists $R_1 > 0$ such that for all $R > R_1$, $M > K_1(R)$ (where $K_1(R)$ is some constant we define that depends only on R) we have the following: For $i = 1, \dots, n-1$ let $g_i \in E_+$ and $g_n = g_\pi \in \pi_{\mathscr{H}} E_+$ such that $\operatorname{Log}(g_i) \in B(l_i(M), R)$ and $\operatorname{Log}(g_\pi) \in B(l_0(M), R)$. Then*

- *$\langle g_1, \dots, g_{n-1} \rangle \subseteq E_+$ is a finite index subgroup, and*
- *For $\tau \in S_n$ we have $\delta([g_{\tau(1)} \mid \dots \mid g_{\tau(n-1)}]) = \operatorname{sign}(\tau)$.*

Proof. This proof largely follows the ideas of Colmez in his proof of [4, Lemma 2.1]. First, note that both $\operatorname{Log}(E_+)$ and $\operatorname{Log}(\pi_{\mathscr{H}} E_+)$ are lattices inside $\mathscr{H}$. There exists a constant $R_1 \coloneqq R(E_+, \pi)$ such that for all $M > 0$ and any $r > R(E_+, \pi)$ there exist $g_1, \dots, g_{n-1} \in E_+$ and $g_\pi \in \pi_{\mathscr{H}} E_+$ such that $\operatorname{Log}(g_i) \in B(l_i(M), r)$ for $i = 1, \dots, n-1$ and $\operatorname{Log}(g_\pi) \in B(l_0(M), r)$. The existence of R_1 follows from Dirichlet's Unit Theorem and, in particular, the non-vanishing of the regulator of a number field. Since the $l_i(M)$ form a basis of $\mathscr{H}$, the $\operatorname{Log}(g_i)$ form a free family of finite index in $\operatorname{Log}(E_+)$, if M is large enough relative to r, say $M > k(r)$.

Now take M satisfying:

i) $M \geq 2(n-1)^4 r$,

ii) $M > (n-1)^2 \log(n!)$,

iii) $M > k(r)$.

For simplicity, let $K_1(r) = \max(2(n-1)^4 r, (n-1)^2 \log(n!), k(r))$ so that we only require $M > K_1(r)$.

Let $\Delta = \det([g_1 \mid \dots \mid g_{n-1}])$. Put $E_i = \exp(M(1 - \frac{i-2}{n-1}))$ and $F_i = \exp(-M(\frac{i-1}{n-1}))$. Hence, the matrix given by $[g_1 \mid \dots \mid g_{n-1}]$ is written

$$\begin{pmatrix}
1 & \beta_{1,2}F_2 & \beta_{1,3}F_3 & \dots & \beta_{1,n}F_n \\
1 & \beta_{2,2}E_2 & \beta_{2,3}E_3 & \dots & \beta_{2,n}E_n \\
1 & \beta_{3,2}F_2 & \beta_{3,3}E_3 & \dots & \beta_{3,n}E_n \\
1 & \beta_{4,2}F_2 & \beta_{4,3}F_3 & \dots & \beta_{3,n}E_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \beta_{n,2}F_2 & \beta_{n,3}F_3 & \dots & \beta_{n,n}E_n
\end{pmatrix},$$

where by i),

$$e^{\frac{-M}{2(n-1)^3}} < \beta_{i,j} < e^{\frac{M}{2(n-1)^3}}.$$

Expand Δ and isolate the diagonal term; using the bounds we defined previously we obtain

$$\mid \Delta - e^{\frac{nM}{2}} \prod_{i=2}^{n} \beta_{i,i} \mid \leq (n!-1) e^{\frac{M}{2(n-1)^2}} e^{M(\frac{n}{2}-\frac{n}{n-1})}$$

and so

$$\Delta \geq e^{\frac{nM}{2}} (e^{\frac{-M}{2(n-1)^2}} - (n!-1) e^{(\frac{M}{2(n-1)^2} - \frac{nM}{n-1})}) > 0$$

according to ii). We then show the other required sign properties in the same way. □

It is required in our later calculations to make the following sign calculation.

Lemma 4.6. *For $i = 1, \dots, n-1$ let $g_i \in E_+$ be chosen as in Lemma 4.5. Write S for the $n \times n$ matrix with rows $\operatorname{Log}(g_1), \dots, \operatorname{Log}(g_{n-1}), v_0$ where $v_0 = (1, \dots, 1) \in \mathbb{R}^n$. Then, if $M > 4(n!-1)R$, we have*

$$\operatorname{sign}(\det(S)) = (-1)^{n-1}.$$

Proof. We have

$$S = \begin{pmatrix} -\frac{M}{n-1} + \beta_{1,1} & M + \beta_{1,2} & -\frac{M}{n-1} + \beta_{1,3} & \dots & -\frac{M}{n-1} + \beta_{1,n} \\ -\frac{M}{n-1} + \beta_{2,1} & -\frac{M}{n-1} + \beta_{2,2} & M + \beta_{2,3} & \dots & -\frac{M}{n-1} + \beta_{2,n} \\ -\frac{M}{n-1} + \beta_{3,1} & -\frac{M}{n-1} + \beta_{3,2} & -\frac{M}{n-1} + \beta_{3,3} & \dots & -\frac{M}{n-1} + \beta_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{M}{n-1} + \beta_{n-1,1} & -\frac{M}{n-1} + \beta_{n-1,2} & -\frac{M}{n-1} + \beta_{n-1,3} & \dots & M + \beta_{n-1,n} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix},$$

where $-R < \beta_{i,j} < R$. We now subtract the first column from each of the other columns and expand the determinant along the bottom row. This gives, after letting $B_{i,j} = \beta_{i,j} - \beta_{i,1}$,

$$\det S = (-1)^{n-1} \det \begin{pmatrix} \frac{nM}{n-1} + B_{1,2} & B_{1,3} & \dots & B_{1,n} \\ B_{2,2} & \frac{nM}{n-1} + B_{2,3} & \dots & B_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n-1,2} & B_{n-1,2} & \dots & \frac{nM}{n-1} + B_{n-1,n} \end{pmatrix}. \tag{18}$$

Write S' for the matrix in (18) and note that $-2R < B_{i,j} < 2R$, for all $i, j = 1, \dots, n-1$. When expanding the determinant of S' and isolating the diagonal terms using the bounds from before, we observe:

$$\mid \det(S') - \prod_{i=1}^{n-1} (\frac{nM}{n-1} + B_{i,i+1}) \mid \leq (n!-1) 2R \left(\frac{nM}{n-1} + 2R \right)^{n-2}.$$

Thus,

$$\det S' \geq \left(\frac{nM}{n-1} - 2R \right)^{n-1} - (n!-1) 2R \left(\frac{nM}{n-1} + 2R \right)^{n-2}.$$

Since we have assumed $M > 4(n!-1)R$ we have

$$\det S' > \left(\frac{nM}{n-1} - \frac{M}{2(n!-1)} \right)^{n-1} - \frac{M}{2} \left(\frac{nM}{n-1} + \frac{M}{2(n!-1)} \right)^{n-2}.$$

It thus remains to show that for $n \geq 2$ the following holds

$$\left(\frac{n}{n-1} - \frac{1}{2(n!-1)}\right)^{n-1} - \frac{1}{2}\left(\frac{n}{n-1} + \frac{1}{2(n!-1)}\right)^{n-2} > 0. \tag{19}$$

Firstly, one can see by calculating that the inequality holds for $n = 2$. Remarking that as n increases the difference between the two terms in brackets decreases, gives that the value of the left hand side of (19) must increase with n. Thus (19) holds. $\square$

We now let $K_2(R) = \max(K_1(R), 4(n!-1)R)$ so that both Lemma 4.5 and Lemma 4.6 hold if $M > K_2(R)$.

Corollary 4.7. *Let $r > 0$ be an integer, D_+ an $r \times r$ diagonal matrix with positive entries, $A \in M_{n\times r}(\mathbb{R})$ and S as in Lemma 4.6. Then the block matrix*

$$B = \left(\begin{array}{c:c} A & D_+ \\ \hdashline S & 0 \end{array}\right),$$

has determinant of sign $(-1)^{n-1}(-1)^{r(n+r-1)}$.

Proof. Write $d_1, \ldots, d_r \in \mathbb{R}_{>0}$ for the diagonal entries of D_+. Using cofactor expansion with the last r columns of B one can see that the determinant of B is equal to

$$\det(S) \prod_{i=1}^{r} d_i (-1)^{(n+r-i)+(i-1))} = \det(S)(-1)^{r(n+r-1)} \prod_{i=1}^{r} d_i.$$

Using Lemma 4.6 and the fact that the entries of D_+ are positive, the result follows. $\square$

We recall the definition of $k(r)$ from the proof of Lemma 4.5 and note the following lemma.

Lemma 4.8. *We can choose $k(r) = Kr$ where K is some constant that does not depend on r. I.e., suppose $r > R_1$ and $M > Kr$, if for $i = 1, \ldots, n-1$, we have $g_i \in E_+$ with $\mathrm{Log}(g_i) \in B(r, l_i(M))$ then the $\mathrm{Log}(g_i)$ form a free family of finite index in $\mathrm{Log}(E_+)$.*

Proof. We claim that it is enough to take $K = 2(n-1)$. For each $i = 1, \ldots, n-1$, let $\mathrm{Log}(g_i) \in B(r, l_i(M))$. We then write

$$\mathrm{Log}(g_i) = (\alpha_i(1), \ldots, \alpha_i(n)) \in \mathscr{H}.$$

It is enough to show that the $\mathrm{Log}(g_i)$ are linearly independent under the projection

$$\begin{aligned} \varphi : \mathscr{H} &\to \mathbb{R}^{n-1} \\ (\alpha_1, \ldots, \alpha_n) &\mapsto (\alpha_1, \ldots, \alpha_{n-1}). \end{aligned}$$

By the definition of $l_i(M)$ and our choice of r it is clear that

$$\alpha_i(j) > 0 \text{ if } j = i+1 \quad \text{and} \quad \alpha_i(j) < 0 \text{ otherwise.}$$

We note that $\alpha_{n-1}(j) < 0$ for all j. It follows immediately that the vectors

$$\varphi(\mathrm{Log}(g_1)), \ldots, \varphi(\mathrm{Log}(g_{n-1}))$$

are linearly independent. Thus the $\mathrm{Log}(g_i)$ for a free family of finite index in $\mathrm{Log}(E_+)$.

$\square$

It follows from the above lemma that if $M > K_2(R)$ then for any $\lambda > 0$ we have that $\lambda M > K_2(\lambda R)$.

Lemma 4.9. *There exists*

1. $R_f, R_g > R_1$,
2. $M_f > K_2(R_f)$ *and*
3. $M_g > K_2(R_g)$,

such that we have the following. Firstly, for $i = 1, \dots, n-1$ *we can choose* $f_i, g_i \in E_+$ *such that* $\mathrm{Log}(f_i) \in B(l_i(M_f), R)$ *and* $\mathrm{Log}(g_i) \in B(l_i(M_g), R)$. *Furthermore, after writing*

$$V_f = \langle f_1, \dots, f_{n-1} \rangle \quad \textit{and} \quad V_g = \langle g_1, \dots, g_{n-1} \rangle$$

we have that $[E_+ : V_f]$ *is coprime to* $[E_+ : V_g]$.

Proof. We firstly choose the $f_i \in E_+$ via Lemma 4.5 and Lemma 4.6, and let $V_f = \langle f_1, \dots, f_{n-1} \rangle$. I.e., we have $\mathrm{Log}(f_i) \in B(l_i(M_f), R_f)$ for some $R_f > R_1$ and $M_f > K_2(R_f)$.

By writing the matrix representing the generators we have chosen for V_f in an upper triangular form, we can make the following choice of generators of E_+. Let $\langle \delta_1, \dots, \delta_{n-1} \rangle = E_+$ such that for some $\tau \in S_{n-1}$ we have, for $i = 1, \dots, n-1$,

$$f_{\tau(i)} = \delta_i^{a_i} \prod_{j=1}^{i-1} \delta_j^{b_{i,j}},$$

and $[E_+ : V_f] = \prod_{i=1}^{n-1} \mid a_i \mid$. By changing the sign if necessary we choose $a_1 > 0$. Furthermore, we note that changing the values of the $b_{i,j}$ in the choice of V_f does not change the index of the subgroup.

For ease of notation, let $a = \prod_{i=1}^{n-1} \mid a_i \mid$. For $i = 2, \dots, n-1$ there exists $R_{g,i} > 0$ and $M_{g,i} > 0$ such that for all $M > M_{g,i}$, there exists $\alpha \in E_+$ with $\mathrm{Log}(\alpha) \in B(l_{\tau(i)}(M), R_{g,i})$ and

$$\alpha = \delta_i^{q_i} \prod_{j=1}^{i-1} \delta_j^{k_j},$$

with q_i a nonzero integer with absolute value coprime to a. We note that this is only possible for $i > 2$ since we require the freedom of having at least one additional component we can vary.

We now consider $i = 1$. We have $\mathrm{Log}(f_{\tau(1)}) = \mathrm{Log}(\delta_1^{a_1}) \in B(l_1(M_f), R_f)$. Therefore any $q_1 > a_1$ we have $\mathrm{Log}(\delta_1^{q_1}) \in B(\frac{q_1}{a_1} l_1(M_f), \frac{q_1}{a_1} R_f)$.

Now let $R_g' = \max(R_1, R_{g,2}, \dots, R_{g,n-1})$ and $M_g' = \max(M_{g,2}, \dots, M_{g,n-1})$. We now find $q_1 > a_1$ which is coprime to a and such that $\frac{q_1}{a_1} M_f > M_g'$ and $\frac{q_1}{a_1} R_f > R_g'$.

We now fix $R_g = \frac{q_1}{a_1} R_f$ and $M_g = \frac{q_1}{a_1} M_f$. Clearly $R_g > R_1$ and it follows from Lemma 4.8 that $M_g > K_2(R_g)$. We then choose $g_{\tau(1)} = \delta_1^{q_1}$, it is immediate that $\mathrm{Log}(g_{\tau(1)}) \in B(l_1(M_g), R_g)$. For $i = 2, \dots, n-1$, we have shown that there exist $g_{\tau(i)} \in E_+$ with $\mathrm{Log}(g_{\tau(i)}) \in B(l_{\tau(i)}(M_g), R_g)$ and

$$g_{\tau(i)} = \delta_i^{q_i} \prod_{j=1}^{i-1} \delta_j^{k_j},$$

with q_i a nonzero integer with absolute value coprime to a. Let $V_g = \langle g_1, \dots, g_{n-1} \rangle$, the result follows. $\square$

4.5 1-cocycles attached to homomorphisms

Let $g : F_{\mathfrak{p}}^* \to A$ be a continuous homomorphism, where A is a locally profinite group. We want to define a cohomology class $c_g \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, A))$ attached to g. We define an $F_{\mathfrak{p}}^*$-action on $C_c(F_{\mathfrak{p}}^*, \mathbb{Z})$ by $(xf)(y) = f(x^{-1}y)$. The following definition is due to Spieß and first appears in [14, Lemma 2.11]. This definition is crucial in making the constructions of the first author and Spieß's cohomological formulas work and we also remark that the definition is unusual in that it appears as though the cocycle z_g should be a coboundary. However, it may not be a coboundary since g does not necessarily extend to a continuous function on $F_{\mathfrak{p}}$.

Definition 4.10. *Let $g : F_{\mathfrak{p}}^* \to A$ be a continuous homomorphism, where A is a locally profinite group. Let $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$ such that $f(0) = 1$. We define c_g to be the class of the cocycle*

$$z_{f,g} : F_{\mathfrak{p}}^* \to C_c(F_{\mathfrak{p}}, A)$$

defined by $z_{f,g}(x) =$ "$(1-x)(g \cdot f)$", or more precisely

$$z_{f,g}(x)(y) = (xf)(y) \cdot g(x) + ((f - xf) \cdot g)(y) \tag{20}$$

for $x \in F_{\mathfrak{p}}^*$ and $y \in F_{\mathfrak{p}}$.

The second term in (20) is allowed to be evaluated at $0 \in F_{\mathfrak{p}}$ since we can extend continuously the function from $F_{\mathfrak{p}}^*$ to $F_{\mathfrak{p}}$ as

$$(f - xf)(0) = 0.$$

The class $c_g = [z_{f,g}] \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, A))$ is independent of the choice of $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$ with $f(0) = 1$. In particular, we can consider the class $c_{\mathrm{id}} \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}))$. For more details on this construction, see [8, §3.2] and [9, §3.1].

4.6 Homology of a group of units

Recall that S_p is the set of primes of F above p that split completely in H. We label the elements of S_p by $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, i.e., we let $r = \#S_p$. Now let $0 \le k \le r$ be an integer and write

$$S_k = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}.$$

Here we have $S_0 = \varnothing$. By Dirichlet's unit theorem, the group of totally positive S_k-units $E_{S_k,+}$ is free abelian of rank $n + k - 1$. Thus the homology group $H_{n+k-1}(E_{S_k,+}, \mathbb{Z})$ is free abelian of rank 1.

In the comological formulas u_2 and u_3 we are required to choose a generator of the homology group with the correct choice of sign. Let $E_+ = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$. Recall we have written $G_{\mathfrak{f}}$ for the narrow ray class group of conductor $\mathfrak{f}$. For $i = 1, \dots, r$, let e_i be the order of $\mathfrak{p}_i$ in $G_{\mathfrak{f}}$, and write $\mathfrak{p}_i^{e_i} = (\pi_i)$ with $\pi_i \equiv 1 \pmod{\mathfrak{f}}$ and π_i totally positive.

For ease of notation, write $\pi_i = \varepsilon_{n-1+i}$. We then choose the following generator for the group $H_{n+k-1}(E_{S_k,+}, \mathbb{Z})$,

$$\eta_{S_k} = \mu \sum_{\tau \in S_{n+k-1}} \mathrm{sign}(\tau) [\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n+k-1)}] \otimes 1. \tag{21}$$

Here $\mu = \pm1$ and is equal to the sign of the determinant of a specific matrix. For $k = 0, \dots, r$, let $L_k = \mathrm{Log} \times \mathrm{ord}_{\mathfrak{p}_1} \times \dots \times \mathrm{ord}_{\mathfrak{p}_k}$. I.e., for $x \in E_{S_k,+}$ we define

$$L_k(x) = (\log(\sigma_1(x)), \dots, \log(\sigma_n(x)), \mathrm{ord}_{\mathfrak{p}_1}(x), \dots, \mathrm{ord}_{\mathfrak{p}_k}(x)).$$

Then μ is the sign of the determinant of the matrix with rows

$$L_k(\pi_1), \dots, L_k(\pi_k), L_k(\varepsilon_1), \dots, L_k(\varepsilon_{n-1}), L(1).$$

Here $L(1) \in \mathbb{R}^{n+k-1}$ is the vector with 1 in the first $n-1$ components and 0 in the last r components. This choice is as given in [15, Remark 2.1] in the case $k = 0$.

Let $V \subset E_+$ be a finite index subgroup. Let $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$ and write

$$V_{S_k} = V \oplus \langle \pi_1, \dots, \pi_k \rangle.$$

Similarly to (21) we choose a generator

$$\eta_{S_k,V} \in H_{n+k-1}(E_{V_k}, \mathbb{Z}). \tag{22}$$

5 Cohomological formula I (u_2)

This section follows the construction given in [8, §3.1]. Throughout this section we use the notation established in §4.1. Let $\eta_{\mathfrak{p}}$ be the generator of $H_n(E_{\mathfrak{p},+}, \mathbb{Z})$ as defined in (21).

Let $\mathscr{F}$ be a fundamental domain for the action of $F^*/E_{\mathfrak{p},+}$ on $(\mathbb{A}_F^{\mathfrak{p}})^*/U^{\mathfrak{p}}$. Then $\mathbb{1}_{\mathscr{F}}$ is an element of $H^0(E_{\mathfrak{p},+}, C(\mathscr{F}, \mathbb{Z})) \cong (C(\mathscr{F}, \mathbb{Z}))^{E_{\mathfrak{p},+}}$. Taking the cap product gives $\mathbb{1}_{\mathscr{F}} \cap \eta_{\mathfrak{p}} \in H_n(E_{\mathfrak{p},+}, C(\mathscr{F}, \mathbb{Z}))$. We now define $\vartheta^{\mathfrak{p}} \in H_n(F^*, \mathscr{C}_c(\varnothing, \mathbb{Z})^{\mathfrak{p}})$ as the homology class corresponding to $\mathbb{1}_{\mathscr{F}} \cap \eta_{\mathfrak{p}}$ under the isomorphism

$$H_n(E_{\mathfrak{p},+}, C(\mathscr{F}, \mathbb{Z})) \cong H_n(F^*, C_c((\mathbb{A}_F^{\mathfrak{p}})^*/U^{\mathfrak{p}}, \mathbb{Z})) \tag{23}$$

that is induced by $C_c((\mathbb{A}_F^{\mathfrak{p}})^*/U^{\mathfrak{p}}, \mathbb{Z}) \cong \mathrm{Ind}_{E_{\mathfrak{p},+}}^{F^*} C(\mathscr{F}, \mathbb{Z})$.

We now follow the construction of [8, §6]. Since the local norm residue symbol for H/F at $\mathfrak{p}$ is trivial we omit it from the reciprocity map, i.e. we consider the homomorphism

$$\mathrm{rec}_{H/F}^{\mathfrak{p}} : (\mathbb{A}_F^{\mathfrak{p}})^* \to G \hookrightarrow \mathbb{Z}[G]^*, \quad x = (x_v)_{v \neq \mathfrak{p}} \mapsto \prod_{v \neq \mathfrak{p}} (x, H/F)_v.$$

Let $R' = R - R_\infty$. We can view $\mathrm{rec}_{H/F}^{\mathfrak{p}}$ as an element of $H^0(F^*, \mathscr{C}(R', \mathbb{Z}[G])^{\mathfrak{p}})$ and denote by

$$\rho_{H/F} \in H_n(F^*, \mathscr{C}_c(R', \mathbb{Z}[G])^{\mathfrak{p}})$$

its image under the map

$$H^0(F^*, \mathscr{C}^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}}) \to H_n(F^*, \mathscr{C}_c^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}}), \quad \psi \mapsto \psi \cap \vartheta^{\mathfrak{p}}.$$

Here the cap product is induced by the map

$$\mathscr{C}^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}} \times \mathscr{C}_c(\varnothing, \mathbb{Z})^{\mathfrak{p}} \to \mathscr{C}_c^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}}, \quad (\psi, \phi) \mapsto \psi \cdot \phi, \tag{24}$$

here $\psi \cdot \phi$ denotes the function $xU^{R' \cup \mathfrak{p}} \mapsto \psi(xU^{R' \cup \mathfrak{p}})\phi(xU^{\mathfrak{p}})$.

For a locally profinite abelian group A we have a canonical map

$$C_c^{\diamond}(F_{\mathfrak{p}}, A) \otimes \mathscr{C}_c(R', \mathbb{Z}[G])^{\mathfrak{p}} \to \mathscr{C}_c^{\diamond}(\mathfrak{p}, R', A \otimes \mathbb{Z}[G]), \quad (f, g) \mapsto f \otimes g,$$

which induces a cap-product pairing

$$H^1(F^*, C_c^{\diamond}(F_{\mathfrak{p}}, A)) \times H_n(F^*, \mathscr{C}_c(R', \mathbb{Z}[G])^{\mathfrak{p}}) \to H_{n-1}(F^*, \mathscr{C}_c^{\diamond}(\mathfrak{p}, R', A \otimes \mathbb{Z}[G])).$$

In particular we can consider

$$c_{\mathrm{id}} \cap \rho_{H/F} \in H_{n-1}(F^*, \mathscr{C}_c^{\diamond}(\mathfrak{p}, R', F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])).$$

Here c_{id} is as defined in Definition 4.10. Now choose $v \in R_\infty$ to be the infinite place corresponding to the standard basis element $e_1 \in \mathbb{R}^n$. Write $R_\infty^v = R_\infty - \{v\}$. Recall that we write W for F considered as a $\mathbb{Q}$-vector space. In [8, §5.3], the following map is defined.

$$\Delta_* : H_{n-1}(F^*, \mathscr{C}_c^{\diamond}(\mathfrak{p}, R', F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])(\delta)) \to H_{n-1}(F^{\ell,v}, C_c^{\diamond}(W_{\widehat{\mathbb{Z}}^\lambda}, F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])(\delta)).$$

We postpone giving the definition of Δ_* until the next section.

Now consider the canonical pairing, where we recall the definition of $\mu_{F_{\mathfrak{p}}^*}$ from §4.2,

$$\mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\lambda}, \mathbb{Z}), \mathbb{Z}) \times C_c^{\diamond}(W_{\widehat{\mathbb{Z}}^\lambda}, F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]) \to F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G], \quad (\mu, f) \mapsto \mu_{F_{\mathfrak{p}}^*}(f). \tag{25}$$

Noting that $F^{\ell,v}$ is acting trivially on $F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ we see that (25) induces, via cap-product, a pairing

$$\cap : H^{n-1}(F^{\ell,v}, \mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\lambda}, \mathbb{Z}), \mathbb{Z})(\delta)) \times H_{n-1}(F^{\ell,v}, C_c^{\diamond}(W_{\widehat{\mathbb{Z}}^\lambda}, F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])(\delta)) \to F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]. \tag{26}$$

Recall the Eisenstein cocycle, $\mathrm{Eis}^0_{F,\lambda}$, from Proposition 4.4. Applying (26) with the Eisenstein cocycle $\mathrm{Eis}^0_F = \mathrm{Eis}^0_{F,\lambda}$ and $\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F})$ we obtain an element $u_2 = u_{S,\lambda} \in F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$,

$$u_2 = u_{S,\lambda} = \sum_{\sigma \in G} u_2(\sigma) \otimes [\sigma^{-1}] = \mathrm{Eis}^0_F \cap \Delta_*(c_{\mathrm{id}} \cap \rho_{H/F}). \tag{27}$$

The first author and Spieß then conjecture that the element $u_2(\sigma)$ is equal to the image of the Gross-Stark unit in $F_{\mathfrak{p}}^*$ under σ. We end this section by stating some known properties of this construction.

Proposition 5.1 (Proposition 6.3, [8]). *a) For $\sigma \in G$ we have $\mathrm{ord}_{\mathfrak{p}}(u_2(\sigma)) = \zeta_{R,T}(\sigma, 0)$.*

b) Let L/F be an abelian extension with $L \supseteq H$ and put $\mathfrak{g} = \mathrm{Gal}(L/F)$. Assume that L/F is unramified outside S and that $\mathfrak{p}$ splits completely in L. Then we have

$$u_2(\sigma) = \prod_{\tau \in \mathfrak{g}, \tau|_H = \sigma} u_2(L/F, \tau).$$

c) Let $\mathfrak{r}$ be a nonarchimedean place of F with $\mathfrak{r} \notin S \cup \overline{T}$ where $\overline{T}$ is as defined in (4). Then we have

$$u_2(S \cup \{\mathfrak{r}\}, \sigma) = u_2(S, \sigma) u_2(S, \sigma_{\mathfrak{r}}^{-1}\sigma)^{-1}.$$

d) Assume that H has a real archimedean place $w \nmid v$. Then $u_2(\sigma) = 1$ for all $\sigma \in G$.

e) Let L/F be a finite abelian extension of F containing H and unramified outside S. Then we have

$$\mathrm{rec}_{\mathfrak{p}}(u_2(\sigma)) = \prod_{\substack{\tau \in \mathrm{Gal}(L/F) \\ \tau|_H = \sigma^{-1}}} \tau^{\zeta_{S,T}(L/F, \tau^{-1}, 0)}.$$

Remark 5.2. In the proposition above we correct a small typo in [8, Proposition 6.3, c)] by replacing $\sigma_{\mathfrak{r}}$ with $\sigma_{\mathfrak{r}}^{-1}$.

5.1 The map Δ_*

We now define the map Δ_*. For more information and the more general construction we refer to [8, §5.3].

Throughout this section we let $A = F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ to ease notation. For sets X_1, X_2 and a map $\psi : X_1 \times X_2 \to A$, we write

$$\mathrm{Supp}(X_1, X_2, \psi) \coloneqq \{x_1 \in X_1 \mid \exists x_2 \in X_2 \text{ with } (x_1, x_2) \in \mathrm{supp}(\psi)\}.$$

Where $\mathrm{supp}(\psi)$ is the support of ψ.

Proposition 5.3. *Let X_1, X_2 be totally disconnected topological Hausdorff spaces, with X_1 discrete. Let A be a locally profinite group. The map*

$$C_c(X_1, \mathbb{Z}) \otimes_{\mathbb{Z}} C_c^{\diamond}(X_2, A) \to C_c^{\diamond}(X_1 \times X_2, A), \tag{28}$$

$$f \otimes g \mapsto ((x_1, x_2) \mapsto f(x_1) \cdot g(x_2))$$

is an isomorphism.

Proof. We calculate the inverse map as follows. Let $Y_1 = \mathrm{Supp}(X_1, X_2, \psi) \subseteq X_1$. Note that Y_1 is finite since ψ has compact support. Then

$$\psi \mapsto \sum_{y \in Y_1} \mathbb{1}_y \otimes_{\mathbb{Z}} \psi(y, \cdot) \in C_c(X_1, \mathbb{Z}) \otimes_{\mathbb{Z}} C_c^{\diamond}(X_2, A)$$

provides an inverse to (28). □

Corollary 5.4. *Let S_1, S_2 be finite disjoint sets of finite places and let S_3 be a set of infinite places. Then there exists an isomorphism*

$$\mathscr{C}_c^{\diamond}(S_1, S_2, A) \to C(F_{S_3}^*/U_{S_3}, \mathbb{Z}) \otimes \mathscr{C}_c^{\diamond}(S_1, S_2, A)^{S_3}. \tag{29}$$

Proof. Since we have

$$\prod_{\mathfrak{p} \in S_1} F_{\mathfrak{p}} \times (\mathbb{A}_F^{S_1})^*/U^{S_1 \cup S_2} = \left(\prod_{\mathfrak{p} \in S_1} F_{\mathfrak{p}} \times (\mathbb{A}_F^{S_1 \cup S_3})^*/U^{S_1 \cup S_2 \cup S_3} \right) \times F_{S_3}^*/U_{S_3}$$

and $F_{S_3}^*/U_{S_3}$ is finite, we are able to apply Proposition 5.3.

□

For every $w \in R_\infty$ we write $\delta_w : F_w^* \to \{\pm 1\}$ for the sign map. We also put $F_{R_\infty^v} = \prod_{w \in R_\infty^v} F_w$ and define

$$\delta_{R_\infty^v} : F_{R_\infty^v}^* \to \{\pm 1\} \quad \text{by} \quad (x_w)_{w \in R_\infty^v} \mapsto \prod_{w \in R_\infty^v} \delta_w(x_w).$$

Recall that for a subgroup $H \subseteq F_{R_\infty^v}^*$ and an H-module M, we write $M(\delta_{R_\infty^v}) = M \otimes \mathbb{Z}(\delta_{R_\infty^v})$. Thus $M(\delta)$ equals M but with twisted H-action given by $x \cdot m = \delta_{R_\infty^v}(x) x m$ for $x \in H$ and $m \in M$. Tensoring the ($F_{R_\infty^v}^*$-equivariant) homomorphism

$$C(F_{R_\infty^v}^* / U_{R_\infty^v}, \mathbb{Z}) \to \mathbb{Z}(\delta_{R_\infty^v}), \quad f \mapsto \sum_{x \in F_{R_\infty^v}^* / U_{R_\infty^v}} \delta_{R_\infty^v}(x) f(x) \tag{30}$$

with $\mathrm{id}_{\mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{R_\infty^v}}$ we obtain, via Corollary 5.4, an $(\mathbb{A}_F^{\mathfrak{p}})^*$-equivariant map

$$\mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A) \to \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{R_\infty^v}(\delta_{R_\infty^v}). \tag{31}$$

We now calculate this map explicitly.

Proposition 5.5. *Let $\psi \in \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)$. The image of ψ under (31) is given by*

$$\sum_{y \in Y_1} \delta_{R_\infty^v}(y) \psi(y, \cdot).$$

Here $\psi(y, \cdot) \in \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{R_\infty^v}$.

Proof. The result follows from Corollary 5.4 and (30). □

Proposition 5.6. *We have*

$$H_{n-1}(F^*, \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{R_\infty^v}(\delta_{R_\infty^v})) \cong H_{n-1}(F^{\overline{\lambda} \cup v}, \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{\overline{\lambda}, \infty}(\delta)). \tag{32}$$

Here δ is as defined in (12). Furthermore, if we write

$$\Psi = \sum_{i=1}^k [g_{i,1} \mid \ldots \mid g_{i,n-1}] \otimes \psi_i \in H_{n-1}(F^*, \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{R_\infty^v}(\delta_{R_\infty^v}))$$

then the image of Ψ under the isomorphism in (32) is represented by

$$\sum_{i=1}^k \sum_{f \in F^* / F^{\overline{\lambda} \cup v}} f \cdot [g_{i,1} \mid \ldots \mid g_{i,n-1}] \otimes \delta_v(f_v) \psi_i(f_{\overline{\lambda} \cup v}, \cdot).$$

Here $f_{\overline{\lambda} \cup v}$ is the image of f in $\prod_{w \in \overline{\lambda} \cup v} F_w^*$ and $\psi_i(f_{\overline{\lambda} \cup v}, \cdot) \in \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{\overline{\lambda}, \infty}(\delta)$.

Proof. It is easy to see that

$$\mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{R_\infty^v}(\delta_{R_\infty^v}) \cong \mathrm{Ind}_{(\mathbb{A}_F^{\overline{\lambda} \cup \mathfrak{p}, \infty})^*}^{(\mathbb{A}_F^{R_\infty^v \cup \mathfrak{p}})^*} \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{\overline{\lambda}, \infty}(\delta) \cong \mathrm{Ind}_{(\mathbb{A}_F^{v \cup \overline{\lambda}})^*}^{(\mathbb{A}_F)^*} \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{\overline{\lambda}, \infty}(\delta).$$

Thus, by weak approximation we have

$$\mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{R_\infty^v}(\delta_{R_\infty^v}) \cong \mathrm{Ind}_{F^{\overline{\lambda} \cup v}}^{F^*} \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{\overline{\lambda}, \infty}(\delta).$$

The result follows by Shapiro's lemma. The explicit description of the map follows simply by tracing through the definitions. □

The last map we need to construct before giving the definition of Δ_* is the $(\mathbb{A}_F^{\overline{\lambda},\infty})^*$ -equivariant map

$$\Delta_{S'}^{\overline{\lambda}} : \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{\overline{\lambda},\infty} \to C_c^{\diamond}(\mathbb{A}_F^{\overline{\lambda},\infty}, A) \cong C_c^{\diamond}(W_{\widehat{\mathbb{Z}}^\lambda}, A). \tag{33}$$

Recall that we have written $S' = R' \cup \{\mathfrak{p}\}$ and $\mathbb{A}_F^{\overline{\lambda},\infty} \cong W_{\widehat{\mathbb{Z}}^\lambda}$. There exist canonical homomorphisms

$$C_c^{\diamond}(F_\mathfrak{p} \times \prod_{\mathfrak{q}\in R'} F_\mathfrak{q}^*, A) \otimes \mathscr{C}_c(\varnothing, \mathbb{Z})^{S'\cup\overline{\lambda},\infty} \to \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{\overline{\lambda},\infty}, \tag{34}$$

$$C_c^{\diamond}(\prod_{\mathfrak{q}\in S'} F_\mathfrak{q}, A) \otimes C_c(\mathbb{A}_F^{S'\cup\overline{\lambda},\infty}, \mathbb{Z}) \to C_c^{\diamond}(\mathbb{A}_F^{\overline{\lambda},\infty}, A). \tag{35}$$

By Proposition 5.3 the map (34) is an isomorphism. Let $\mathcal{J}^{S'\cup\overline{\lambda}}$ denote the group of fractional ideals of F that are coprime to $S' \cup \overline{\lambda}$. Since $(\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty}$ is isomorphic to $\mathcal{J}^{S'\cup\overline{\lambda}}$, the ring $\mathscr{C}_c^0(\varnothing, \mathbb{Z})^{S'\cup\overline{\lambda},\infty}$ can be identified with the group ring $\mathbb{Z}[\mathcal{J}^{S'\cup\overline{\lambda}}]$. We define (33) as the tensor product $\Delta_{S'}^{\overline{\lambda}} = i \otimes I^{S\cup\overline{\lambda}}$ where $i : C_c^{\diamond}(F_\mathfrak{p} \times \prod_{\mathfrak{q}\in R'} F_\mathfrak{q}^*, A) \to C_c^{\diamond}(\prod_{\mathfrak{q}\in S'} F_\mathfrak{q}, A)$ is the inclusion map induced by extension by 0 and $I^{S'\cup\overline{\lambda}} : \mathbb{Z}[\mathcal{J}^{S'\cup\overline{\lambda}}] \to C_c(\mathbb{A}_F^{S'\cup\overline{\lambda},\infty}, \mathbb{Z})$ maps a fractional ideal $\mathfrak{a} \in \mathcal{J}^{S'\cup\overline{\lambda}}$ to the characteristic function of $\widehat{\mathfrak{a}}^{S'\cup\overline{\lambda}} = \mathfrak{a}(\prod_{\mathfrak{p}\notin S'\cup\overline{\lambda}} \mathcal{O}_\mathfrak{p})$, which we denote by $\mathrm{char}(\mathfrak{a}(\prod_{\mathfrak{p}\notin S'\cup\overline{\lambda}} \mathcal{O}_\mathfrak{p}))$.

For functions $f : X_1 \to A$ and $g : X_2 \to \mathbb{Z}$ we define the function $f \odot g : X_1 \times X_2 \to A$ by $(f \odot g)(x_1, x_2) = f(x_1)g(x_2)$.

Proposition 5.7. *If* $\psi \in \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{\overline{\lambda},\infty}$ *then*

$$\Delta_{S'}^{\overline{\lambda}}(\psi) = \sum_{z\in Z} \psi(z,\cdot)_! \odot \mathrm{char}\left(\prod_{w \text{ finite}} \mathfrak{q}_w^{ord_w(z_w)} (\prod_{\mathfrak{p}\notin S'\cup\overline{\lambda}} \mathcal{O}_\mathfrak{p}) \right) \in C_c^{\diamond}(\mathbb{A}_F^{\overline{\lambda},\infty}, A),$$

where the sum ranges over $Z = \mathrm{Supp}((\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty}, F_\mathfrak{p} \times \prod_{\mathfrak{q}\in R'} F_\mathfrak{q}^*, \psi)$.

Proof. Let $\psi \in \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{\overline{\lambda},\infty}$, and define Z as above. Then the image of ψ under the inverse map of the isomorphism (34) is

$$\psi \mapsto \sum_{z\in Z} \psi(z,\cdot) \otimes \mathbb{1}_z.$$

To calculate the effect of $I^{S'\cup\overline{\lambda}}$, first note that the isomorphism $\mathcal{J}^{S'\cup\overline{\lambda}} \to (\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty}$ is given by

$$\mathfrak{m} \mapsto \prod_{\mathfrak{q}\in S^\mathfrak{m}} \pi_\mathfrak{q}^{\mathfrak{m}(\mathfrak{q})},$$

where $S_\mathfrak{m}$ is the set of places that divide $\mathfrak{m}$ and $\mathfrak{m}(\mathfrak{q})$ is the integer such that the fractional ideal $\mathfrak{q}^{-\mathfrak{m}(\mathfrak{q})}\mathfrak{m}$ is coprime to $\mathfrak{q}$ and $\pi_\mathfrak{q}$ is a uniformiser associated to the prime ideal $\mathfrak{q}$. We then view the image as an element of $(\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty}$ by imposing that at the places away from $S_\mathfrak{m}$ the value is 1. Thus when we identify $\mathscr{C}_c^0(\varnothing, \mathbb{Z})^{S'\cup\overline{\lambda},\infty}$ with $\mathbb{Z}[\mathcal{J}^{S'\cup\overline{\lambda}}]$ we map

$$\phi \mapsto \sum_{\mathfrak{m}\in\mathcal{J}^{S'\cup\overline{\lambda}}} \phi(\prod_{\mathfrak{q}\in S_\mathfrak{m}} \pi_\mathfrak{q}^{\mathfrak{m}(\mathfrak{q})})\mathfrak{m} \in \mathbb{Z}[\mathcal{J}^{S'\cup\overline{\lambda}}].$$

Applying $I^{S'\cup\overline{\lambda}}$ we have

$$\sum_{\mathfrak{m}\in\mathcal{J}^{S'\cup\overline{\lambda}}}\phi(\prod_{\mathfrak{q}\in S_{\mathfrak{m}}}\pi_{\mathfrak{q}}^{\mathfrak{m}(\mathfrak{q})})\mathfrak{m}\mapsto\sum_{\mathfrak{m}\in\mathcal{J}^{S\cup\overline{\lambda}}}\phi(\prod_{\mathfrak{q}\in S_{\mathfrak{m}}}\pi_{\mathfrak{q}}^{\mathfrak{m}(\mathfrak{q})})\mathbb{1}_{\mathfrak{m}(\prod_{\mathfrak{p}\notin S'\cup\overline{\lambda}}\mathcal{O}_{\mathfrak{p}})}.$$

Returning to ψ we have under the map $\Delta_S^{\overline{\lambda}}$

$$\begin{aligned}\psi&\mapsto\sum_{z\in Z}\psi(z,\cdot)_!\otimes\sum_{\mathfrak{m}\in\mathcal{J}^{S'\cup\overline{\lambda}}}\mathbb{1}_z(\prod_{\mathfrak{q}\in S_{\mathfrak{m}}}\pi_{\mathfrak{q}}^{\mathfrak{m}(\mathfrak{q})})\mathbb{1}_{\mathfrak{m}(\prod_{\mathfrak{p}\notin S'\cup\overline{\lambda}}\mathcal{O}_{\mathfrak{p}})}\\&=\sum_{z\in Z}\psi(z,\cdot)_!\otimes\operatorname{char}\left(\prod_{v\text{ finite}}\mathfrak{q}_v^{\operatorname{ord}_v(z_v)}(\prod_{\mathfrak{p}\notin S'\cup\overline{\lambda}}\mathcal{O}_{\mathfrak{p}})\right).\end{aligned}$$

Lastly, the image of the above under the map (35) is:

$$\sum_{z\in Z}\psi(z,\cdot)_!\odot\operatorname{char}\left(\prod_{v\text{ finite}}\mathfrak{q}_v^{\operatorname{ord}_v(z_v)}(\prod_{\mathfrak{p}\notin S'\cup\overline{\lambda}}\mathcal{O}_{\mathfrak{p}})\right).$$

□

We are now able to define Δ_* via the composition

$$\begin{aligned}\Delta_*:H_{n-1}(F^*,\mathcal{C}_c^{\diamond}(\{\mathfrak{p}\},R',F_{\mathfrak{p}}^*\otimes\mathbb{Z}[G]))&\xrightarrow{(31)_*}H_{n-1}(F^*,\mathcal{C}_c^{\diamond}(\{\mathfrak{p}\},R',F_{\mathfrak{p}}^*\otimes\mathbb{Z}[G])^{R_\infty^v}(\delta_{R_\infty^v}))\\&\xrightarrow{(32)}H_{n-1}(F^{\overline{\lambda}\cup v},\mathcal{C}_c^{\diamond}(\{\mathfrak{p}\},R',F_{\mathfrak{p}}^*\otimes\mathbb{Z}[G])^{\overline{\lambda},\infty}(\delta))\\&\xrightarrow{(33)_*}H_{n-1}(F^{\overline{\lambda}\cup v},C_c^{\diamond}(W_{\widehat{\mathbb{Z}}^\lambda},F_{\mathfrak{p}}^*\otimes\mathbb{Z}[G])(\delta)).\end{aligned}$$

5.2 Transferring to a subgroup

Let V be a finite index subgroup of E_+. Let $\eta_{\mathfrak{p},V}$ be the generator of $H_n(V\oplus\langle\pi\rangle,\mathbb{Z})$ as defined in (22). Let $\mathcal{F}_V$ be a fundamental domain for the action of $F^*/(V\oplus\langle\pi\rangle)$ on $(\mathbb{A}_F^{\mathfrak{p}})^*/U^{\mathfrak{p}}$. Then $\mathbb{1}_{\mathcal{F}_V}$ is an element of $H^0(V\oplus\langle\pi\rangle,C(\mathcal{F}_V,\mathbb{Z}))\cong(C(\mathcal{F}_V,\mathbb{Z}))^{V\oplus\langle\pi\rangle}$. Taking the cap product then gives

$$\mathbb{1}_{\mathcal{F}_V}\cap\eta_{\mathfrak{p},V}\in H_n(V\oplus\langle\pi\rangle,C(\mathcal{F}_V,\mathbb{Z})).$$

We now define $\vartheta_V^{\mathfrak{p}}\in H_n(F^*,\mathcal{C}_c(\varnothing,\mathbb{Z})^{\mathfrak{p}})$ as the homology class corresponding to $\mathbb{1}_{\mathcal{F}_V}\cap\eta_{\mathfrak{p},V}$ under the isomorphism

$$H_n(V\oplus\langle\pi\rangle,C(\mathcal{F}_V,\mathbb{Z}))\cong H_n(F^*,C_c((\mathbb{A}_F^{\mathfrak{p}})^*/U^{\mathfrak{p}},\mathbb{Z}))\tag{36}$$

that is induced by $C_c((\mathbb{A}_F^{\mathfrak{p}})^*/U^{\mathfrak{p}},\mathbb{Z})\cong\operatorname{Ind}_{V\oplus\langle\pi\rangle}^{F^*}C(\mathcal{F}_V,\mathbb{Z})$. As before we view $\operatorname{rec}_{H/F}^{\mathfrak{p}}$ as an element of $H^0(F^*,\mathcal{C}(R',\mathbb{Z}[G])^{\mathfrak{p}})$ and denote by

$$\rho_{H/F,V}\in H_n(F^*,\mathcal{C}_c(R',\mathbb{Z}[G])^{\mathfrak{p}})$$

its image under the map

$$H^0(F^*,\mathcal{C}^{\diamond}(R',\mathbb{Z}[G])^{\mathfrak{p}})\to H_n(F^*,\mathcal{C}_c^{\diamond}(R',\mathbb{Z}[G])^{\mathfrak{p}}),\quad\psi\mapsto\psi\cap\vartheta_V^{\mathfrak{p}}.$$

Here the cap product is induced by the map (24). We then define

$$u_2(V)=\operatorname{Eis}_F^0\cap\Delta_*(c_{\text{univ}}\cap\rho_{H/F,V}).$$

Proposition 5.8. *Let V be a finite index subgroup of E_+. Write*

$$u_2 = \sum_{\sigma\in G} u_2(\sigma)\otimes\sigma^{-1}.$$

Then

$$u_2(V) = u_2^{[E_+:V]} = \sum_{\sigma\in G} u_2(\sigma)^{[E_+:V]}\otimes\sigma^{-1}.$$

Note that, since $F_\mathfrak{p}^*\otimes\mathbb{Z}[G]$ only has an operation as a group, we have $(\sum_{\sigma\in G} a_\sigma\otimes\sigma)^n = \sum_{\sigma\in G} a_\sigma^n\otimes\sigma$.

Proof. We mimic the proof of [3, Theorem 1.5]. For ease of notation we define

$$A = F_\mathfrak{p}^*\otimes\mathbb{Z}[G] \quad\text{and}\quad \mathrm{Meas}(W_{\widetilde{\mathbb{Z}}^\lambda},\mathbb{Z}) = \mathrm{Hom}(C_c(W_{\widetilde{\mathbb{Z}}^\lambda},\mathbb{Z}),\mathbb{Z})(\delta).$$

General properties of group cohomology (see [1, pp. 112-114]) yield the following commutative diagram.

$$\begin{array}{ccccc} H^0(V\oplus\langle\pi\rangle, C(\mathcal{F}_V,\mathbb{Z})) & \times & H_n(V\oplus\langle\pi\rangle,\mathbb{Z}) & \xrightarrow{\cap} & H_n(F^*, C_c((\mathbb{A}_F^\mathfrak{p})^*/U^\mathfrak{p},\mathbb{Z})) \\ \mathrm{res}\uparrow & & \downarrow\mathrm{cores} & & \downarrow\mathrm{id} \\ H^0(E_{+,\mathfrak{p}}, C(\mathcal{F},\mathbb{Z})) & \times & H_n(E_{+,\mathfrak{p}},\mathbb{Z}) & \xrightarrow{\cap} & H_n(F^*, C_c((\mathbb{A}_F^\mathfrak{p})^*/U^\mathfrak{p},\mathbb{Z})) \end{array} \tag{37}$$

The cap-products in the top and bottom rows above include applying the isomorphisms (23) and (36), respectively. By Proposition 9.5 in [1, §3], we have following identities,

$$\mathrm{cores}(\eta_{\mathfrak{p},V}) = [E_{+,\mathfrak{p}}:V]\eta_\mathfrak{p},$$
$$\mathrm{res}(\mathbb{1}_\mathcal{F}) = \mathbb{1}_{\mathcal{F}_V}.$$

Applying these identities with diagram (37) gives

$$\vartheta_V^\mathfrak{p} = [E_{+,\mathfrak{p}}:V]\vartheta^\mathfrak{p}.$$

The proposition follows. □

5.3 Explicit expression for u_2

Let $V\subseteq E_+$ be a finite index subgroup such that if $V = \langle\varepsilon_1,\dots,\varepsilon_{n-1}\rangle$ the ε_i and π satisfy Lemma 4.5 and Lemma 4.6. For ease of notation we write $\varepsilon_n = \pi$. In this section we calculate the value of $u_2(V) = \mathrm{Eis}_F^0\cap\Delta_*(c_{\mathrm{id}}\cap\rho_{H/F,V})$ as an explicit multiplicative integral.

Following §4.6 and Corollary 4.7, we choose the following generator for $H_n(V\oplus\langle\pi\rangle,\mathbb{Z})$,

$$\eta_{\mathfrak{p},V} = (-1)\sum_{\tau\in S_n}\mathrm{sign}(\tau)[\varepsilon_{\tau(1)}\mid\dots\mid\varepsilon_{\tau(n)}]\otimes 1.$$

We then calculate

$$\vartheta_V^\mathfrak{p} = \mathbb{1}_{\mathcal{F}_V}\cap\eta_{\mathfrak{p},V} = (-1)\sum_{\tau\in S_n}\mathrm{sign}(\tau)[\varepsilon_{\tau(1)}\mid\dots\mid\varepsilon_{\tau(n)}]\otimes\mathbb{1}_{\mathcal{F}_V}.$$

Thus,

$$\rho_{H/F,V} = \mathrm{rec}_{H/F}^\mathfrak{p}\cap\vartheta_V^\mathfrak{p} = (-1)\sum_{\tau\in S_n}\mathrm{sign}(\tau)[\varepsilon_{\tau(1)}\mid\dots\mid\varepsilon_{\tau(n)}]\otimes(\mathrm{rec}_{H/F}^\mathfrak{p}\cdot\mathbb{1}_{\mathcal{F}_V}).$$

Here $\mathrm{rec}^{\mathfrak{p}}_{H/F} \cdot \mathbb{1}_{\mathcal{F}_V}$ is as defined in (24). It then follows that $c_{\mathrm{id}} \cap \rho_{H/F,V}$ is equal to

$$c_{\mathrm{id}} \cap \rho_{H/F,V} = (-1)^n(-1) \sum_{\tau \in S_n} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \\ \Big((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes (\mathrm{rec}^{\mathfrak{p}}_{H/F} \cdot \mathbb{1}_{\mathcal{F}_V}) \Big). \quad (38)$$

Note that we have the action $(x \cdot f)(y) = f(yx^{-1})$ for a continuous map f and unit x. We also recall the definition of $z_{\mathrm{id}} = z_{\mathbb{1}_{\mathcal{O}_\mathfrak{p}},\mathrm{id}}$ from §4.5. Then $z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \in C_c^{\diamond}(F_\mathfrak{p}, F_\mathfrak{p}^*)$. We now apply the map Δ_* to this quantity. In §4.2.1 Δ_* is defined via the composition of three maps namely $(31)_*$, (32) and $(33)_*$. By Proposition 5.5 we have that the image of $c_{\mathrm{id}} \cap \rho_{H/F,V}$ under $(31)_*$ is given by

$$(-1)^{n+1} \sum_{\tau \in S_n} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \\ \left((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes \sum_{y \in Y_{\tau(n)}} \delta_{R_\infty^v}(y)(\mathrm{rec}^{\mathfrak{p}}_{H/F}(y, \cdot) \cdot \mathbb{1}_{\mathcal{F}_V}(y, \cdot)) \right), \quad (39)$$

where

$$Y_{\tau(n)} = \mathrm{Supp}(F^*_{R^v_\infty}/U_{R^v_\infty}, F_\mathfrak{p} \times (\mathbb{A}_F^{\mathfrak{p} \cup R^v_\infty})^*/U^{S \cup R^v_\infty}, \psi_{\tau(n)}).$$

Here, for ease of notation we have written

$$\psi_{\tau(n)} = (\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes (\mathrm{rec}^{\mathfrak{p}}_{H/F} \cdot \mathbb{1}_{\mathcal{F}_V}).$$

It is now convenient for us to make a choice for $\mathcal{F}_V$. Let G_V denote the group of fractional ideals of $\mathcal{O}_{F,\mathfrak{p}}$ modulo the group of fractional principal ideals generated by elements of V, where $\mathcal{O}_{F,\mathfrak{p}}$ denotes the ring of $\mathfrak{p}$ integers of F. Let $\{\mathfrak{b}_1, \dots, \mathfrak{b}_h\}$ be a set of integral ideals prime to $R' \cup \overline{\lambda}$ representing G_V. We may then choose

$$\mathcal{F}_V = \{b_1 U^\mathfrak{p}, \dots, b_h U^\mathfrak{p}\}$$

where $b_1, \dots, b_h \in (\mathbb{A}_F^\mathfrak{p})^*$ are ideles whose associated fractional $\mathcal{O}_{F,\mathfrak{p}}$-ideals are $\mathfrak{b}_1 \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}, \dots,$ $\mathfrak{b}_h \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}$. For $i = 1, \dots, h$ we can choose the b_i to be totally positive and prime to $R' \cup \overline{\lambda}$. This description of $\mathcal{F}_V$ is similar to a construction given in [9, Page 14]. From this description of $\mathcal{F}_V$ we have that $Y_{\tau(n)}$ is trivial for all n. Thus (39) is equal to

$$(-1)^{n+1} \sum_{\tau \in S_n} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \Big((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes (\mathrm{rec}^{\mathfrak{p} \cup R^v_\infty}_{H/F} \cdot \mathbb{1}^{R^v_\infty}_{\mathcal{F}_V}) \Big). \quad (40)$$

We now apply (32). By Proposition 5.6 we have that the image of (40) under (32) is equal to

$$(-1)^{n+1} \sum_{\tau \in S_n} \sum_{f \in F^*/F^{\overline{\lambda} \cup v}} \mathrm{sign}(\tau) f[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \\ \Big((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes \delta_v(f_v)(\mathrm{rec}^{\mathfrak{p} \cup R^v_\infty}_{H/F}(f_{\overline{\lambda} \cup v}, \cdot) \cdot \mathbb{1}^{R^v_\infty}_{\mathcal{F}_V}(f_{\overline{\lambda} \cup v}, \cdot)) \Big). \quad (41)$$

By our choice of $\mathcal{F}_V$ we have that $\mathbb{1}^{R^v_\infty}_{\mathcal{F}_V}(f_{\overline{\lambda} \cup v}, \cdot) = 0$ unless $f \equiv 1 \pmod{F^{\overline{\lambda} \cup v}}$. Hence (41) is equal to

$$(-1)^{n+1} \sum_{\tau \in S_n} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \Big((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes (\mathrm{rec}^{\mathfrak{p} \cup \overline{\lambda} \cup \infty}_{H/F} \cdot \mathbb{1}^{\overline{\lambda} \cup \infty}_{\mathcal{F}_V}) \Big). \quad (42)$$

We can now finish calculating the effect of Δ_* on $c_{\mathrm{id}} \cap \rho_{H/F,V}$ by applying (33) to (42) and using Proposition 5.7 to calculate that $\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})$ is equal to

$$(-1)^{n+1} \sum_{\tau \in S_n} \operatorname{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \Bigg((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes \sum_{z \in Z} (\mathrm{rec}_{H/F}^{\mathfrak{p} \cup \overline{\lambda} \cup \infty}(z, \cdot) \cdot \mathbb{1}_{\mathscr{F}_V}^{\overline{\lambda} \cup \infty}(z, \cdot)) \odot \mathrm{char}\Bigg(\prod_{w \text{ finite}} \mathfrak{q}_w^{\mathrm{ord}_w(z_w)} \big(\prod_{\mathfrak{p} \notin S' \cup \overline{\lambda}} \mathbb{O}_{\mathfrak{p}} \big) \Bigg) \Bigg). \tag{43}$$

Here

$$Z = \operatorname{Supp}((\mathbb{A}_F^{S' \cup \overline{\lambda}, \infty})^* / U^{S' \cup \overline{\lambda}, \infty}, F_{\mathfrak{p}} \times \prod_{\mathfrak{q} \in R'} F_{\mathfrak{q}}^*, \psi_{\tau(n)}).$$

Also, $\psi_{\tau(n)} = (\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes (\mathrm{rec}_{H/F}^{\mathfrak{p} \cup \overline{\lambda} \cup \infty} \cdot \mathbb{1}_{\mathscr{F}_V}^{\overline{\lambda} \cup \infty})$. We now apply the measure Eis_F^0 to $\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})$. Recall that the measure is defined in (26). We write $\mu_{F_{\mathfrak{p}}^*}$ for the measure with values in $F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ induced from the Eisenstein series Eis_F^0. We now consider the function $(\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)})$. For ease of notation we define, for $\tau \in S_n$ and $z \in Z$,

$$\phi_{\tau(n),z} = (\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes \sum_{z \in Z} (\mathrm{rec}_{H/F}^{\mathfrak{p} \cup \overline{\lambda} \cup \infty}(z, \cdot) \cdot \mathbb{1}_{\mathscr{F}_V}^{\overline{\lambda} \cup \infty}(z, \cdot)) \odot \mathrm{char}\Bigg(\prod_{w \text{ finite}} \mathfrak{q}_w^{\mathrm{ord}_w(z_w)} \big(\prod_{\mathfrak{p} \notin S' \cup \overline{\lambda}} \mathbb{O}_{\mathfrak{p}} \big) \Bigg) \Bigg).$$

We are then able to calculate, recalling that $z_{\mathrm{id}} = z_{\mathbb{1}_{\mathbb{O}_{\mathfrak{p}}}, \mathrm{id}}$,

$$((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) = \begin{cases} \mathbb{1}_{\mathbb{O}} \cdot \mathrm{id}_{F_{\mathfrak{p}}^*} + \mathbb{1}_{\pi \mathbb{O}_{\mathfrak{p}}} \cdot \pi & \text{if } \tau(n) = n, \\ \mathbb{1}_{\pi \mathbb{O}_{\mathfrak{p}}} \cdot \varepsilon_{\tau(n)} & \text{if } \tau(n) \neq n. \end{cases} \tag{44}$$

To calculate the measure we first note that $F_{\mathfrak{p}}^* \cong \langle \pi \rangle \oplus \mathbb{O}$ and that $\mathbb{O} \cong \lim_{m \to \infty} \mathbb{O}/1 + \mathfrak{p}^m \mathbb{O}_{\mathfrak{p}}$. By (44) we are able to calculate the value at π and $\mathbb{O}$ separately. Let $m \geq 0$ and $\alpha \in \mathbb{O}/1 + \mathfrak{p}^m \mathbb{O}_{\mathfrak{p}}$. We write $U_\alpha = \alpha(1 + \mathfrak{p}^m \mathbb{O}_{\mathfrak{p}})$. For $\sigma \in G$, we define the following maps

$$\phi_{n,z}^{\pi \otimes \sigma^{-1}} : W_{\widehat{\mathbb{Z}}^\lambda} \to \mathbb{Z}, \quad \text{and} \quad \phi_{n,z}^{U_\alpha \otimes \sigma^{-1}} : W_{\widehat{\mathbb{Z}}^\lambda} \to \mathbb{Z},$$

by

$$\phi_{n,z}^{\pi \otimes \sigma^{-1}}(x) = \begin{cases} 1 & \text{if } \phi_{n,z}(x) = \pi \otimes \sigma^{-1}, \\ 0 & \text{else,} \end{cases} \quad \text{and } \phi_{n,z}^{U_\alpha \otimes \sigma^{-1}}(x) = \begin{cases} 1 & \text{if } \phi_{n,z}(x) \in U_\alpha \otimes \sigma^{-1}, \\ 0 & \text{else.} \end{cases}$$

For $\tau \in S_n$ with $\tau(n) \neq n$ we also define

$$\phi_{n,z}^{\varepsilon_{\tau(n)} \otimes \sigma^{-1}} : W_{\widehat{\mathbb{Z}}^\lambda} \to \mathbb{Z}$$

by

$$\phi_{\tau(n),z}^{\varepsilon_{\tau(n)} \otimes \sigma^{-1}}(x) = \begin{cases} 1 & \text{if } \phi_{\tau(n),z}(x) = \varepsilon_{\tau(n)} \otimes \sigma^{-1}, \\ 0 & \text{else.} \end{cases}$$

The construction of the measure given in §4.2 now allows us to calculate

$$\begin{aligned}
&\mathrm{Eis}_F^0 \cap (\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})) = (-1)^{n+1}(-1)^{(n-1)(n-1)} \\
&\sum_{z\in Z}\sum_{\sigma\in G}\left(\sum_{\substack{\tau\in S_n\\ \tau(n)=n}} \mathrm{sign}(\tau)\lim_{m\to\infty}\left(\sum_{\alpha\in\mathbb{O}/(1+\mathfrak{p}^m\mathcal{O}_\mathfrak{p})} \mathrm{Eis}_\tau^0(\phi_{n,z}^{U_\alpha\otimes\sigma^{-1}})(\alpha\otimes\sigma^{-1})\right) + \mathrm{Eis}_\tau^0(\phi_{n,z}^{\pi\otimes\sigma^{-1}})(\pi\otimes\sigma^{-1})\right. \\
&\qquad\left. + \sum_{\substack{\tau\in S_n\\ \tau(n)\neq n}} \mathrm{sign}(\tau)\mathrm{Eis}_\tau^0(\phi_{\tau(n),z}^{\varepsilon_{\tau(n)}\otimes\sigma^{-1}})(\varepsilon_{\tau(n)}\otimes\sigma^{-1})\right). \qquad (45)
\end{aligned}$$

Here we have written $\mathrm{Eis}_\tau^0 = \mathrm{Eis}_F^0([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}])$. Let

$$\phi \in \{\phi_{n,z}^{U_\alpha\otimes\sigma^{-1}}, \phi_{n,z}^{\pi\otimes\sigma^{-1}}, \phi_{n,z}^{\varepsilon_{\tau(n)}\otimes\sigma^{-1}}\}.$$

Then, by Proposition 4.4 we have that

$$\mathrm{Eis}_F^0([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}])(\phi) = \delta([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}])L_\lambda(\overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]), \Phi; 0)$$

where $\Phi = \phi \otimes \phi_\lambda$ and ϕ_λ is as defined in (16). Let

$$\overline{C}_\tau = \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]).$$

Recall from (17) that for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ we have

$$L_\lambda(\overline{C}_\tau, \Phi; s) = \sum_{v\in W\cap\overline{C}_\tau} \Phi(v)\mathrm{N}v^{-s}.$$

For $i = 1, \dots, n$ we define

$$\mathcal{B}_i \coloneqq \bigcup_{\substack{\tau\in S_n\\ \tau(n)=i}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]). \qquad (46)$$

We also write $\mathcal{B} = \mathcal{B}_n$. Since we have chosen the ε_i as in Lemma 4.5 we have

$$\mathrm{sign}(\tau)\delta([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]) = 1. \qquad (47)$$

Applying (47) and the definition in (46) to (45) we have, after noting $(-1)^{n+1}(-1)^{(n-1)(n-1)} = 1$,

$$\begin{aligned}
&\mathrm{Eis}_F^0 \cap (\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})) = \\
&\sum_{z\in Z}\sum_{\sigma\in G}\left(\lim_{m\to\infty}\left(\sum_{\alpha\in\mathbb{O}/(1+\mathfrak{p}^m\mathcal{O}_\mathfrak{p})} L_\lambda(\mathcal{B}, \Phi_{n,z}^{U_\alpha\otimes\sigma^{-1}}; 0)(\alpha\otimes\sigma^{-1})\right) + L_\lambda(\mathcal{B}, \Phi_{n,z}^{\pi\otimes\sigma^{-1}}; 0)(\pi\otimes\sigma^{-1})\right. \\
&\qquad\left. + \sum_{i=1}^{n-1} L_\lambda(\mathcal{B}_i, \Phi_{i,z}^{\varepsilon_i\otimes\sigma^{-1}}; 0)(\varepsilon_i\otimes\sigma^{-1})\right). \qquad (48)
\end{aligned}$$

We now calculate each term in the above expression, beginning with the limit term. Fix $m \geq 0$ and $\sigma \in G$. Let $\alpha \in \mathbb{O}/1 + \mathfrak{p}^m\mathcal{O}_\mathfrak{p}$. We also let $\mathfrak{b}$ be a fractional ideal of F, coprime to $S \cup \overline{\lambda}$, and such that

$\sigma_{\mathfrak{b}} = \sigma$. We need to find the elements $z \in Z$ such that $\phi_{n,z}^{U_\alpha \otimes \sigma^{-1}}$ is not trivial. For this we require that for some $x \in \prod_{\mathfrak{q} \in R'} F_{\mathfrak{q}}^*$,

$$\sigma^{-1} = (\mathrm{rec}_{H/F}^{\mathfrak{p} \cup \overline{\lambda} \cup \infty}(z, x) \cdot \mathbb{1}_{\mathscr{F}_V}^{\mathfrak{p} \cup \overline{\lambda} \cup \infty}(z, x)).$$

By the definition of $\mathscr{F}_V$ and the reciprocity map, the above equation is non-trivial only if $z \in \mathscr{F}_V$ and $\prod_{v \text{ finite}} \mathfrak{q}_v^{-\mathrm{ord}_v(z_v)} = \mathfrak{b}^{-1}(\alpha)\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\alpha)}$ for some $(\alpha) \in P^{\mathfrak{f},1}$. Recall that $\mathfrak{f}$ is the conductor of H/F and we have written $P^{\mathfrak{f},1} = \{(\alpha) \mid \alpha \in F_+^*,\ \alpha \equiv 1 \pmod{\mathfrak{f}}\}$. By the description of $\mathscr{F}_V$ we note that for each $\sigma \in G$ there is a unique $z \in Z$ that satisfies the above equation. Since

$$\prod_{v \text{ finite}} \mathfrak{q}_v^{\mathrm{ord}_v(z_v)} = \mathfrak{b}^{-1}(\alpha)\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)}$$

for $(\alpha) \in P^{\mathfrak{f},1}$ we have that $(\alpha)\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)}$ must be coprime to $\mathfrak{p} \cup \overline{R} \cup \overline{\lambda}$ since z and $\mathfrak{b}^{-1}$ are. Thus, for all $r \in (\alpha)\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)}$ we have $r^{-1} \in \prod_{\mathfrak{r} \notin S' \cup \overline{\lambda}} \mathscr{O}_{\mathfrak{r}}$. We therefore have

$$F \cap \mathfrak{b}^{-1}(\alpha)\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)}(\prod_{\mathfrak{r} \notin S' \cup \overline{\lambda}} \mathscr{O}_{\mathfrak{r}}) = \mathfrak{b}^{-1}.$$

We now define, for a Shintani set A, $U \subseteq F_{\mathfrak{p}}$ compact open, fractional ideal $\mathfrak{b}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$

$$L_{R,\lambda}(\mathfrak{b}, A, U, s) \coloneqq \sum_{\substack{x \in W \cap A,\ x \in U, \\ x \in \mathfrak{b}^{-1},\ (x,R)=1}} \phi_\lambda(x) \mathrm{N}(x)^{-s}.$$

Then

$$\begin{aligned}\lim_{m \to \infty} \left(\sum_{\alpha \in \mathbb{O}/(1+\mathfrak{p}^m \mathscr{O}_{\mathfrak{p}})} L_\lambda(\mathscr{B}, \Phi_{n,z}^{U_\alpha \otimes \sigma_{\mathfrak{b}}^{-1}}; 0)(\alpha \otimes \sigma^{-1}) \right) &= \lim_{m \to \infty} \left((\prod_{\alpha \in \mathbb{O}/(1+\mathfrak{p}^m \mathscr{O}_{\mathfrak{p}})} \alpha^{L_{R,\lambda}(\mathfrak{b}, \mathscr{B}, U_\alpha; 0)} \otimes \sigma_{\mathfrak{b}}^{-1}) \right) \\ &= (\times\!\!\!\!\!\!\int_{\mathbb{O}} x \, dL_{R,\lambda}(\mathfrak{b}, \mathscr{B}, x; 0)) \otimes \sigma_{\mathfrak{b}}^{-1}.\end{aligned}$$

Here the multiplicative integral is as in Definition 3.1. We can apply similar calculations for the other terms in (48) and thus deduce following the explicit expression for $\mathrm{Eis}_F^0 \cap (\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V}))$:

$$\begin{aligned}&\mathrm{Eis}_F^0 \cap (\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})) = \\ &\sum_{\sigma_{\mathfrak{b}} \in G} \left((\prod_{i=1}^{n-1} \varepsilon_i^{L_{R,\lambda}(\mathfrak{b}, \mathscr{B}_i, \pi \mathscr{O}_{\mathfrak{p}}; 0)}) \pi^{L_{R,\lambda}(\mathfrak{b}, \mathscr{B}, \pi \mathscr{O}_{\mathfrak{p}}; 0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \, dL_{R,\lambda}(\mathfrak{b}, \mathscr{B}, x; 0)) \otimes \sigma_{\mathfrak{b}}^{-1} \right). \quad (49)\end{aligned}$$

6 Cohomological formula II (u_3)

In [9] the first author and Spieß give two equivalent constructions for their formula. Since we require each of them in the later sections we give both here. We denote them by u_3 and u_3'. In this work we adopt slightly different conventions from [9], namely we have $u_3 = u_3(DS)^{\#}$. Here # denotes the involution on $\mathbb{Z}[G]$ given by $g \mapsto g^{-1}$ for $g \in G$ and $u_3(DS)$ is the construction in [9]. This is done by adjusting the definitions of κ_λ and $\omega_{\mathfrak{f},\lambda}^{\mathfrak{p}}$ in §4.3. The key adjustment we have is to apply $\mathrm{rec}_{H/F}((\eta))^{-1}$ rather than $\mathrm{rec}_{H/F}((\eta))$ in (15).

We begin with u_3. Recall that in §4.5 and §4.3 we have defined the following objects:

$$c_g \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}^*)) \quad \text{and} \quad \kappa_\lambda \in H^{n-1}(E_{S_p,+}, \mathrm{Hom}(C_c(F_{S_p}, \mathbb{Z}), F_{\mathfrak{p}} \otimes \mathbb{Z}[G])).$$

Let $r = \#S_p$ and $\eta_{S_p} \in H_{n+r-1}(E_{S_p,+}, \mathbb{Z})$ be as defined in (21). Label the elements of S_p by $\mathfrak{p}, \mathfrak{p}_2 \ldots, \mathfrak{p}_r$. We now define a class

$$c_{\mathrm{id},\mathfrak{p}} \in H^r(F_{S_p}^*, C_c(F_{S_p}, F_{\mathfrak{p}}^*))$$

by

$$c_{\mathrm{id},\mathfrak{p}} = c_{\mathrm{id}} \cup c_{O_{\mathfrak{p}_2}} \cup \cdots \cup c_{O_{\mathfrak{p}_r}}.$$

Here the cup product is induced by the canonical map

$$C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}^*) \otimes \cdots \otimes C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}^*) \to C_c(F_{S_p}, F_{\mathfrak{p}}^*)$$

defined by

$$\bigotimes_{\mathfrak{q} \in S_p} f_{\mathfrak{q}} \mapsto \left((x_{\mathfrak{q}})_{\mathfrak{q} \in S_p} \mapsto \prod_{\mathfrak{q} \in S_p} f_{\mathfrak{q}}(x_{\mathfrak{q}}) \right).$$

Definition 6.1. *We define*

$$u_3 = c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_\lambda \cap \eta_{S_p}) \in F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G].$$

Adapted from [9, Conjecture 3.1] we have the following conjecture.

Conjecture 6.2. *We have $u_3 = u_{\mathfrak{p}}$.*

We now give the definition for u_3'. Recall that in §4.3 we have defined

$$\omega_{\mathfrak{f},\lambda}^{\mathfrak{p}} \in H^{n-1}(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathrm{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z}[G])).$$

Definition 6.3. *Let $\eta_{\mathfrak{p},E_+(\mathfrak{f})} \in H_n(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathbb{Z})$ be the generator defined in (22). Then, we define*

$$u_3' \coloneqq c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\lambda}^{\mathfrak{p}} \cap \eta_{\mathfrak{p},E_+(\mathfrak{f})}). \tag{50}$$

The following Proposition follows from [9, Proposition 3.6].

Proposition 6.4. *We have $u_3 = u_3'$.*

6.1 Transferring to a subgroup

Let V be a finite index subgroup of $E_+(\mathfrak{f})$. Let $\eta_{\mathfrak{p},V} \in H_n(V \oplus \langle \pi \rangle, \mathbb{Z})$ be the generator defined in (22). For $x_1, \ldots, x_n \in V \oplus \langle \pi \rangle$ and compact open $U \subset F_{\mathfrak{p}}$ we put

$$\nu_{\mathfrak{b},\lambda,V}^{\mathfrak{p}}(x_1, \ldots, x_n)(U) \coloneqq \delta(x_1, \ldots, x_n) \zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}(x_1, \ldots, x_n), U, 0).$$

As before, it follows from [3, Theorem 2.6] that $\nu_{\mathfrak{b},\lambda,V}^{\mathfrak{p}}$ is a homogeneous $(n-1)$-cocycle on $V \oplus \langle \pi \rangle$ with values in the space of $\mathbb{Z}$-distribution on $F_{\mathfrak{p}}$. Hence we obtain a class

$$\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}} \coloneqq [\nu_{\mathfrak{b},\lambda,V}^{\mathfrak{p}}] \in H^{n-1}(V \oplus \langle \pi \rangle, \mathrm{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z})).$$

We then define

$$u_3'(V) = c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}} \cap \eta_{\mathfrak{p},V}).$$

The next proposition shows the relation between u_3' and $u_3'(V)$.

Proposition 6.5 (Proposition 6.12, [12])**.** *Let V be a finite index subgroup of $E_+(\mathfrak{f})$. Then we have*

$$u_3'(V) = (u_3')^{[E_+(\mathfrak{f}):V]}. \tag{51}$$

Let $V' \subseteq E_+$ be a finite index subgroup. We write V' here to differentiate between $V' \subseteq E_+$ and $V \subseteq E_+(\mathfrak{f})$. We now define $u_3(V')$ similarly and note its relation to $u_3'(V)$. Write

$$V'_{S_p} = V' \oplus \langle \pi_1, \dots, \pi_r \rangle.$$

Let $\eta_{S_p,V'} \in H_{n+r-1}(V'_{S_p}, \mathbb{Z})$ be the generator defined in (22). For $x_1, \dots, x_n \in V'_{S_p}$ and compact open $U \subset F_\mathfrak{p}$ we put

$$\mu_{\chi,\mathfrak{b},\lambda,V}(x_1,\dots,x_n)(U) \coloneqq \delta(x_1,\dots,x_n)\mathfrak{L}_{R,\lambda}(\overline{C}_{e_1}(x_1,\dots,x_n),\mathfrak{b},U,0).$$

As before, it follows from [3, Theorem 2.6] that $\mu_{\chi,\mathfrak{b},\lambda,V}$ is a homogeneous $(n-1)$-cocycle on V'_{S_p} with values in the space of $\mathbb{Z}$-distribution on F_{S_p}. Hence, we obtain a class

$$\kappa_{\mathfrak{b},\lambda,V'} \coloneqq \sum_{i=1}^{h} \mathrm{rec}(\mathfrak{b}_i)^{-1}[\mu_{\mathfrak{b},\lambda,V'}] \in H^{n-1}(V'_{S_p}, \mathrm{Hom}(C_c(F_{S_p},\mathbb{Z}),\mathbb{Z}[G])).$$

We then define

$$u_3(V') = c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V'} \cap \eta_{S_p,V'}).$$

Proposition 6.6. *Let V be a finite index subgroup of $E_+(\mathfrak{f})$. Let V' be any finite index subgroup of E_+ such that $V \subseteq V'$ and $[E_+ : V'] = [E_+(\mathfrak{f}) : V]$. Then*

$$u_3(V') = u_3'(V).$$

Proof. This proposition follows from the proof of [9, Proposition 3.6]. □

Following from this proposition we have a simple corollary which gives the result of Proposition 6.5, but for u_3'.

Corollary 6.7. *Let V' be a finite index subgroup of E_+. Then,*

$$u_3(V') = u_3^{[E_+:V']}.$$

Proof. We calculate

$$u_3(V') = u_3'(V) = (u_3')^{[E_+(\mathfrak{f}):V]} = u_3^{[E_+:V']}.$$

Here the first and last equality follow from Proposition 6.6 and the middle equality follows from Proposition 6.5. □

6.2 Explicit expression for u_3

Let $V = \langle\varepsilon_1,\dots,\varepsilon_{n-1}\rangle \subseteq E_+$ where $\varepsilon_1,\dots,\varepsilon_{n-1}$ and $\pi = \varepsilon_n$ are chosen to satisfy Lemma 4.5 and Lemma 4.6. Write $\varepsilon_n = \pi$. As before we have the notation $V_{S_p} = \langle\varepsilon_1,\dots,\varepsilon_{n-1},\pi_1,\dots,\pi_r\rangle$. Here we have $\pi = \pi_1$. As in (46), for $i = 1,\dots,n$ we define

$$\mathscr{B}_i \coloneqq \bigcup_{\substack{\tau\in S_n\\ \tau(n)=i}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]).$$

As before, we write $\mathscr{B} = \mathscr{B}_n$. We now calculate u_3. We begin by calculating the value of $c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V)$. For ease of notation, for $i = 1,\dots,r$, we let $\varepsilon_{n+i-1} = \pi_i$. Following §4.6 and Corollary 4.7, we choose the following generator for $H_{n+r-1}(V_{S_p},\mathbb{Z})$,

$$\eta_{S_p,V} = (-1)^{n-1}(-1)^{r(n+r-1)} \sum_{\tau\in S_{n+r-1}} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n+r-1)}] \otimes 1.$$

We can now calculate, after noting that $(-1)^{n-1}(-1)^{r(n+r-1)}(-1)^{(n+r-1)(n-1)} = (-1)^r$,

$$\kappa_{\mathfrak{b},\lambda,V} \cap \eta_{S_p,V} = (-1)^r \sum_{\tau\in S_{n+r-1}} \mathrm{sign}(\tau)\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]) \otimes [\varepsilon_{\tau(n)} \mid \dots \mid \varepsilon_{\tau(n+r-1)}].$$

Recall from §4.5 that we can choose as a representative of c_{id} the inhomogeneous 1-cocycle $z_{\mathrm{id}} = z_{\mathbb{1}_{\pi\mathcal{O}_\mathfrak{p}},\mathrm{id}}$, i.e., we take $f = \mathbb{1}_{\pi\mathcal{O}_\mathfrak{p}}$ in Definition 4.10. One can easily compute, as is done in the proof of [9, Proposition 4.6], that for $i = 1,\dots,n+r-1$, $i \neq n$

$$\varepsilon_i^{-1} z_{\mathrm{id}}(\varepsilon_i) = \mathbb{1}_{\pi\mathcal{O}_\mathfrak{p}} \cdot \varepsilon_i, \tag{52}$$

and

$$\pi^{-1} z_{\mathrm{id}}(\pi) = \mathbb{1}_{\mathbb{O}} \cdot \mathrm{id}_{F_\mathfrak{p}^*} + \mathbb{1}_{\mathcal{O}_\mathfrak{p}} \cdot \pi. \tag{53}$$

Returning to our main calculation, we have

$$\begin{aligned} c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \eta_{S_p,V}) = (-1)^r(-1)^{r^2} \sum_{\tau\in S_{n+r-1}} \int_{F_{S_p}} & c_{\mathrm{id},\mathfrak{p}}([\varepsilon_{\tau(n)} \mid \dots \mid \varepsilon_{\tau(n+r-1)}])(x) \\ & d(\mathrm{sign}(\tau)([\varepsilon_{\tau(n)} \mid \dots \mid \varepsilon_{\tau(n+r-1)}])\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]))(x). \end{aligned}$$

Now we note that for $i = 2,\dots,r$ we have that $c_{o_{\mathfrak{p}_i}}(\varepsilon_j) = 0$ unless $i = j$. Hence, we only get non-zero terms when $\tau(k) = k$ for $k = n+1,\dots,n+r-1$. Therefore, since $(-1)^r(-1)^{r^2} = 1$, we have

$$\begin{aligned} c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \eta_{S_p,V}) = \sum_{\tau\in S_n} \int_{F_{S_p}} & c_{\mathrm{id},\mathfrak{p}}([\varepsilon_{\tau(n)} \mid \varepsilon_{n+1} \mid \dots \mid \varepsilon_{n+r-1}])(x) \\ & d(\mathrm{sign}(\tau)([\varepsilon_{\tau(n)} \mid \varepsilon_{n+1} \mid \dots \mid \varepsilon_{n+r-1}])\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]))(x). \end{aligned}$$

Then, since for $i = 2,\dots,r$ and $\tau \in S_n$ we can calculate $(\varepsilon_{\tau(n)}\varepsilon_{n+1}\dots\varepsilon_{n+i-1})^{-1} \cdot c_{o_{\mathfrak{p}_i}}(\varepsilon_i) = \mathbb{1}_{\mathcal{O}_{\mathfrak{p}_i}}$, we have

$$\begin{aligned} c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \eta_{S_p,V}) = \sum_{\tau\in S_n} \int_{F_\mathfrak{p}} & c_{\mathrm{id}}([\varepsilon_{n+1} \mid \dots \mid \varepsilon_{n+r-1}])(x) \\ & d(\mathrm{sign}(\tau)\varepsilon_{\tau(n)}\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]))(x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}). \end{aligned}$$

We recall the definition of $\kappa_{\mathfrak{b},\lambda,V}$ from §6.1. Using that we have chosen V and π through Lemma 4.5 we note that for $\tau \in S_n$ and a compact open $U \subseteq \mathcal{O}_{\mathfrak{p}}$, we have by definition:

$$\operatorname{sign}(\tau)\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]) = \mathfrak{L}_{R,\lambda}(\overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]), \mathfrak{b}, U, 0).$$

Here $\mathfrak{L}$ is as defined in (15). Thus,

$$c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \eta_{S_p,V}) = \sum_{\tau \in S_n} \int_{F_{\mathfrak{p}}} \varepsilon_{\tau(n)}^{-1} z_{\mathrm{id}}(\varepsilon_{\tau(n)})(x) \\ d(\mathfrak{L}_{R,\lambda}(\overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]), \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)).$$

Applying (52) and (53), and piecing together the appropriate Shintani sets we further deduce

$$c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \eta_{S_p,V}) = \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}, \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)) \times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}}} \pi \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}, \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)) \\ \prod_{i=1}^{n-1} \times\!\!\!\!\!\!\int_{\pi\mathcal{O}_{\mathfrak{p}}} \varepsilon_i \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}_i, \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)). \quad (54)$$

Considering the first two terms on the right hand side of (54) it is clear that

$$\times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}, \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)) \times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}}} \pi \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}, \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)) \\ = \pi^{\mathfrak{L}_{R,\lambda}(\mathscr{B},\mathfrak{b},\mathcal{O}_{S_p},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}, \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)),$$

where $\mathcal{O}_{S_p} = \prod_{i=1}^{r} \mathcal{O}_{\mathfrak{p}_i} \subset F_{S_{\mathfrak{p}}}$. We now consider the sum on the right hand side of (54). It is straightforward to see that

$$\prod_{i=1}^{n-1} \times\!\!\!\!\!\!\int_{\pi\mathcal{O}_{\mathfrak{p}}} \varepsilon_i \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}_i, \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)) = \prod_{i=1}^{n-1} \varepsilon_i^{\mathfrak{L}_{R,\lambda}(\mathscr{B}_i,\mathfrak{b},\pi\mathcal{O}_{\mathfrak{p}} \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i},0)}.$$

Hence

$$c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \eta_{S_p,V}) \\ = (\prod_{i=1}^{n-1} \varepsilon_i^{\mathfrak{L}_{R,\lambda}(\mathscr{B}_i,\mathfrak{b},\pi\mathcal{O}_{\mathfrak{p}} \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i},0)}) \pi^{\mathfrak{L}_{R,\lambda}(\mathscr{B},\mathfrak{b},\mathcal{O}_{S_p},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}, \mathfrak{b}, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)).$$

Thus we have

$$u_3(V) = \sum_{k=1}^{h} \mathrm{rec}_{H/F}(\mathfrak{b}_k)^{-1} \Bigg((\prod_{i=1}^{n-1} \varepsilon_i^{\mathfrak{L}_{R,\lambda}(\mathscr{B}_i,\mathfrak{b}_k,\pi\mathcal{O}_{\mathfrak{p}} \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i},0)}) \pi^{\mathfrak{L}_{R,\lambda}(\mathscr{B},\mathfrak{b}_k,\mathcal{O}_{S_p},0)} \\ \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d(\mathfrak{L}_{R,\lambda}(\mathscr{B}, \mathfrak{b}_k, x \times \prod_{i=2}^{r} \mathcal{O}_{\mathfrak{p}_i}, 0)) \Bigg).$$

6.3 Explicit expression for u_3'

For later calculations we also require an explicit expression for u_3'. Let V be a finite index subgroup of $E_+(\mathfrak{f})$ such that $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$ where $\varepsilon_1, \dots, \varepsilon_{n-1}$ and $\pi = \varepsilon_n$ are chosen to satisfy Lemma 4.5 and Lemma 4.6. For $i = 1, \dots, n$ write

$$\mathscr{B}_i \coloneqq \bigcup_{\substack{\tau \in S_n \\ \tau(n)=i}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]).$$

Let $\mathscr{B} = \mathscr{B}_n$. Recall we write $V_\mathfrak{p} = V \oplus \langle \pi \rangle$. Following §4.6 and Corollary 4.7, we choose the following generator for $H_n(V_\mathfrak{p}, \mathbb{Z})$,

$$\eta_{\mathfrak{p},V} = (-1) \sum_{\tau \in S_n} \operatorname{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n)}] \otimes 1.$$

We can now calculate

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \eta_{\mathfrak{p},V} = (-1)^{n(n-1)}(-1) \sum_{i=1}^{n} \sum_{\substack{\tau \in S_n \\ \tau(n)=i}} \operatorname{sign}(\tau) \omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]) \otimes [\varepsilon_i].$$

We recall the definition of $\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}$ from §4.3. For $\tau \in S_n$ and a compact open $U \subseteq \mathcal{O}_\mathfrak{p}$, we have

$$\operatorname{sign}(\tau)\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]) = \zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]), U, 0). \tag{55}$$

Returning to our main calculation, using (55) we have

$$c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \eta_{\mathfrak{p},V}) = \sum_{i=1}^{n} \sum_{\substack{\tau \in S_n \\ \tau(n)=i}} \int_{F_\mathfrak{p}} z_{\mathrm{id}}(\varepsilon_i)(x)\ d(\varepsilon_i \zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]), x, 0)).$$

We note that taking the cap product gives another factor of (-1) which cancels the factor from before. Applying (52) and (53) and piecing together the appropriate Shintani sets we further deduce

$$\begin{aligned} c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \eta_{\mathfrak{p},V}) = \times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d(\zeta_{R,\lambda}(\mathfrak{b}, \mathscr{B}, x, 0)) \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}} \pi\ d(\zeta_{R,\lambda}(\mathfrak{b}, \mathscr{B}, x, 0)) \\ \prod_{i=1}^{n-1} \times\!\!\!\!\!\!\int_{\pi\mathcal{O}_\mathfrak{p}} \varepsilon_i\ d(\zeta_{R,\lambda}(\mathfrak{b}, \mathscr{B}_i, x, 0)). \end{aligned} \tag{56}$$

It is clear that we can then write

$$u_3'(V, \sigma) = c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \eta_{\mathfrak{p},V}) = \prod_{i=1}^{n-1} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b}, \mathscr{B}_i, \pi\mathcal{O}_\mathfrak{p}, 0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b}, \mathscr{B}, \mathcal{O}_\mathfrak{p}, 0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d(\zeta_{R,\lambda}(\mathfrak{b}, \mathscr{B}, x, 0))(x).$$

7 Equality of u_2 and u_3

In this section we prove the following theorem.

Theorem 7.1. *We have $u_2 = u_3$.*

Proof. Let V be finite index subgroup of E_+ which satisfies Lemma 4.5. We will show

$$u_2(V) = u_3(V).$$

Then by Proposition 5.8 and Proposition 6.5 we have that

$$u_2(V) = u_2^{[E_+:V]} \quad \text{and} \quad u_3(V) = u_3^{[E_+:V]}.$$

We note that since $F_\mathfrak{p}^* \otimes \mathbb{Z}[G]$ only has an operation as a group we have $(\sum_{\sigma\in G} a_\sigma \otimes \sigma)^n = \sum_{\sigma\in G} a_\sigma^n \otimes \sigma$. It then follows that $u_2(\sigma)^{[E_+:V]} = u_3(\sigma)^{[E_+:V]}$. Considering Lemma 4.9, one can make two choices for V in Lemma 4.5 and Lemma 4.6, say V_1, V_2 such that $[E_+ : V_1]$ is coprime to $[E_+ : V_2]$. This then gives the result of the theorem.

We now recall the explicit calculations for $u_2(V)$ and $u_3(V)$ as given in §5.3 and §6.2 respectively.

$$u_2(V) = \mathrm{Eis}_F^0 \cap (\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})) = \sum_{\sigma_\mathfrak{b}\in G} \left((\prod_{i=1}^{n-1} \varepsilon_i^{L_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,\pi\mathbb{O}_\mathfrak{p};0)}) \pi^{L_{R,\lambda}(\mathfrak{b},\mathscr{B},\pi\mathbb{O}_\mathfrak{p};0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \, dL_{R,\lambda}(\mathfrak{b},\mathscr{B},x;0)) \otimes \sigma_\mathfrak{b}^{-1} \right),$$

and

$$u_3(V) = \sum_{k=1}^{h} \mathrm{rec}_{H/F}(\mathfrak{b}_k)^{-1} \left((\prod_{i=1}^{n-1} \varepsilon_i^{\mathfrak{L}_{R,\lambda}(\mathscr{B}_i,\mathfrak{b}_k,\pi\mathbb{O}_\mathfrak{p}\times\prod_{i=2}^r \mathbb{O}_{\mathfrak{p}_i},0)}) \pi^{\mathfrak{L}_{R,\lambda}(\mathscr{B},\mathfrak{b}_k,\mathbb{O}_{S_p},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \, d(\mathfrak{L}_{R,\lambda}(\mathscr{B},\mathfrak{b}_k,x\times\prod_{i=2}^r \mathbb{O}_{\mathfrak{p}_i},0)) \right).$$

Note that for $U \subseteq \mathbb{O}_\mathfrak{p}$, $i = 1,\dots,n$, $k = 1,\dots,h$ and $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ that

$$\begin{aligned}
\mathfrak{L}_R(\mathscr{B}_i,\mathfrak{b}_k,U\times\prod_{i=2}^r \mathbb{O}_{\mathfrak{p}_i},s) &= (\mathrm{N}\mathfrak{b}_k)^{-s} \sum_{\substack{\xi\in\mathscr{B}_i\cap(\mathfrak{b}_k)_{S_p}^{-1},\ \xi\in U\times\prod_{i=2}^r \mathbb{O}_{\mathfrak{p}_i} \\ (\xi,R)=1}} \frac{\mathrm{rec}_{H/F}((\xi))^{-1}}{\mathrm{N}\xi^s} \\
&= (\mathrm{N}\mathfrak{b}_k)^{-s} \sum_{\substack{\xi\in F\cap\mathscr{B}_i,\ \xi\in U \\ \xi\in\mathfrak{b}_k^{-1}\ (\xi,R)=1}} \frac{\mathrm{rec}_{H/F}((\xi))^{-1}}{\mathrm{N}\xi^s}.
\end{aligned}$$

We write

$$\mathfrak{L}_{R,\lambda}(\mathscr{B}_i,\mathfrak{b}_k,U\times\prod_{i=2}^r \mathbb{O}_{\mathfrak{p}_i},0) = \sum_{\sigma\in G} \mathfrak{L}_{R,\lambda}(\sigma,\mathscr{B}_i,\mathfrak{b}_k,U,0)\otimes\sigma^{-1}$$

where $\mathfrak{L}_{R,\lambda}(\sigma,\mathscr{B}_i,\mathfrak{b}_k,U,0) \in \mathbb{Z}$. Recall that in §4.2 we defined for $L = \sum_{\sigma\in G} a_\sigma \otimes \sigma \in \mathbb{Z}[G]$ and $\alpha \in F_\mathfrak{p}^*$,

$$\alpha^L = \sum_{\sigma\in G} \alpha^{a_\sigma} \otimes \sigma.$$

It then follows from the definitions of $\mathfrak{L}_{R,\lambda}$ and $L_{R,\lambda}$ that for $\alpha \in F_\mathfrak{p}^*$,

$$\sum_{k=1}^{h} \sum_{\sigma_\mathfrak{b}\in G} \alpha^{\mathfrak{L}_{R,\lambda}(\sigma_\mathfrak{b},\mathscr{B}_i,\mathfrak{b}_k,U,0)} \otimes \sigma_{\mathfrak{b}_i}^{-1}\sigma_\mathfrak{b}^{-1} = \sum_{\sigma_\mathfrak{b}\in G} \alpha^{L_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,U;0)} \otimes \sigma_\mathfrak{b}^{-1}.$$

This completes the proof that $u_2(V) = u_3(V)$. $\square$

8 Equality of u_1 and u_2

In this section we prove the following theorem.

Theorem 8.1. *We have $u_1 = u_2$.*

We prove the theorem by showing that for each $\sigma \in G$ we have $u_1(\sigma) = u_2(\sigma)$. This is done by using a strong enough compatibility property which forces the formulas to be equal. A special argument will be required in the case that R contains no finite places, i.e., $R = R_\infty$.

We are given a CM abelian extension H/F of conductor $\mathfrak{f}$ such that $\mathfrak{p}$ splits completely in H. Let $\mathfrak{f}'$ be an auxiliary ideal of $\mathcal{O}_F$ that is divisible only by primes dividing $\mathfrak{f}$. Let H' be another finite abelian CM extension of F in which $\mathfrak{p}$ splits completely, such that the conductor of H'/F divides $\mathfrak{f}\mathfrak{f}'$. In particular, the extension H'/F is unramified outside R.

Let $\sigma \in G$. Write $u_1(\sigma, H)$ and $u_2(\sigma, H)$ for σ components of the formulas u_1 and u_2, for the extension H/F and Galois group element σ. We show that, for $i = 1, 2$,

$$u_i(\sigma, H) = \prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} u_i(\tau, H'). \tag{57}$$

We refer to (57) as norm compatibility.

Proposition 8.2. *We have*

$$u_1(\sigma, H) \equiv u_2(\sigma, H) \pmod{E_+(\mathfrak{f})}$$

Proof. Let V be a finite index subgroup of $E_+(\mathfrak{f})$ of rank $n-1$ satisfying the conditions given in the statement of Proposition 3.8. Furthermore, we choose V such that if $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$ then the ε_i along with π satisfy Lemma 4.5. Let V' be a finite index subgroup of E_+ of rank $n-1$, containing V, such that $[E_+ : V'] = [E_+(\mathfrak{f}) : V]$.

By Theorem 7.1 and Proposition 6.6 we have

$$u_2(V', \sigma) = u_3(V', \sigma) = u_3'(V, \sigma).$$

We recall from §6.3 the explicit description of $u_3'(V, \sigma)$,

$$u_3'(V, \sigma) = c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \eta_{\mathfrak{p},V}) = \prod_{i=1}^{n-1} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b},\mathcal{B}_i,\pi\mathcal{O}_\mathfrak{p},0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b},\mathcal{B},\mathcal{O}_\mathfrak{p},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d(\zeta_{R,\lambda}(\mathfrak{b},\mathcal{B},x,0))(x).$$

We have defined

$$u_1(V, \mathcal{B}, \sigma) = \prod_{\epsilon \in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathcal{B}\cap\pi^{-1}\mathcal{B},\mathcal{O}_\mathfrak{p},0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b},\mathcal{B},\mathcal{O}_\mathfrak{p},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathfrak{b}, \mathcal{B}, x),$$

where

$$\mathcal{B} = \bigcup_{\tau \in S_{n-1}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]).$$

Thus, $u_1(V, \mathcal{B}, \sigma) \equiv u_3'(V, \sigma) \pmod{E_+(\mathfrak{f})}$ and hence

$$u_1(V, \mathcal{B}, \sigma) \equiv u_2(V', \sigma) \pmod{E_+(\mathfrak{f})}. \tag{58}$$

It follows from (58), Proposition 3.8 and Proposition 5.8 that there exists $\alpha \in F_{\mathfrak{p}}^*$ such that

$$\alpha u_1(\sigma, H) = u_2(\sigma, H) \quad \text{and} \quad \alpha^{[E_+:V]} \in E_+(\mathfrak{f}). \tag{59}$$

Note that here we are also using Proposition 3.7. By Lemma 4.9, we can choose W to be a finite index subgroup of $E_+(\mathfrak{f})$ satisfying the same conditions as V but with $[E_+(\mathfrak{f}) : W]$ coprime to $[E_+(\mathfrak{f}) : V]$. Thus we also have $\alpha^{[E_+(\mathfrak{f}):W]} \in E_+(\mathfrak{f})$, which combines with (59) to yield $\alpha \in E_+(\mathfrak{f})$ as desired. □

Assuming that (57) holds we can prove the following.

Proposition 8.3. *Suppose that (57) holds and that $R \neq R_\infty$. Then,*

$$u_1 = u_2.$$

Proof. From Proposition 8.2 we have that for each $\tau \in G'$,

$$u_1(\tau, H') \equiv u_2(\tau, H') \pmod{E_+(\mathfrak{f}\mathfrak{f}')}.$$

Our assumption that (57) holds then gives that for each $\sigma \in G$,

$$u_1(\sigma, H) \equiv u_2(\sigma, H) \pmod{E_+(\mathfrak{f}\mathfrak{f}')}.$$

Since $R \neq R_\infty$, we have

$$\bigcap_{\mathfrak{f}'} E_+(\mathfrak{f}\mathfrak{f}') = \{1\}.$$

Here the intersection is taken over all possible ideals $\mathfrak{f}'$ divisible only by primes dividing $\mathfrak{f}$. Thus we have

$$u_1(\sigma, H) = u_2(\sigma, H).$$

□

Remark 8.4. If $R = R_\infty$ then $\mathfrak{f} = \mathfrak{f}\mathfrak{f}' = 1$ for all possible extensions. Hence, the proof of Proposition 8.3 does not apply.

To handle the case $R = R_\infty$ we extend the definition of u_1 to work with the trivial extension. For a Shintani set $\mathcal{D}$ and compact open $U \subseteq \mathcal{O}_{\mathfrak{p}}$, we define

$$\nu(\mathcal{D}, U) = \zeta_{R,T}(\mathcal{O}_F, \mathcal{D}, U, 0).$$

It is clear that

$$\nu(\mathcal{D}, U) = \sum_{\sigma_{\mathfrak{b}} \in G} \nu(\mathfrak{b}, \mathcal{D}, U). \tag{60}$$

We then define, for a Shintani domain $\mathcal{D}$,

$$u_1(F) = \left(\prod_{\epsilon \in E_+} \epsilon^{\nu(\epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})} \right) \pi^{\nu(\mathcal{D}, \mathcal{O}_{\mathfrak{p}})} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \, d\nu(\mathcal{D}, x). \tag{61}$$

By (60) and since $\mathfrak{f} = 1$ we have

$$u_1(F) = \prod_{\sigma \in G} u_1(\sigma, H).$$

Lemma 8.5. *We have*

$$u_1(F) = 1.$$

Proof. Since $\mathscr{D}$ is a Shintani domain we have

$$\nu(\mathscr{D}, \mathbb{O}_{\mathfrak{p}}) = \zeta_{R,T}(F/F, \mathbb{O}_F, 0) = 0. \tag{62}$$

Therefore the π-power term in (61) vanishes. Next, we write

$$\times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathscr{D}, x) = \frac{\times\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x \; d\nu(\mathscr{D}, x)}{\times\!\!\!\!\!\int_{\pi\mathbb{O}_{\mathfrak{p}}} x \; d\nu(\mathscr{D}, x)}. \tag{63}$$

By Lemma 2.11 we calculate

$$\times\!\!\!\!\!\!\int_{\pi\mathbb{O}_{\mathfrak{p}}} x \; d\nu(\mathscr{D}, x) = \pi^{\nu(\pi\mathscr{D}, \pi\mathbb{O}_{\mathfrak{p}})} \times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x \; d\nu(\pi^{-1}\mathscr{D}, x) = \times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x \; d\nu(\pi^{-1}\mathscr{D}, x) \tag{64}$$

since $\nu(\pi\mathscr{D}, \pi\mathbb{O}_{\mathfrak{p}}) = 0$ as in (62). Since $\mathscr{D}$ is a Shintani domain we can write

$$\pi^{-1}\mathscr{D} = \bigcup_{\epsilon \in E_+} (\epsilon\mathscr{D} \cap \pi^{-1}\mathscr{D}).$$

We then have

$$\begin{aligned} \times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x \; d\nu(\pi^{-1}\mathscr{D}, x) &= \prod_{\epsilon \in E_+} \left(\times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x \; d\nu(\epsilon\mathscr{D} \cap \pi^{-1}\mathscr{D}, x) \right) \\ &= \left(\prod_{\epsilon \in E_+} \epsilon^{\nu(\epsilon\mathscr{D} \cap \pi^{-1}\mathscr{D}, \mathbb{O}_{\mathfrak{p}})} \right) \times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x \; d\nu(\mathscr{D}, x). \end{aligned} \tag{65}$$

Combining (63), (64), and (65) yields

$$\times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathscr{D}, x) = \left(\prod_{\epsilon \in E_+} \epsilon^{\nu(\epsilon\mathscr{D} \cap \pi^{-1}\mathscr{D}, \mathbb{O}_{\mathfrak{p}})} \right)^{-1}$$

Applying the definition of $u_1(F)$ yields the desired result. □

Proposition 8.6. *Suppose that (57) holds and that $R = R_\infty$. Then,*

$$u_1 = u_2.$$

Proof. Let $\sigma \in G$. By Proposition 8.2 there exists $\varepsilon(\sigma) \in E_+$ such that

$$u_1(\sigma) = \varepsilon(\sigma) u_2(\sigma).$$

Let $\mathfrak{r}$ be a prime of F. From the equation

$$\zeta_{R \cup \{\mathfrak{r}\}}(\mathfrak{b}, \mathscr{D}, U, s) = \zeta_R(\mathfrak{b}, \mathscr{D}, U, s) - \mathrm{N}\mathfrak{r}^{-s} \zeta_R(\mathfrak{b}\mathfrak{r}^{-1}, \mathscr{D}, U, s),$$

it follows that

$$u_1(S \cup \{\mathfrak{r}\}, \sigma) = u_1(S, \sigma) u_1(S, \sigma_{\mathfrak{r}}^{-1}\sigma)^{-1}.$$

Proposition 5.1 *c)* gives the same result for u_2. We then have:

$$\begin{aligned} u_2(S,\sigma)u_2(S,\sigma_{\mathfrak{r}}^{-1}\sigma)^{-1} &= u_2(S\cup\{\mathfrak{r}\},\sigma) \\ &= u_1(S\cup\{\mathfrak{r}\},\sigma) \\ &= u_1(S,\sigma)u_1(S,\sigma_{\mathfrak{r}}^{-1}\sigma)^{-1} \\ &= \varepsilon(\sigma)\varepsilon(\sigma\sigma_{\mathfrak{r}}^{-1})^{-1}u_2(S,\sigma)u_2(S,\sigma_{\mathfrak{r}}^{-1}\sigma)^{-1}. \end{aligned} \tag{66}$$

Here, (66) is given by Proposition 8.3, which can be applied since we have added $\mathfrak{r}$ to the set R. It follows that $\varepsilon(\sigma) = \varepsilon(\sigma\sigma_{\mathfrak{r}}^{-1})$. Repeating this for all such $\mathfrak{r}$ we see that $\varepsilon(\sigma)$ is independent of $\sigma \in G$. Write $\varepsilon = \varepsilon(\sigma)$. Then

$$1 = u_1(F) = \prod_{\sigma\in G} u_1(\sigma,H) = \varepsilon^{|G|}\prod_{\sigma\in G} u_2(\sigma,H) = \varepsilon^{|G|}.$$

The last equality follows since $\prod_{\sigma\in G} u_2(\sigma,H) = 1$ by 5.1 *b),d)*. Since $\varepsilon \in E_+$, it follows that $\varepsilon = 1$. This gives the desired result. □

Theorem 8.1, under the assumption that (57) holds, then follows from the combination of Proposition 8.3 and Proposition 8.6. In the next section we prove the norm compatibility property, (57), for u_1 and u_2.

9 Norm compatibility relations

In this appendix we prove norm compatibility properties for u_1 and u_2. These results appear in [5] and [8] without proof.

9.1 Norm compatibility for u_1

The reciprocity map identifies $\mathrm{Gal}(H'/H)$ with

$$\{\beta \in (\mathcal{O}_F/\mathfrak{f}\mathfrak{f}')^* \mid \beta \equiv 1 \pmod{\mathfrak{f}}\}/E_+(\mathfrak{f})_{\mathfrak{p}}. \tag{67}$$

We let $\mathcal{D}_{\mathfrak{f}}$ be a Shintani domain for $E_+(\mathfrak{f})$ and define

$$\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma\mathcal{D}_{\mathfrak{f}},$$

where the union is over a set of representatives $\{\gamma\}$ for $E_+(\mathfrak{f}\mathfrak{f}')$ in $E_+(\mathfrak{f})$. Let e' be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}\mathfrak{f}'}$, and suppose that $\mathfrak{p}^{e'} = (\pi')$ with π' totally positive and $\pi' \equiv 1 \pmod{\mathfrak{f}\mathfrak{f}'}$. We can choose π' such that $\pi' = \pi^\alpha$ for some $\alpha \geq 1$. We then define $\mathbb{O}' = \mathcal{O}_{\mathfrak{p}} - \pi'\mathcal{O}_{\mathfrak{p}}$.

Let B denote a set of totally positive elements of $\mathcal{O}_F$ which are relatively prime to S and $\overline{\lambda}$ and whose images in $(\mathcal{O}_F/\mathfrak{f}\mathfrak{f}')^*$ are a set of distinct representatives for (67).

The following theorem is stated without proof by the first author in [5, Theorem 7.1]. For completeness we include a proof of this result here.

Theorem 9.1 (Theorem 7.1, [5]). *We have*

$$u_T(\mathfrak{b},\mathcal{D}_{\mathfrak{f}}) = \prod_{\beta\in B} u_T(\mathfrak{b}(\beta),\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}).$$

The key to the proof of Theorem 9.1 is to use translation properties of Shintani sets. For a subset A of equivalence classes of (67), let $\nu_A(\mathfrak{b}, \mathcal{D}, U) = \zeta_{R,\lambda}^A(\mathfrak{b}, \mathcal{D}, U, 0)$, where ζ_R^A is the zeta function

$$\zeta_R^A(\mathfrak{b}, \mathcal{D}, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in \mathfrak{b}^{-1} \cap \mathcal{D},\ \alpha \in U \\ \alpha \in A,\ (\alpha, R) = 1}} \mathrm{N}\alpha^{-s}.$$

This definition extends to $\zeta_{R,\lambda}^A$ as in (7). Throughout this section we will use the following simple equality:

$$\nu_{\{\pi^{-1}\}}(\mathfrak{b}, \mathcal{D}, U) = \nu_{\{1\}}(\mathfrak{b}, \pi\mathcal{D}, \pi U).$$

This follows from Lemma 2.11. Recall the following definition. For $\beta \in B$,

$$u_T(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}) = \epsilon(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi')(\pi')^{\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0)} \times\!\!\!\!\!\!\int_{\mathbb{O}'} x \; d\nu(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x).$$

It is clear from the definition of B that Theorem 9.1 follows from the following proposition.

Proposition 9.2. *Let $\beta \in B$. We have*

$$\begin{aligned} &u_T(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}) \\ &\quad = \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_B(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} \right) \pi^{\nu_B(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_B(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}}, x). \end{aligned}$$

The proof of Theorem 9.2 is largely an exercise in explicit calculation. We begin by considering the multiplicative integral in $u_T(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'})$.

Lemma 9.3. *We have*

$$\begin{aligned} &\times\!\!\!\!\!\!\int_{\mathbb{O}'} x \; d\nu(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x) \\ &\quad = \left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i \mathbb{O})} \right) \left(\prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})} \right) \\ &\qquad \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})} \right) \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_B(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}}, x). \end{aligned}$$

Proof. Since $\pi' = \pi^\alpha$ and $\mathbb{O}' = \mathcal{O}_{\mathfrak{p}} - \pi'\mathcal{O}_{\mathfrak{p}}$ we have $\mathbb{O}' = \bigcup_{i=0}^{\alpha-1} \pi^i \mathbb{O}$. Then

$$\times\!\!\!\!\!\!\int_{\mathbb{O}'} x \; d\nu(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x) = \prod_{i=0}^{\alpha-1} \times\!\!\!\!\!\!\int_{\pi^i\mathbb{O}} x \; d\nu(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x).$$

By changing variable and then factoring out π^i we have

$$\begin{aligned} I(\beta) &:= \left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O})} \right) \prod_{i=0}^{\alpha-1} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i x) \\ &= \left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O})} \right) \prod_{i=0}^{\alpha-1} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x). \end{aligned}$$

We now note that we can write, for $i = 1, \ldots, \alpha - 1$,

$$\pi^{-i}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} (\epsilon\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}).$$

Then,

$$\begin{aligned}
&\prod_{i=0}^{\alpha-1} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, x) \\
&= \prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, x) \\
&= \left(\prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})} \right) \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_A(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, x).
\end{aligned}$$

Here $A = \{1, \pi^{-1}, \ldots, \pi^{\alpha-1}\}$. Then since $\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma\mathscr{D}_{\mathfrak{f}}$ we can write

$$\times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_A(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, x) = \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right) \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_{\langle A, E\rangle}(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}}, x)$$

where $E = E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')$. Thus we have, noting that $B = AE = \{ae \mid a \in A, e \in E\}$,

$$\begin{aligned}
I(\beta) = &\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O})} \right) \left(\prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})} \right) \\
&\left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right) \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_B(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}}, x).
\end{aligned}$$

□

We now consider the powers of π given in the definition of $u_T(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'})$ and arising in the statement of Lemma 9.3. Recall that $\pi' = \pi^\alpha$.

Lemma 9.4.

$$\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O})} \right) \pi^{\alpha\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0)} = \pi^{\nu_B(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})}.$$

Proof. Since $\pi^i\mathbb{O} = \pi^i\mathcal{O}_{\mathfrak{p}} - \pi^{i+1}\mathcal{O}_{\mathfrak{p}}$ we have by a telescope argument

$$\sum_{i=1}^{\alpha-1} i\nu(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O}) = -(\alpha - 1)\nu(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^\alpha\mathcal{O}_{\mathfrak{p}}) + \sum_{i=1}^{\alpha-1} \nu(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathcal{O}_{\mathfrak{p}}).$$

Recalling the definition of $\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}$ we also note that for $i = 0, \ldots, \alpha - 1$ we have

$$\nu(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathcal{O}_{\mathfrak{p}}) = \nu_E(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}}, \pi^i\mathcal{O}_{\mathfrak{p}}).$$

Thus we can calculate, using the fact that $\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0) = \nu(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O}_{\mathfrak{p}})$,

$$\begin{aligned}
&\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O})}\right)\pi^{\alpha\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F,\mathfrak{b}(\beta),0)}\\
&=\left(\prod_{i=1}^{\alpha-1} \pi^{\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O}_{\mathfrak{p}})}\right)\pi^{-(\alpha-1)\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^\alpha\mathbb{O}_{\mathfrak{p}})}\pi^{\alpha\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\mathbb{O}_{\mathfrak{p}})}\\
&=\left(\prod_{i=1}^{\alpha-1} \pi^{\nu_E(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\pi^i\mathbb{O}_{\mathfrak{p}})}\right)\pi^{-(\alpha-1)\nu_E(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\pi^\alpha\mathbb{O}_{\mathfrak{p}})}\pi^{\alpha\nu_E(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\mathbb{O}_{\mathfrak{p}})}.
\end{aligned}$$

By Lemma 2.11 we have for $i = 1, \dots, \alpha$,

$$\nu_E(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}}, \pi^i\mathbb{O}_{\mathfrak{p}}) = \nu_{E,\{\pi^{-i}\}}(\mathfrak{b}(\beta), \pi^{-i}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}}).$$

We can then write $\pi^i\mathscr{D}_{\mathfrak{f}} = \bigcup_{\delta\in E_+(\mathfrak{f})} \delta\mathscr{D}_{\mathfrak{f}} \cap \pi^{-i}\mathscr{D}_{\mathfrak{f}}$. Then

$$\begin{aligned}
\nu_{E,\{\pi^{-i}\}}(\mathfrak{b}, \pi^{-i}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}}) &= \sum_{\delta\in E_+(\mathfrak{f})} \nu_{E,\{\pi^{-i}\}}(\mathfrak{b}, \delta\mathscr{D}_{\mathfrak{f}}\cap\pi^{-i}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}})\\
&= \sum_{\delta\in E_+(\mathfrak{f})} \nu_{E,\{\pi^{-i}\}}(\mathfrak{b}, \mathscr{D}_{\mathfrak{f}}\cap\delta\pi^{-i}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}})\\
&= \nu_{E,\{\pi^{-i}\}}(\mathfrak{b}, \mathscr{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}}).
\end{aligned}$$

Remarking that $\{\pi^{-\alpha}\} = \{1\}$, we deduce that

$$\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O})}\right)\pi^{\alpha\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F,\mathfrak{b}(\beta),0)} = \prod_{i=0}^{\alpha-1}\pi^{\nu_{E,\{\pi^{-i}\}}(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\mathbb{O}_{\mathfrak{p}})} = \pi^{\nu_{\langle A,E\rangle}(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\mathbb{O}_{\mathfrak{p}})}.$$

Noting that $B = AE$ completes the proof. □

We now consider the error term in the definition of $u_T(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'})$ and the products of elements of $E_+(\mathfrak{f})$ that arise in Lemma 9.3. Considering Lemma 9.3 and Lemma 9.4 we can see that to prove Proposition 9.2 it is enough to prove the following.

Proposition 9.5. *Let*

$$\begin{aligned}
\mathrm{Err}(\beta) = \epsilon(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi')&\left(\prod_{i=0}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}\cap\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\mathbb{O})}\right)\\
&\times\left(\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_A(\mathfrak{b}(\beta),\gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O})}\right).
\end{aligned}$$

Then

$$\mathrm{Err}(\beta) = \prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{\nu_B(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O}_{\mathfrak{p}})}.$$

For clarity, we shall perform the calculations required for this proposition in a few lemmas.

Lemma 9.6. *We have*

$$\mathrm{Err}(\beta) = \left(\prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{\nu_E(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O}_{\mathfrak{p}})}\right)\left(\prod_{i=1}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O})}\right).$$

Proof. Considering the definition of $\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}$ we calculate

$$\epsilon(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi') \tag{68}$$

$$= \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O}_\mathfrak{p})} \tag{69}$$

$$= \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O}_\mathfrak{p})} \tag{70}$$

$$= \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{-\nu(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_\mathfrak{p})} \right) \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O}_\mathfrak{p})} \right). \tag{71}$$

Similarly we have

$$\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O}_\mathfrak{p})}$$

$$= \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu(\mathfrak{b}(\beta), \gamma\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_\mathfrak{p})} \right) \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_E(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_\mathfrak{p})} \right). \tag{72}$$

We also calculate for $i = 1, \ldots, \alpha - 1$

$$\prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})}$$

$$= \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{-\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}) + \nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right)$$
$$\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})}. \tag{73}$$

We now note the following equalities, both of which hold via telescoping sum arguments.

1.
$$\prod_{i=1}^{\alpha-1} \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{-\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right) \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right)$$
$$= \prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})}$$

2.
$$\prod_{i=1}^{\alpha-1} \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right)$$
$$= \prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \pi\mathbb{O}_\mathfrak{p}) - \nu_{\{\pi^{-\alpha}\}}(\mathfrak{b}(\beta), \gamma\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_\mathfrak{p})}$$
$$= \prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \pi\mathbb{O}_\mathfrak{p}) - \nu(\mathfrak{b}(\beta), \gamma\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}_\mathfrak{p})}.$$

Combining these two equalites with the calculations in (71), (72) and (73) gives the result. □

If $\alpha = 1$ then Lemma 9.6 is equivalent to Proposition 9.5 and thus we are finished in the case $\alpha = 1$. From this point on we assume that $\alpha > 1$.

Lemma 9.7. *If* $\alpha > 1$ *then*

$$\mathrm{Err}(\beta) = \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_B(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}})}\right)\left(\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}})}\right) \prod_{i=1}^{\alpha-1}\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \pi\mathbb{O}_{\mathfrak{p}})} \prod_{i=2}^{\alpha-1}\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})}.$$

Proof. For $i = 2, \ldots, \alpha$ we have

$$\pi^i\mathcal{D}_{\mathfrak{f}} = \bigcup_{\delta \in E_+(\mathfrak{f})} \pi^{-1}\delta\mathcal{D}_{\mathfrak{f}} \cap \pi^{-i}\mathcal{D}_{\mathfrak{f}}.$$

Thus, applying this to the result of Lemma 9.6, we have

$$\mathrm{Err}(\beta) = \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_E(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}})}\right)\left(\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}})}\right) \prod_{i=1}^{\alpha-1}\left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})}\right).$$

Remarking that $\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-1}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})} = 1$, since $\epsilon\beta^{-1}\pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathcal{D}_{\mathfrak{f}} = \varnothing$, gives the result. ☐

If $\alpha = 2$ it is straightforward to see that Lemma 9.7 is equivalent to Proposition 9.5 and thus we are also finished in the case $\alpha = 2$. From this point on we assume that $\alpha > 2$. From Lemma 9.7 one can see that to prove Proposition 9.5 it is enough for us to show

$$1 = \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}})} \prod_{i=1}^{\alpha-1}\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \pi\mathbb{O}_{\mathfrak{p}})} \prod_{i=2}^{\alpha-1}\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})}. \quad (74)$$

To do this we first show the following lemma.

Lemma 9.8. *We have that for $j = 1, \dots, \alpha-1$ the right hand side of (74) is equal to*

$$e(j) = \left(\prod_{\delta\in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-j}\mathscr{D}_\mathfrak{f}\cap\pi^{-\alpha}\beta^{-1}\mathscr{D}_\mathfrak{f},\mathscr{O}_\mathfrak{p})}\right)$$
$$\prod_{i=j}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-(j-1)}\mathscr{D}_\mathfrak{f}\cap\pi^{-j}\beta^{-1}\mathscr{D}_\mathfrak{f},\pi\mathscr{O}_\mathfrak{p})}$$
$$\prod_{i=j+1}^{\alpha-1}\prod_{\delta\in E_+(\mathfrak{f})} \delta^{\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-j}\mathscr{D}_\mathfrak{f}\cap\pi^{-i}\beta^{-1}\mathscr{D}_\mathfrak{f},\mathbb{O})}.$$

Note that for $j = \alpha-1$ the last product is empty. We also remark that it is implicit in the statement of this lemma that $e(1) = \cdots = e(\alpha-1)$.

Proof. We prove this by induction. The case $j = 1$ holds trivially. We now assume it holds for j and prove the result for $j+1$, i.e., we show $e(j) = e(j+1)$. To do this we note that for $i = j+2, \dots, \alpha$, we have

$$\pi^{-i}\mathscr{D}_\mathfrak{f} = \bigcup_{\kappa\in E_+(\mathfrak{f})} \pi^{-(j+1)}\kappa\mathscr{D}_\mathfrak{f} \cap \pi^{-i}\mathscr{D}_\mathfrak{f}.$$

Thus, $e(j)$ is equal to the product of the following elements:

$$\left(\prod_{\delta\in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-j}\mathscr{D}_\mathfrak{f}\cap\pi^{-(j+1)}\beta^{-1}\mathscr{D}_\mathfrak{f},\mathscr{O}_\mathfrak{p})}\right)\left(\prod_{\kappa\in E_+(\mathfrak{f})} \kappa^{\nu_E(\mathfrak{b}(\beta),\kappa\beta^{-1}\pi^{-(j+1)}\mathscr{D}_\mathfrak{f}\cap\pi^{-\alpha}\beta^{-1}\mathscr{D}_\mathfrak{f},\mathscr{O}_\mathfrak{p})}\right) \tag{75}$$

$$\prod_{i=j}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-(j-1)}\mathscr{D}_\mathfrak{f}\cap\pi^{-j}\beta^{-1}\mathscr{D}_\mathfrak{f},\pi\mathscr{O}_\mathfrak{p})} \tag{76}$$

$$\prod_{i=j+1}^{\alpha-1}\prod_{\delta\in E_+(\mathfrak{f})} \delta^{\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-j}\mathscr{D}_\mathfrak{f}\cap\pi^{-(j+1)}\beta^{-1}\mathscr{D}_\mathfrak{f},\mathbb{O})} \tag{77}$$

$$\prod_{i=j+2}^{\alpha-1}\prod_{\kappa\in E_+(\mathfrak{f})} \kappa^{\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\kappa\beta^{-1}\pi^{-(j+1)}\mathscr{D}_\mathfrak{f}\cap\pi^{-i}\beta^{-1}\mathscr{D}_\mathfrak{f},\mathbb{O})}. \tag{78}$$

We remark that the first bracketed term in (75), and (78) are already products in $e(j+1)$. We now consider (77) and calculate that it is equal to

$$\left(\prod_{i=j+1}^{\alpha-1}\prod_{\delta\in E_+(\mathfrak{f})} \delta^{\nu_{\langle E,\{\pi^{-(i-1)}\}\rangle}(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-(j-1)}\mathscr{D}_\mathfrak{f}\cap\pi^{-j}\beta^{-1}\mathscr{D}_\mathfrak{f},\pi\mathscr{O}_\mathfrak{p})}\right)$$
$$\left(\prod_{i=j+1}^{\alpha-1}\prod_{\delta\in E_+(\mathfrak{f})} \delta^{-\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-j}\mathscr{D}_\mathfrak{f}\cap\pi^{-(j+1)}\beta^{-1}\mathscr{D}_\mathfrak{f},\pi\mathscr{O}_\mathfrak{p})}\right). \tag{79}$$

We now consider the way the terms in (79) interact with (76). Multiplying (79) by (76) gives

$$\left(\prod_{i=j+1}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-j}\mathscr{D}_\mathfrak{f}\cap\pi^{-(j+1)}\beta^{-1}\mathscr{D}_\mathfrak{f},\pi\mathscr{O}_\mathfrak{p})}\right)$$
$$\left(\prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E,\{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-(j-1)}\mathscr{D}_\mathfrak{f}\cap\pi^{-j}\beta^{-1}\mathscr{D}_\mathfrak{f},\pi\mathscr{O}_\mathfrak{p})}\right). \tag{80}$$

The first term in (80) is the term we were missing from $e(j+1)$. Thus it only remains to show that the second bracketed term in (75) multiplied by the second bracketed term in (80) is equal to 1. This is shown by the following calculation,

$$\begin{aligned}&\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-j}\mathcal{D}_\mathfrak{f}\cap\pi^{-(j+1)}\beta^{-1}\mathcal{D}_\mathfrak{f},\mathcal{O}_\mathfrak{p})}\\ &=\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_{\langle E,\{\pi\}\rangle}(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-(j-1)}\mathcal{D}_\mathfrak{f}\cap\pi^{-j}\beta^{-1}\mathcal{D}_\mathfrak{f},\pi\mathcal{O}_\mathfrak{p})}\\ &=\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_{\langle E,\{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-(j-1)}\mathcal{D}_\mathfrak{f}\cap\pi^{-j}\beta^{-1}\mathcal{D}_\mathfrak{f},\pi\mathcal{O}_\mathfrak{p})}.\end{aligned}$$

We therefore deduce that

$$e(j)=e(j+1)$$

as claimed. This completes the proof of the lemma. ☐

We are now ready to prove Proposition 9.5.

Proof of Proposition 9.5. We consider $e(\alpha-1)$. From Lemma 9.8, we have that $e(\alpha-1)$ is equal to the right hand side of (74). Then

$$e(\alpha-1)=\left(\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-(\alpha-1)}\mathcal{D}_\mathfrak{f}\cap\pi^{-\alpha}\beta^{-1}\mathcal{D}_\mathfrak{f},\mathcal{O}_\mathfrak{p})}\right)$$
$$\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{-\nu_{\langle E,\{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-(\alpha-2)}\mathcal{D}_\mathfrak{f}\cap\pi^{-(\alpha-1)}\beta^{-1}\mathcal{D}_\mathfrak{f},\pi\mathcal{O}_\mathfrak{p})}.$$

Since $\{\pi\}=\{\pi^{-(\alpha-1)}\}$, it is clear that

$$e(\alpha-1)=1.$$

This completes the proof of Proposition 9.5 and thus proves Theorem 9.1. ☐

9.2 Norm compatibility for u_2

We recall the definition

$$u_2=\sum_{\sigma\in G}u_2(\sigma)\otimes[\sigma^{-1}]=\mathrm{Eis}_F^0\cap\Delta_*(c_{\mathrm{id}}\cap\rho_{H/F}).$$

Theorem 9.9. *We have for any $\sigma\in G$,*

$$u_2(\sigma,H)=\prod_{\substack{\tau\in G'\\ \tau|_H=\sigma}}u_2(\tau,H').$$

Remark 9.10. This theorem has been stated without proof by the first author with Spieß in Proposition 5.1. We include the proof for completeness. We note also that the proof of the norm compatibility for u_2 is much simpler than that for u_1. This is a result of the additional structure we have due to the cohomological nature of the construction.

Proof of Theorem 9.9. We consider the natural map

$$\psi : F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G'] \to F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$$
$$\sum_{\tau \in G'} n_\tau \otimes [\tau] \mapsto \sum_{\sigma \in G} \Big(\prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} n_\tau \Big) \otimes [\sigma].$$

Then, on the one hand,

$$\psi(u_2(H')) = \sum_{\sigma \in G} \Big(\prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} u_2(\tau, H') \Big) \otimes [\sigma].$$

On the other hand

$$\begin{aligned} \psi(u_2(H')) &= \psi(\mathrm{Eis}_F^0 \cap \Delta_*(c_{\mathrm{id}} \cap \rho_{H'/F})) \\ &= \mathrm{Eis}_F^0 \cap \psi_* \Delta_*(c_{\mathrm{id}} \cap \rho_{H'/F}) \\ &= \mathrm{Eis}_F^0 \cap \Delta_*(c_{\mathrm{id}} \cap \psi_* \rho_{H'/F}). \end{aligned}$$

The only equality of note here is the final one. This follows since we can commute ψ_* with Δ_*, which follows from the calculations of §5.1 and §5.3. Then since $\psi_* \rho_{H'/F} = \rho_{H/F}$, the desired result follows. □

References

[1] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[2] Pierrette Cassou-Noguès. Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques. *Invent. Math.*, 51(1):29–59, 1979.

[3] Pierre Charollois, Samit Dasgupta, and Matthew Greenberg. Integral Eisenstein cocycles on $\mathbf{GL}_n$, II: Shintani's method. *Comment. Math. Helv.*, 90(2):435–477, 2015.

[4] Pierre Colmez. Résidu en $s = 1$ des fonctions zêta p-adiques. *Invent. Math.*, 91(2):371–389, 1988.

[5] Samit Dasgupta. Shintani zeta functions and Gross-Stark units for totally real fields. *Duke Math. J.*, 143(2):225–279, 2008.

[6] Samit Dasgupta and Mahesh Kakde. Brumer-Stark units and Hilbert's 12th problem, preprint. `https://arxiv.org/pdf/2103.02516.pdf`.

[7] Samit Dasgupta and Mahesh Kakde. On the Brumer-Stark conjecture, preprint. `https://arxiv.org/pdf/2010.00657.pdf`.

[8] Samit Dasgupta and Michael Spieß. Partial zeta values, Gross's tower of fields conjecture, and Gross-Stark units. *J. Eur. Math. Soc. (JEMS)*, 20(11):2643–2683, 2018.

[9] Samit Dasgupta and Michael Spiess. On the characteristic polynomial of the Gross regulator matrix. *Trans. Amer. Math. Soc.*, 372(2):803–827, 2019.

[10] Pierre Deligne and Kenneth A. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

[11] Benedict H. Gross. On the values of abelian L-functions at $s = 0$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 35(1):177–197, 1988.

[12] Matthew H. Honnor. Comparing two formulas for the Gross-Stark units. `https://arxiv.org/pdf/2108.06402.pdf`.

[13] Takuro Shintani. On evaluation of zeta functions of totally real algebraic number fields at non-positive integers. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 23(2):393–417, 1976.

[14] Michael Spieß. On special zeros of p-adic L-functions of Hilbert modular forms. *Invent. Math.*, 196(1):69–138, 2014.

[15] Michael Spieß. Shintani cocycles and the order of vanishing of p-adic Hecke L-series at $s = 0$. *Math. Ann.*, 359(1-2):239–265, 2014.

[16] John Tate. On Stark's conjectures on the behavior of $L(s, \chi)$ at $s = 0$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(3):963–978 (1982), 1981.