A SHINTANI-TYPE FORMULA FOR GROSS–STARK UNITS OVER FUNCTION FIELDS

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Abstract. Let F be a totally real number field of degree n, and let H be a finite abelian extension of F. Let $\mathfrak{p}$ denote a prime ideal of F that splits completely in H. Following Brumer and Stark, Tate conjectured the existence of a $\mathfrak{p}$-unit u in H whose $\mathfrak{p}$-adic absolute values are related in a precise way to the partial zeta-functions of the extension H/F. Gross later refined this conjecture by proposing a formula for the $\mathfrak{p}$-adic norm of the element u. Recently, using methods of Shintani, the first author refined the conjecture further by proposing an exact formula for u in the $\mathfrak{p}$-adic completion of H.

In this article we state and prove a function field analogue of this Shintani-type formula. The role of the totally real field F is played by the function field of a curve over a finite field in which n places have been removed. These places represent the "real places" of F. Our method of proof follows that of Hayes, who proved Gross's conjecture for function fields using the theory of Drinfeld modules and their associated exponential functions.

1. Introduction

Let F be a totally real number field of degree n, and let $\mathfrak{p}$ be a prime ideal of F. Let K be a finite abelian extension of F such that $\mathfrak{p}$ splits completely in K. Tate [10] stated an analogue of the rank one abelian Stark conjecture that predicts the existence of a $\mathfrak{p}$-unit $u \in K$ whose valuations at the places above $\mathfrak{p}$ are determined in a precise fashion by the values at 0 of the partial zeta-functions of K/F (see Conjecture 3.1 below). He called the conjecture the Brumer–Stark conjecture because it generalized Stark's conjectures for archimedean primes as well as work of Brumer on ideal class groups. Later, Gross refined the Brumer–Stark conjecture by predicting a formula for the image of the Stark unit u under the local reciprocity map of class field theory attached to certain auxiliary extensions L/F containing K [5]. Recently, the first author stated a further refinement that proposes an exact formula for u in the completion of K at a prime ideal lying above $\mathfrak{p}$ ([2, Conjecture 3.21], restated as Conjecture 4.2 below). This formula was inspired by work of Shintani, who used a geometry of numbers approach towards calculating the special values of the partial zeta functions of F [9].

To be precise, suppose the n real embeddings of F are labelled $\iota_1, \dots, \iota_n$. Define

$$\iota : F \to \mathbf{R}^n \tag{1}$$

by $\iota(x) = (\iota_i(x))_{i=1}^n$. The group $F^\times$ acts on $\mathbf{R}^n$ via application of ι and componentwise multiplication. Let $\mathcal{O}$ be the ring of integers of F, and let Q be the positive orthant $(\mathbf{R}^{>0})^n \subset \mathbf{R}^n$. For any ideal $\mathfrak{f} \subset \mathcal{O}$, the group $E(\mathfrak{f})$ of totally positive units of $\mathcal{O}$ congruent to 1 modulo $\mathfrak{f}$ acts properly discontinuously on Q. Shintani proved that there exists a fundamental domain $\mathcal{D}$ for the action of $E(\mathfrak{f})$ on Q that consists of a union of simplicial cones. Let $H_\mathfrak{f}$ be the

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narrow ray class field of F of conductor $\mathfrak{f}$, and let H be the maximal subextension of F in which $\mathfrak{p}$ splits completely. Conjecture 4.2 gives a $\mathfrak{p}$-adic formula for the Stark unit $u \in H$ in terms of the Shintani zeta functions attached to the domain $\mathcal{D}$.

The main result of the present article is the proof of a function field analogue of Conjecture 4.2. The Brumer–Stark conjecture itself was already proven in the function field context by Deligne ([3], see also [10, Chapitre V]), and Gross's refinement was proven in this setting by Hayes [7].

Let F be the function field of a smooth projective algebraic curve over $\mathbf{F}_q$. Fix arbitrary places $\infty_1, \ldots, \infty_n$ of F, which will be viewed as the "infinite places" of F.[1] Let $\mathcal{O}$ be the ring of elements of F that are integral away from $\infty_1, \ldots, \infty_n$. Let $\mathfrak{f} \subset \mathcal{O}$ be an ideal and let $H_\mathfrak{f}$ be the narrow ray class field of F of conductor $\mathfrak{f}$, defined in Section 2 following Hayes [8]. The field $H_\mathfrak{f}$ is a finite abelian extension of F unramified outside $\mathfrak{f}\infty$, where ∞ denotes the product of the infinite places. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ not dividing $\mathfrak{f}$, and let H be the maximal subextension of $H_\mathfrak{f}/F$ in which $\mathfrak{p}$ splits completely. Our analogue of Conjecture 4.2, stated in Theorem 5.2 below, is an explicit formula for the Stark unit u in H.

Let d_{∞_i} denote the degree of the infinite place ∞_i. Let F_{∞_i} be the completion of F at ∞_i. Let $\mathcal{O}_{\infty_i}$ be the ring of integers of F_{∞_i} and let k_{∞_i} be its residue field. We normalize the absolute value on F_{∞_i} by

$$|x|_{\infty_i} = q^{-d_{\infty_i} v_{\infty_i}(x)},$$

where $v_{\infty_i} : F_{\infty_i}^\times \to \mathbf{Z}$ is the surjective valuation associated to ∞_i. Choose a uniformizer π_{∞_i} for F_{∞_i}. Define a sign function

$$\operatorname{sgn}_{\infty_i} : F_{\infty_i}^\times \to k_{\infty_i}^\times \tag{2}$$

as the unique homomorphism such that the restriction $\operatorname{sgn}_{\infty_i}|_{\mathcal{O}_{\infty_i}^\times}$ is the reduction map

$$\mathcal{O}_{\infty_i}^\times \to k_{\infty_i}^\times,$$

and such that $\operatorname{sgn}_{\infty_i}(\pi_{\infty_i}) = 1$. We say that $x \in F$ is totally positive and write $x \gg 0$ if $\operatorname{sgn}_{\infty_i}(x) = 1$ for each i. Let

$$\iota : F^\times \to \prod_{i=1}^{n} (\mathbf{R}^{>0} \times k_{\infty_i}^\times) \tag{3}$$

denote the map given by $\iota(x) = (|x|_{\infty_i}, \operatorname{sgn}_{\infty_i}(x))$. Note that unlike the number field setting, the map ι is not injective. Once again $F^\times$ acts on the codomain of ι through the application of ι and componentwise multiplication. Let

$$Q := \prod_{i=1}^{n} (\mathbf{R}^{>0} \times 1)$$

[1] In many applications, one considers the function field $\mathbf{F}_q(T)$ with the single infinite place ∞ corresponding to the point at infinity on $\mathbf{P}^1$ and associated valuation

$$v_\infty(a) = -\deg a.$$

One then lets F be a finite extension of $\mathbf{F}_q(T)$ and chooses the infinite places $\infty_1, \ldots, \infty_n$ to be the places of F lying above the place ∞ of $\mathbf{F}_q(T)$. This family of examples is the one most closely analogous to the case of number fields; however, the results in this article are valid for any arbitrary choice of places at infinity.

be the "positive orthant" inside $\prod_{i=1}^n(\mathbf{R}^{>0}\times k_{\infty_i}^\times)$. If $E(\mathfrak{f})$ denotes the group of totally positive units of $\mathcal{O}$ congruent to 1 modulo $\mathfrak{f}$, then we define a Shintani domain $\mathcal{D}$ to be a fundamental domain for the action of $E(\mathfrak{f})$ on Q.

Theorem 5.2 gives a formula for the Stark unit u in H in terms of the Shintani zeta functions attached to $\mathcal{D}$. Our method of proof is to relate the $\mathfrak{p}$-adic integral appearing in this formula to the infinite products considered by Hayes in [7]. These infinite products arise from the exponential functions associated to certain rank one Drinfeld modules. Hayes had proven the algebraicity of his infinite products using the algebraicity properties of the special values of exponential functions, which in turn follow from the moduli theory of Drinfeld modules.

From the point of view of Stark's conjectures, the introduction of places at infinity is somewhat artificial in the function field context. Indeed, the theorems of Deligne and Hayes work entirely at $\mathfrak{p}$ and do not employ these auxiliary places. The reason we have introduced the infinite places in this article is to place ourselves in a setting of a function field extension H/F for which Gross's conjecture does not provide an explicit $\mathfrak{p}$-adic formula for the Stark unit u in H. Rather, the group of units in $E(\mathfrak{f})$ provides a kernel in the local reciprocity map, which thereby introduces an ambiguity in Gross's formula for u. The idea of introducing a domain $\mathcal{D}$ is used to eliminate this ambiguity. Hayes considers certain extensions of F (denoted $K_\mathfrak{m}$ below) for which Gross's conjecture does provide a $\mathfrak{p}$-adic formula for the Stark unit without ambiguity. In fact, his extensions $K_\mathfrak{m}$ contain our extensions H, and one aspect of our method of proof is that the Stark unit for H arises as the norm of the Stark unit from $K_\mathfrak{m}$ (see Proposition 3.2).

While we hope that the notion of Shintani domain for function fields may have broader application for other problems in which places at infinity occur naturally, our primary motivation has been to provide theoretical evidence for Conjecture 4.2. We hope that the methods herein will shed some light on the number field setting. In particular, analogues of Propositions 8.1 through 8.3 seem highly desirable.

We now outline the contents of the article. In Section 2 we define the class fields that will play a role in this article. In Section 3 we state the Brumer–Stark conjecture. In Sections 4 and 5 we state Shintani-type formulas for Stark units in the number field and function field cases, respectively. In Section 6 we recall Hayes's product formula for Stark units over $K_\mathfrak{m}$, and in Section 7 we derive an analogous formula over the subfield H. Finally, in Section 8 we prove the Shintani-type formula for the Stark unit over H in the function field setting using the product formula from Section 7.

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2. Definitions and class fields

In this section, we define the class fields that will play an important role in this article. At the outset, F may be a totally real field or a function field. In former case, each real place ∞_i gives rise to a canonical sign homomorphism

$$\operatorname{sgn}_{\infty_i} : F_{\infty_i}^\times = \mathbf{R}^\times \to \{\pm 1\}.$$

In the function field case, the sign homomorphism sgn_{∞_i} defined in (2) depends on a choice of uniformizer π_{∞_i}.

Let $\mathfrak{f} \subset \mathcal{O}$ be an ideal. For each place v of F, let $\mathcal{O}_v$ denote the ring of integers of the completion F_v, and let $U_{v,\mathfrak{f}}$ denote the group of elements of $\mathcal{O}_v^\times$ that are congruent to 1 modulo $\mathfrak{f}\mathcal{O}_v$.

Definition. The *narrow ray class field of conductor* $\mathfrak{f}$ is defined via class field theory by

$$\mathrm{Gal}(H_\mathfrak{f}/F) = \mathbf{A}_F^\times / (F^\times \prod_{\infty_i} \ker \mathrm{sgn}_{\infty_i} \prod_{v \mid \mathfrak{f}} U_{v,\mathfrak{f}} \prod_{v \nmid \infty \mathfrak{f}} \mathcal{O}_v^\times). \tag{4}$$

The right side of (4) is canonically isomorphic to the narrow ray class group

$$\mathrm{Cl}_\mathfrak{f}^+(\mathcal{O}) := I_\mathfrak{f}(\mathcal{O})/P_\mathfrak{f}(\mathcal{O}),$$

where $I_\mathfrak{f}(\mathcal{O})$ is the group of fractional ideals of $\mathcal{O}$ relatively prime to $\mathfrak{f}$, and $P_\mathfrak{f}(\mathcal{O})$ is the group of principal fractional ideals generated by totally positive elements of $\mathcal{O}$ congruent to 1 modulo $\mathfrak{f}$.

Let $\mathfrak{p}$ be a prime of $\mathcal{O}$ not dividing $\mathfrak{f}$. Let $H = H(\mathfrak{f};\mathfrak{p})$ be the maximal subextension of $H_\mathfrak{f}/F$ in which $\mathfrak{p}$ splits completely, with Galois group

$$\begin{aligned} \mathrm{Gal}(H/F) &= \mathbf{A}_F^\times / (F^\times \prod_{\infty_i} \ker \mathrm{sgn}_{\infty_i} \prod_{v \mid \mathfrak{f}} U_{v,\mathfrak{f}} \prod_{v \nmid \infty \mathfrak{f}\mathfrak{p}} \mathcal{O}_v^\times \times F_\mathfrak{p}^\times) \\ &\cong \mathrm{Cl}_\mathfrak{f}^+(\mathcal{O})/\langle \mathfrak{p} \rangle. \end{aligned} \tag{5}$$

If F is a totally real field, class field theory implies that any finite abelian extension of F is contained in the narrow ray class field $H_\mathfrak{f}$ for some ideal $\mathfrak{f} \subset \mathcal{O}$. Therefore every such extension in which $\mathfrak{p}$ splits completely is contained in some $H(\mathfrak{f};\mathfrak{p})$. This is no longer true in the function field setting. This is because $\prod_{\infty_i} \ker \mathrm{sgn}_{\infty_i}$ is the connected component of the identity in $\mathbf{A}_F^\times$ in the number field setting, but there are abelian extensions not fixed by $\prod_{\infty_i} \ker \mathrm{sgn}_{\infty_i}$ in the function field setting.

For example, let A be the ring of elements of F integral away from $\mathfrak{p}$. Let $\mathfrak{m}$ be an ideal of A. We define a finite abelian extension $K_\mathfrak{m}$ of F via class field theory:

$$\begin{aligned} \mathrm{Gal}(K_\mathfrak{m}/F) &= \mathbf{A}_F^\times / (F^\times \prod_{v \mid \mathfrak{m}} U_{v,\mathfrak{m}} \prod_{v \nmid \mathfrak{m}\mathfrak{p}} \mathcal{O}_v^\times \times F_\mathfrak{p}^\times) \\ &\cong I_\mathfrak{m}(A)/P_\mathfrak{m}(A), \end{aligned} \tag{6}$$

where $I_\mathfrak{m}(A)$ is the group of fractional ideals of A relatively prime to $\mathfrak{m}$, and $P_\mathfrak{m}(A)$ is the group of principal fractional ideals of A generated by elements congruent to 1 modulo $\mathfrak{m}$.

Each place of F outside the infinite places and $\mathfrak{p}$ gives rise to both a prime ideal of $\mathcal{O}$ and a prime ideal of A. Thus there is a canonical bijection between the set of ideals of $\mathcal{O}$ relatively prime to $\mathfrak{p}$ and the set of ideals of A relatively prime to ∞. Furthermore, each infinite place ∞_i corresponds to a prime ideal of A. We may therefore consider the field $K_\mathfrak{m}$ with $\mathfrak{m} = \mathfrak{f}\infty$. Since the kernel of sgn_{∞_i} contains $U_{\infty_i,\mathfrak{f}\infty}$, it follows that $K_{\mathfrak{f}\infty}$ contains H. Let $E_\mathfrak{p}(\mathfrak{f})$ denote the group of totally positive $\mathfrak{p}$-units of $\mathcal{O}$ congruent to 1 modulo $\mathfrak{f}$. We have:

Proposition 2.1. *There is a canonical isomorphism*

$$\mathrm{Gal}(K_{\mathfrak{f}\infty}/H) \cong (\iota(F^\times) \cap Q)/\iota(E_\mathfrak{p}(\mathfrak{f})).$$

Proof. By inspecting (5) and (6), we observe that any element of $\mathrm{Gal}(K_{\mathfrak{f}\infty}/H)$ has an adelic representative of the form $\{x_v\}$, where $x_{\infty_i} \in \ker \mathrm{sgn}_{\infty_i}$ for each i and $x_v = 1$ for $v \nmid \infty$. We map this representative to

$$\prod_{\infty_i} |x_{\infty_i}|_{\infty_i} \times 1 \in \iota(F^\times) \cap Q. \tag{7}$$

The element in (7) is well-defined modulo $\iota(E_{\mathfrak{p}}(\mathfrak{f}))$. The resulting map

$$\mathrm{Gal}(K_{\mathfrak{f}\infty}/H) \to (\iota(F^\times) \cap Q)/\iota(E_{\mathfrak{p}}(\mathfrak{f}))$$

is surjective since the x_{∞_i} can be chosen to have arbitrary absolute value, and is easily checked to be injective. □

Proposition 2.1 will be useful in calculating the norm of the Stark unit from $K_{\mathfrak{f}\infty}$ to H.

3. Stark's conjecture

As in the previous section, let F be either a totally real number field or a function field. Let K be an abelian extension of F such that $\mathfrak{p}$ splits completely in K. Let S be a set of primes of F that contains $\infty_1, \dots, \infty_n$, $\mathfrak{p}$, and all primes ramifying in K/F. Assume that $\#S \geq 3$ and let $R = S - \{\mathfrak{p}\}$. For an ideal $\mathfrak{a} \subset \mathcal{O}$ relatively prime to R, denote by $\sigma_{\mathfrak{a}}$ the Frobenius element attached to $\mathfrak{a}$ in $\mathrm{Gal}(K/F)$. Let $\mathrm{N}\mathfrak{a} = [\mathcal{O} : \mathfrak{a}\mathcal{O}]$. In the function field setting we have

$$\mathrm{N}\mathfrak{a} = q^{\deg_\infty \mathfrak{a}},$$

where

$$\deg_\infty(\mathfrak{a}) := \sum_{\mathfrak{q} \nmid \infty} d_{\mathfrak{q}} v_{\mathfrak{q}}(\mathfrak{a}).$$

Here the sum runs over all places $\mathfrak{q}$ of F not equal to one of the ∞_i. For an element α of $\mathcal{O}$, we have the equality

$$\deg_\infty(\alpha\mathcal{O}) = -\sum_{i=1}^{n} d_{\infty_i} v_{\infty_i}(\alpha)$$

from the product formula. We write simply $\mathrm{N}\alpha = \mathrm{N}(\alpha\mathcal{O})$.

Definition. For each $\sigma \in \mathrm{Gal}(K/F)$, define the partial zeta-function

$$\zeta_R(K/F, \sigma, s) = \sum_{\substack{(\mathfrak{a}, R)=1 \\ \sigma_{\mathfrak{a}} = \sigma}} \mathrm{N}\mathfrak{a}^{-s} \tag{8}$$

for $\mathrm{Re}\, s \geq 1$. The function $\zeta_R(K/F, \sigma, s)$ extends by analytic continuation to a meromorphic function on the entire complex plane, with only a simple pole at $s = 1$.

Observe that

$$\zeta_S(K/F, \sigma, s) = (1 - \mathrm{N}\mathfrak{p}^{-s})\zeta_R(K/F, \sigma, s), \tag{9}$$

and in particular

$$\zeta_S(K/F, \sigma, 0) = 0. \tag{10}$$

It is known that the special value $\zeta_R(K/F, \sigma, 0)$ is always rational. In order to obtain a special value that is integral at 0, we introduce an auxiliary prime ideal $\eta \subset \mathcal{O}$. In the number field case, assume that the residue characteristic of η is at least 2 more than the absolute ramification index of $\mathfrak{p}$. Define the "shifted zeta function"

$$\zeta_{R,\eta}(K/F,\sigma,s) = \zeta_R(K/F,\sigma\sigma_\eta,s) - \mathrm{N}\eta^{1-s}\zeta_R(K/F,\sigma,s). \tag{11}$$

The values $\zeta_{R,\eta}(K/F,\sigma,0)$ are integral. The shifted zeta function $\zeta_{S,\eta}(K/F,\sigma,s)$ is obtained by replacing R with S in (11).

Remark. Our definition of the shifted zeta function agrees with that of [7] and differs with that of [2], where σ is replaced by $\sigma\sigma_\eta^{-1}$ in the right side of (11). This difference is merely notational.

Fix a prime $\mathfrak{P}$ of K above the prime $\mathfrak{p}$ of F. The following is Gross's formulation of the Brumer–Stark conjecture [5, Conjecture 7.4], which was originally stated by Tate in [10].

Conjecture 3.1 (Brumer–Stark–Tate)**.** *There exists a unique element $u_{K,\eta} \in K$ such that:*

- $|u_{K,\eta}|_{\mathfrak{P}'} = 1$ *for all places $\mathfrak{P}'$ of K not above $\mathfrak{p}$, including archimedean $\mathfrak{P}'$.*
- *For all $\sigma \in \mathrm{Gal}(K/F)$, we have* $\log|u_{K,\eta}^\sigma|_{\mathfrak{P}} = -\zeta'_{S,\eta}(K/F,\sigma,0)$.
- $u_{K,\eta} \equiv 1 \pmod{\mathfrak{Q}}$ *for every prime $\mathfrak{Q}$ of K lying above the prime η of F.*

Using (9), the second condition of Conjecture 3.1 can be reformulated as

$$v_{\mathfrak{P}}(u_{K,\eta}^\sigma) = \zeta_{R,\eta}(K/F,\sigma,0). \tag{12}$$

The following norm-compatibility relation of Stark units is well-known and easy to verify:

Proposition 3.2. *Let K' be a finite abelian extension of F containing K. Suppose that K'/F is unramified outside S and that $\mathfrak{p}$ splits completely in K'. If Conjecture 3.1 is true for K'/F, then it is true for K/F as well and we have*

$$u_{K,\eta} = \mathrm{N}_{K'/K}(u_{K',\eta}).$$

4. A conjectural formula for Stark's unit

In this section, F denotes a totally real number field of degree n. We recall the conjectural formula for the Stark unit $u_{H,\eta}$ presented in [2]. Here $H = H(\mathfrak{f};\mathfrak{p})$ as in (5).

We first give some preliminaries on $\mathfrak{p}$-adic integrals. Recall that $\mathcal{O}_{\mathfrak{p}}$ denotes the ring of integers of the completion $F_{\mathfrak{p}}$.

Definition. A **Z**-*valued measure* on $\mathcal{O}_{\mathfrak{p}}$ is a function μ from the set of compact open subsets of $\mathcal{O}_{\mathfrak{p}}$ to $\mathbf{Z}$ such that for disjoint compact open sets U and V, we have $\mu(U \cup V) = \mu(U) + \mu(V)$. If $X \subset \mathcal{O}_{\mathfrak{p}}$ is a compact open subset, $f : X \to F_{\mathfrak{p}}^\times$ is a continuous map, and a μ is a $\mathbf{Z}$-valued measure on $\mathcal{O}_{\mathfrak{p}}$, we define the *multiplicative integral*

$$\times\!\!\!\!\!\!\int_X f\, d\mu = \lim_{m\to\infty} \prod_{x \in F_{\mathfrak{p}}^\times / U_{\mathfrak{p},\mathfrak{p}^m}} x^{\mu(f^{-1}(x))} \in F_{\mathfrak{p}}^\times. \tag{13}$$

The products inside the limit are all finite by the compactness of X, and the limit converges because the sequence of products is evidently Cauchy.

We now define a particular measure using methods of Shintani [9] and Cassou-Nogues [1]. Recall that Q denotes the positive orthant $(\mathbf{R}^{>0})^n$.

Definition. For linearly independent vectors $v_1, \ldots, v_r$ in Q, define the *simplicial cone* $C(v_1, \ldots, v_r) \subset Q$ by

$$C(v_1, \ldots, v_r) = \left\{ \sum_{i=1}^{r} c_i v_i \mid c_i \in \mathbf{R}^{>0} \right\}. \tag{14}$$

When all the v_i belong to $\iota(F) \cap Q$, the simplicial cone $C(v_1, \ldots, v_r)$ is called a *Shintani cone*. A *Shintani set* is a subset of Q that can be written as a finite disjoint union of Shintani cones.

Theorem 4.1 (Shintani, [9])**.** *There exists a Shintani set $\mathcal{D}$ that is a fundamental domain for the action of $E(\mathfrak{f})$ on Q.*

Definition. A *Shintani domain* is a Shintani set $\mathcal{D}$ as in Theorem 4.1.

We now define the zeta functions associated to Shintani sets. First we state some technical conditions that are needed to ensure that the Shintani zeta functions have integral special values at $s = 0$.

Definition. Let η be a prime ideal of $\mathcal{O}$ such that $\mathrm{N}\eta = \ell$ is a prime number with $\ell \geq n+2$. Then η is *good* for a Shintani cone C if C can be written $C = C(v_1, \ldots, v_r)$ with each $v_i \in \mathcal{O}$, $v_i \notin \eta$. The prime η is *good* for a Shintani set $\mathcal{D}$ if $\mathcal{D}$ can be decomposed as the finite disjoint union of Shintani cones C such that η is good for each C.

Let $\mathcal{D}$ be a Shintani set (not necessarily a Shintani domain) such that η is good for $\mathcal{D}$. Assume that no prime of S has residue characteristic ℓ. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S. For each compact open subset $U \subset \mathcal{O}_{\mathfrak{p}}$, define a zeta function $\zeta_R(\mathfrak{b}, \mathcal{D}, U, s)$ by

$$\zeta_R(\mathfrak{b}, \mathcal{D}, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\alpha} \mathrm{N}\alpha^{-s}, \tag{15}$$

where the sum ranges over all $\alpha \in (\mathfrak{b}^{-1}\mathfrak{f} + 1) \cap U$ such that

$$\alpha \equiv 1 \pmod{\mathfrak{f}}, \ \iota(\alpha) \in \mathcal{D}, \text{ and } (\alpha, R) = 1.$$

It follows from [9, Proposition 1, §1.1] that the function $\zeta_R(\mathfrak{b}, \mathcal{D}, U, s)$ extends to a meromorphic function on $\mathbf{C}$ (see [2, Proposition 6.1]). Define a shifted zeta function

$$\zeta_{R,\eta}(\mathfrak{b}, \mathcal{D}, U, s) = \zeta_R(\mathfrak{b}\eta, \mathcal{D}, U, s) - \mathrm{N}\eta^{1-s}\zeta_R(\mathfrak{b}, \mathcal{D}, U, s). \tag{16}$$

If $\mathfrak{b}$ is a fractional ideal of F of the form $\mathfrak{a}\eta^{-1}$ where $\mathfrak{a}$ is relatively prime to S and to $\ell = \mathrm{N}\eta$, and η is good for $\mathcal{D}$, then the special value $\zeta_{R,\eta}(\mathfrak{b}, \mathcal{D}, U, 0)$ is an integer [2, Proposition 6.1]. Hence we can define a $\mathbf{Z}$-valued measure $\nu(\mathfrak{b}, \mathcal{D})$ on $\mathcal{O}_{\mathfrak{p}}$ by

$$\nu(\mathfrak{b}, \mathcal{D}, U) = \zeta_{R,\eta}(\mathfrak{b}, \mathcal{D}, U, 0). \tag{17}$$

Let e denote the order of $\langle \mathfrak{p} \rangle$ in $\mathrm{Cl}_{\mathfrak{f}}^{+}(\mathcal{O})$. Write $\mathfrak{p}^e = (\pi)$ with π totally positive and congruent to 1 modulo $\mathfrak{f}$. Let $\mathbf{O} = \mathcal{O}_{\mathfrak{p}} - \pi\mathcal{O}_{\mathfrak{p}}$. Define

$$\epsilon(\mathfrak{b}, \mathcal{D}, \pi) = \prod_{\epsilon \in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})}. \tag{18}$$

The product in (18) can be shown to be finite by compactness. The chosen place $\mathfrak{P}$ of H above $\mathfrak{p}$ naturally gives rise to an embedding $H \subset H_{\mathfrak{P}} \cong F_{\mathfrak{p}}$. The following conjecture was proposed in [2].

Conjecture 4.2. *Let $\mathcal{D}$ be a Shintani domain such that η is good for $\mathcal{D}$. For any integral ideal $\mathfrak{b}$ of $\mathcal{O}$ that is relatively prime to S and to ℓ, let*

$$u_\eta(\mathfrak{b},\mathcal{D}) = \epsilon(\mathfrak{b},\mathcal{D},\pi)\cdot \pi^{\zeta_{R,\eta}(H_\mathfrak{f}/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbf{O}} x\, d\nu(\mathfrak{b},\mathcal{D},x) \in F_\mathfrak{p}^\times.$$

We have:

- *$u_\eta(\mathfrak{b},\mathcal{D})$ depends only on the class of $\mathfrak{b}$ in $\mathrm{Cl}_\mathfrak{f}^+(\mathcal{O})/\langle\mathfrak{p}\rangle$, and in particular not on $\mathcal{D}$.*
- *$u_\eta(\mathfrak{b},\mathcal{D}) \in H^\times$ and $|u_\eta(\mathfrak{b},\mathcal{D})|_{\mathfrak{P}'} = 1$ for every $\mathfrak{P}'$ not lying above $\mathfrak{p}$, including archimedean places $\mathfrak{P}'$.*
- *$u_\eta(\mathfrak{b},\mathcal{D}) \equiv 1 \pmod{\mathfrak{Q}}$ for every prime ideal $\mathfrak{Q}$ of H lying above η.*
- *(Shimura reciprocity law) For any ideal $\mathfrak{a}$ relatively prime to S and to ℓ, we have*

$$u_\eta(\mathfrak{a}\mathfrak{b},\mathcal{D}) = u_\eta(\mathfrak{a},\mathcal{D})^{\sigma_\mathfrak{b}}.$$

It is proven in [2, Proposition 3.3] that Conjecture 4.2 implies the Brumer–Stark conjecture (Conjecture 3.1) and that

$$u_\eta(\mathfrak{b},\mathcal{D}) = u_{H,\eta}^{\sigma_\mathfrak{b}}.$$

Furthermore, it is proven in [2, Theorem 3.22] that the element $u_\eta(1,\mathcal{D}) \in F_\mathfrak{p}^\times$ satisfies the analytic properties of Gross's conjectures ([4, Conjecture 2.12 and Proposition 3.8] and [5, Conjecture 7.6]).

5. Shintani zeta functions in function fields

Let F be a function field with places $\infty_1,\ldots,\infty_n$ at infinity. In (3) we defined a map

$$\iota : F^\times \to \prod_{i=1}^n \left(\mathbf{R}^{>0}\times k_i^\times\right)$$

by $\iota(x) = (|x|_{\infty_i}, \mathrm{sgn}_{\infty_i}(x))$. The group $F^\times$ acts on $\prod_{i=1}^n(\mathbf{R}^{>0}\times k_i^\times)$ through the application of ι and componentwise multiplication. We defined the totally positive orthant

$$Q = \prod_{i=1}^n \left(\mathbf{R}^{>0}\times 1\right) \subset \prod_{i=1}^n \left(\mathbf{R}^{>0}\times k_i^\times\right),$$

which contains the images under ι of all totally positive elements of $F^\times$.

Definition. A *Shintani domain* is a fundamental domain $\mathcal{D}$ for the action of $E_\mathfrak{f}$ on Q.

Remark. A striking feature of the function field setting is that we need not specify the form of the fundamental domain $\mathcal{D}$. One may take a union of simplicial cones in Q, but this is not necessary. This is related to the fact that ι is not injective in the function field case. In the number field setting, if we are given totally positive elements $a, b \in F^\times$, then the points $\iota(a+nb) \in Q$ for integer $n \geq 0$ form a discrete set of points lying on a line in $\mathbf{R}^n$. Therefore, it is natural to consider simplicial cones when defining a fundamental domain $\mathcal{D}$. However, in the function field setting, it is possible for the elements $a+nb$ to map to the same point under ι, so the shape of the fundamental domain $\mathcal{D}$ becomes irrelevant.

We now define Shintani zeta functions for F in a manner analogous to the number field setting. Let $\mathcal{D}$ be a subset of Q, and let $\mathfrak{b}$ be an integral ideal of $\mathcal{O}$. Let $\eta \notin S$ be any prime ideal of $\mathcal{O}$, and define zeta functions $\zeta_R(\mathfrak{b},\mathcal{D},U,s)$ and $\zeta_{R,\eta}(\mathfrak{b},\mathcal{D},U,s)$ from equations (15) and (16) without change. At the end of this section we prove:

Proposition 5.1. *For any integral ideal $\mathfrak{b}$ of $\mathcal{O}$ relatively prime to S and to η, and any compact open subset $U \subset \mathcal{O}_{\mathfrak{p}}$, the zeta function $\zeta_{R,\eta}(\mathfrak{b}, \mathcal{D}, U, s)$ is a finite Dirichlet series in s with integral coefficients. In particular, its value at 0 is an integer.*

For any $\mathcal{D} \subset Q$, define as before a measure $\nu(\mathfrak{b}, \mathcal{D}, U)$ by equation (17). Define $\epsilon(\mathfrak{b}, \mathcal{D}, \pi)$ by (18). Because we have not specified the shape of the set $\mathcal{D}$, the compactness argument used in the number field setting to show that all but finitely many of the sets $\epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}$ appearing in (18) are empty no longer applies. However, we show in Corollary 5.5 that all but finitely many exponents in (18) equal 0, so the product is well-defined. We can now formulate a function field analogue of Conjecture 4.2. Once again, the chosen place $\mathfrak{P}$ of H above $\mathfrak{p}$ naturally gives rise to an embedding $H \subset H_{\mathfrak{P}} \cong F_{\mathfrak{p}}$.

Theorem 5.2. *Let $\mathcal{D}$ be a Shintani domain. For any integral ideal $\mathfrak{b}$ of $\mathcal{O}$ that is relatively prime to S and to η, let*

$$u_\eta(\mathfrak{b}, \mathcal{D}) = \epsilon(\mathfrak{b}, \mathcal{D}, \pi) \cdot \pi^{\zeta_{R,T}(H/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \; d\nu(\mathfrak{b}, \mathcal{D}, x) \in F_{\mathfrak{p}}^{\times}.$$

We have:

- $u_\eta(\mathfrak{b}, \mathcal{D})$ *depends only on the class of* $\mathfrak{b}$ *in* $\mathrm{Cl}_{\mathfrak{f}}^{+}(\mathcal{O})/\langle \mathfrak{p} \rangle$*, and in particular not on* $\mathcal{D}$.
- $u_\eta(\mathfrak{b}, \mathcal{D}) \in H^{\times}$ *and* $|u_\eta(\mathfrak{b}, \mathcal{D})|_{\mathfrak{P}'} = 1$ *for every* $\mathfrak{P}'$ *not lying above* $\mathfrak{p}$.
- $u_\eta(\mathfrak{b}, \mathcal{D}) \equiv 1 \pmod{\mathfrak{Q}}$ *for every prime ideal* $\mathfrak{Q}$ *of* H *lying above* η.
- *(Shimura reciprocity law) For any ideal* $\mathfrak{a}$ *relatively prime to* S *and to* η*, we have*
$$u_\eta(\mathfrak{a}\mathfrak{b}, \mathcal{D}) = u_\eta(\mathfrak{a}, \mathcal{D})^{\sigma_{\mathfrak{b}}}.$$

We will prove Theorem 5.2 in Section 8. We now proceed to a proof of Proposition 5.1.

Definition. Let $\mathcal{D}$ be a subset of Q. Let $A_N(\mathfrak{b}, \mathcal{D})$ denote the set of totally positive elements $\alpha \in \mathfrak{b}^{-1}\mathfrak{f} + 1$ relatively prime to R such that $\iota(\alpha) \in \mathcal{D}$ and $\deg_\infty(\alpha) = N$. Let

$$S_N(\mathfrak{b}, \mathcal{D}) = \bigcup_{m \le N} A_m(\mathfrak{b}, \mathcal{D}).$$

For $U \subset \mathcal{O}_{\mathfrak{p}}$, let $a_N(\mathfrak{b}, \mathcal{D}, U) = \#A_N(\mathfrak{b}, \mathcal{D}) \cap U$ and $s_N(\mathfrak{b}, \mathcal{D}, U) = \#S_N(\mathfrak{b}, \mathcal{D}) \cap U$. If $\mathcal{D} \subset Q$ is given, we will often drop it from the notation.

Grouping terms by degree, we can rewrite $\zeta_R(\mathfrak{b}, \mathcal{D}, U, s)$ as

$$\begin{aligned}\zeta_R(\mathfrak{b}, \mathcal{D}, U, s) &= \mathrm{N}\mathfrak{b}^{-s} \sum_{N=-\deg_\infty \mathfrak{b}}^{\infty} a_N(\mathfrak{b}, U) q^{-Ns} \\ &= \sum_{N=-\deg_\infty \mathfrak{b}}^{\infty} a_N(\mathfrak{b}, U) q^{-(N+\deg_\infty \mathfrak{b})s} \qquad (19)\end{aligned}$$

We will need the following lemma, which is a version of the Riemann-Roch theorem:

Lemma 5.3. *Let g denote the genus of F and let $\{\mathfrak{q}\}$ denote a finite set of places of F. For each $\mathfrak{q}$ choose an integer $r_{\mathfrak{q}}$ such that $M := \sum r_{\mathfrak{q}} d_{\mathfrak{q}} > 2g - 2$. For each $\mathfrak{q}$, let $x_{\mathfrak{q}}$ be an element of $F_{\mathfrak{q}}$, and let $B_{\mathfrak{q}}(x_{\mathfrak{q}}, -r_{\mathfrak{q}})$ denote the $\mathfrak{q}$-adic ball*

$$\{x \in F_{\mathfrak{q}} \mid x - x_{\mathfrak{q}} \in \mathfrak{q}^{-r_{\mathfrak{q}}}\mathcal{O}_{\mathfrak{q}}\}.$$

Define a subset $\mathcal{A} \subset \mathbf{A}_F$ by

$$\mathcal{A} = \prod_{\mathfrak{q}} B_{\mathfrak{q}}(x_{\mathfrak{q}}, -r_{\mathfrak{q}}) \prod_{v \notin \{\mathfrak{q}\}} \mathcal{O}_v. \tag{20}$$

Then $F \cap \mathcal{A}$ has q^{M-g+1} elements.

Proof. Let

$$\mathcal{L} = \prod_{\mathfrak{q}} \mathfrak{q}^{-r_{\mathfrak{q}}} \mathcal{O}_{\mathfrak{q}} \prod_{v \notin \{\mathfrak{q}\}} \mathcal{O}_v.$$

Since $\mathcal{A}$ is a coset of the $\mathbf{F}_q$-vector space $\mathcal{L}$, it follows that $F \cap \mathcal{A}$ is a coset of the $\mathbf{F}_q$-vector space $F \cap \mathcal{L}$, provided that $F \cap \mathcal{A}$ is nonempty. Since $M > 2g - 2$, the Riemann-Roch theorem implies that the dimension of $F \cap \mathcal{L}$ over $\mathbf{F}_q$ is $M - g + 1$. Hence $F \cap \mathcal{A}$ has q^{M-g+1} elements if it is nonempty.

To show that $F \cap \mathcal{A}$ is nonempty, let $s_{\mathfrak{q}} = \max\{-v_{\mathfrak{q}}(x_{\mathfrak{q}}), r_{\mathfrak{q}}\}$, and define

$$\mathcal{L}' = \prod_{\mathfrak{q}} \mathfrak{q}^{-s_{\mathfrak{q}}} \mathcal{O}_{\mathfrak{q}} \prod_{v \notin \{\mathfrak{q}\}} \mathcal{O}_v.$$

The quotient $F \cap \mathcal{L}'/F \cap \mathcal{L}$ of $\mathbf{F}_q$-vector spaces injects into the quotient

$$\mathcal{L}'/\mathcal{L} \cong \prod_{\mathfrak{q}} \mathfrak{q}^{-s_{\mathfrak{q}}}/\mathfrak{q}^{-r_{\mathfrak{q}}}.$$

The dimension of the latter vector space is $\sum (s_{\mathfrak{q}} - r_{\mathfrak{q}}) d_{\mathfrak{q}}$, and by Riemann-Roch on $F \cap \mathcal{L}'$ and $F \cap \mathcal{L}$, the dimension of $F \cap \mathcal{L}'/F \cap \mathcal{L}$ is exactly the same. Hence we have a bijection $F \cap \mathcal{L}'/F \cap \mathcal{L} \to \mathcal{L}'/\mathcal{L}$, and the preimage under this map of the image of $\prod x_i$ in $\mathcal{L}'/\mathcal{L}$ is exactly $F \cap \mathcal{A}$. Hence $F \cap \mathcal{A}$ is nonempty. □

This lemma is the key fact available in function fields that enables the following proof.

Proof of Proposition 5.1. Every compact open $U \subset \mathcal{O}_{\mathfrak{p}}$ can be written as a finite disjoint union of $\mathfrak{p}$-adic balls

$$B_{\mathfrak{p}}(a, r) = a + \mathfrak{p}^r \mathcal{O}_{\mathfrak{p}},$$

so it suffices to prove the result when U is such a ball. We have

$$\zeta_{R,\eta}(\mathfrak{b}, \mathcal{D}, U, s) = \sum_{N=-\deg_\infty(\mathfrak{b}\eta)}^{\infty} (a_N(\mathfrak{b}\eta, U) - \mathrm{N}\eta \cdot a_N(\mathfrak{b}, U)) q^{-(N+\deg_\infty(\mathfrak{b}\eta))s}. \tag{21}$$

The coefficients of this Dirichlet series are manifestly integral, so it remains to show that for N sufficiently large, we have

$$a_N(\mathfrak{b}\eta, U) = \mathrm{N}\eta \cdot a_N(\mathfrak{b}, U).$$

Partition the set $A_N(\mathfrak{b}) \cap U$ as follows. Let $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m$ be the non-infinite places of R. For every n-tuple of integers $w = (w_1, \ldots, w_n)$ with $\sum_i d_{\infty_i} w_i = N$, and every sequence $x = (x_1, \ldots, x_m)$ of elements $x_j \in k_{\mathfrak{q}_j}^\times = (\mathcal{O}_{\mathfrak{q}_j}/\mathfrak{q}_j \mathcal{O}_{\mathfrak{q}_j})^\times$, let

$$A_{N,w,x}(\mathfrak{b}) = \{\alpha \in A_N(\mathfrak{b}) \mid v_{\infty_i}(\alpha) = -w_i \text{ for } i = 1, \ldots, n \text{ and } \alpha \equiv x_j \pmod{\mathfrak{q}_j} \text{ for } j = 1, \ldots, m\}.$$

Define $A_{N,w,x}(\mathfrak{b}\eta)$ likewise. The sets $A_{N,w,x}(\mathfrak{b})$ and $A_{N,w,x}(\mathfrak{b}\eta)$ partition $A_N(\mathfrak{b})$ and $A_N(\mathfrak{b}\eta)$, respectively. Hence it suffices to show that for N sufficiently large, we have

$$\#A_{N,w,x}(\mathfrak{b}\eta) \cap B_{\mathfrak{p}}(a,r) = \mathrm{N}\eta \cdot \#A_{N,w,x}(\mathfrak{b}) \cap B_{\mathfrak{p}}(a,r). \tag{22}$$

Note that the two conditions that $v_{\infty_i}(\alpha) = -w_i$ and that α is positive at ∞_i are equivalent to the single condition $\alpha \equiv \pi_{\infty_i}^{-w_i} \pmod{\pi_{\infty_i}^{-w_i+1}}$. Thus we can write

$$A_{N,w,x}(\mathfrak{b}) \cap B_{\mathfrak{p}}(a,r) = F \cap \mathcal{A}$$

where $\mathcal{A} \subset \mathbf{A}_K$ is the elementary open subset

$$\mathcal{A} = B_{\mathfrak{p}}(a,r) \prod_{i=1}^{n} B_{\infty_i}(\pi_{\infty_i}^{-w_i}, -w_i+1) \prod_{j=1}^{m} B_{\mathfrak{q}_j}(x_j, 1) \prod_{\mathfrak{q}\notin S} B_{\mathfrak{q}}(1, v_{\mathfrak{q}}(\mathfrak{b}^{-1}\mathfrak{f})).$$

Lemma 5.3 implies that there exists M such that for $N \geq M$

$$\begin{aligned} \#(A_{N,w,x}(\mathfrak{b}) \cap B_{\mathfrak{p}}(a,r)) &= q^{-rd_{\mathfrak{p}} + \sum_{i=1}^{n}(d_{\infty_i}(w_i-1)) - \sum_{i=1}^{m} d_{\mathfrak{q}_i} - \deg_\infty(\mathfrak{b}^{-1}\mathfrak{f}) - g + 1} \\ &= q^{N + \deg_\infty \mathfrak{b} - \kappa}, \end{aligned} \tag{23}$$

where κ is an integer independent of N, w, x and $\mathfrak{b}$. Likewise,

$$\#(A_{N,w,x}(\mathfrak{b}\eta) \cap B_{\mathfrak{p}}(a,r)) = q^{N + \deg_\infty(\mathfrak{b}\eta) - \kappa}$$

for the same κ. Since $\mathrm{N}\eta = q^{\deg_\infty \eta}$, we obtain (22) as desired. ☐

We note that the integer M defined in the proof of the previous propostion depends on R, η, $\mathfrak{b}$, and U, but not on the Shintani domain $\mathcal{D}$. The following corollary then follows immediately, using equation (21).

Corollary 5.4. *There exists an integer M not depending on $\mathcal{D}$ such that for all $N \geq M$,*

$$\zeta_{R,\eta}(\mathfrak{b}, \mathcal{D}, U, 0) = s_N(\mathfrak{b}\eta, \mathcal{D}, U) - \mathrm{N}\eta \cdot s_N(\mathfrak{b}, \mathcal{D}, U).$$

Corollary 5.5. *Let $\mathcal{D}$ be a Shintani domain. For all but finitely many $\epsilon \in E(\mathfrak{f})$, we have*

$$\nu(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}) = 0.$$

Proof. Corollary 5.4 implies that there exists an M such that

$$\nu(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}) = s_N(\mathfrak{b}\eta, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}) - \mathrm{N}\eta \cdot s_N(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}) \tag{24}$$

for all $N \geq M$ and all $\epsilon \in E(\mathfrak{f})$. Fix an N such that $N \geq M$. The sets $S_N(\mathfrak{b}\eta, \pi^{-1}\mathcal{D})$ and $S_N(\mathfrak{b}, \pi^{-1}\mathcal{D})$ are finite, so the right hand side of (24) is nonzero for only finitely many ϵ. ☐

6. Hayes's construction of Stark units

The Brumer–Stark conjecture (Conjecture 3.1) was proven in the function field setting by Deligne using the theory of 1-motives ([3], see also [10, Chapitre V]). Hayes gave an alternate approach using the theory of Drinfeld modules and their associated exponential functions [6]. This point of view allowed him to prove Gross's refinement of the Brumer–Stark conjecture [7]. Hayes's formulas for the Stark unit will serve as the starting point for the proof of Theorem 5.2.

Let $\mathbf{C}_{\mathfrak{p}}$ denote the completion of an algebraic closure of $F_{\mathfrak{p}}$. Let $\varpi \in F_{\mathfrak{p}}$ be a uniformizer and let $\mathrm{sgn}_{\mathfrak{p}} : F_{\mathfrak{p}}^\times \to k_{\mathfrak{p}}^\times$ be the associated sign function. Let $\mathfrak{b}$ and $\mathfrak{m}$ be relatively prime ideals of the ring A of elements integral outside $\mathfrak{p}$. The quotient $\mathbf{C}_\mathfrak{p}/\mathfrak{b}^{-1}\mathfrak{m}$ gives rise to Drinfeld A-module $\phi_{\mathfrak{b}^{-1}\mathfrak{m}}$ over $\mathbf{C}_\mathfrak{p}$ with associated exponential function

$$e_{\mathfrak{b}^{-1}\mathfrak{m}} : \mathbf{C}_\mathfrak{p}/\mathfrak{b}^{-1}\mathfrak{m} \to \mathbf{C}_\mathfrak{p}$$

defined by

$$e_{\mathfrak{b}^{-1}\mathfrak{m}}(z) = z \prod_{\alpha \in \mathfrak{b}^{-1}\mathfrak{m} - \{0\}} \left(1 - \frac{z}{\alpha}\right). \tag{25}$$

Hayes defined a notion of when the Drinfeld module $\phi_{\mathfrak{b}^{-1}\mathfrak{m}}$ is *normalized* with respect to $\mathrm{sgn}_\mathfrak{p}$, and proved that there exists a $\xi \in \mathbf{C}_\mathfrak{p}$ such that the Drinfeld module $\phi_{\xi\mathfrak{b}^{-1}\mathfrak{m}}$ is sign-normalized [7, Section 5]. Let

$$\lambda(\mathfrak{b}) = \xi \cdot e_{\mathfrak{b}^{-1}\mathfrak{m}}(1). \tag{26}$$

The moduli and reduction theory of sign-normalized Drinfeld modules allows one to deduce that $\lambda(\mathfrak{b})$ lies in a certain specific finite abelian extension of F that is unramified outside $\mathfrak{m}$ and $\mathfrak{p}$ (see [7, Section 4] or [8, Section 16]). We therefore define

$$u_{K_\mathfrak{m},\eta}(\mathfrak{b}) = \lambda(\mathfrak{b})^{\sigma_\eta - \mathrm{N}\eta}. \tag{27}$$

The constant ξ is defined uniquely only up to a root of unity, so the same is true of $\lambda(\mathfrak{b})$. However, the element $u_{K_\mathfrak{m},\eta}(\mathfrak{b})$ is uniquely well-defined.

Theorem 6.1 (Hayes). *The elements $u_{K_\mathfrak{m},\eta}(\mathfrak{b})$ lie in the field $K_\mathfrak{m}$. The element $u_{K_\mathfrak{m},\eta}(1)$ is a Stark unit for $K_\mathfrak{m}/F$ and furthermore satisfies Gross's refinement of Stark's conjecture. We have the Shimura Reciprocity Law:*

$$u_{K_\mathfrak{m},\eta}(\mathfrak{a}\mathfrak{b}) = u_{K_\mathfrak{m},\eta}(\mathfrak{b})^{\sigma_\mathfrak{a}}.$$

Our statement of Theorem 6.1 combines the results of [6, §5] with [7, Theorems 4.17 and 5.10]. Central to the proof of Theorem 6.1 is the product formula for $u_{K_\mathfrak{m}}(\mathfrak{b})$ given in Lemma 6.2 below, which Hayes proved by manipulating the product in (25).

Definition. Let $\Sigma_N(\mathfrak{b},\mathfrak{m})$ denote the set of all elements $z \in \mathfrak{b}^{-1}\mathfrak{m}+1$ such that z is relatively prime to R and $\deg_\mathfrak{p} z \leq N$.

Lemma 6.2 ([7], Theorem 5.10). *For any ideal $\mathfrak{b}$ of A that is relatively prime to $\mathfrak{m}$, we have*

$$u_{K_\mathfrak{m},\eta}(\mathfrak{b}) = \lim_{N\to\infty} \left(\prod_{z \in \Sigma_N(\mathfrak{b}\eta,\mathfrak{m})} z \Bigg/ \prod_{z \in \Sigma_N(\mathfrak{b},\mathfrak{m})} z^{\mathrm{N}\eta} \right). \tag{28}$$

7. A product formula for the Stark unit over H

We now apply the results of the previous section with $\mathfrak{m} = \mathfrak{f}\infty$. We will take the norm from $K_{\mathfrak{f}\infty}$ to H of equation (28) to provide a product formula for the Stark unit $u_{H,\eta}$ and its conjugates. Recall the explicit description of $\mathrm{Gal}(K_{\mathfrak{f}\infty}/H)$ given in Proposition 2.1. Let T be a set of elements of $F^\times$ whose images under ι are coset representatives for the finite group $(\iota(F^\times) \cap Q)/\iota(E_\mathfrak{p}(\mathfrak{f}))$. By strong approximation, we may choose the elements of T to lie in the ring A and such that they are congruent to 1 modulo $\mathfrak{f}$.

From Proposition 3.2 we have the following formula for the Stark unit over H:

$$\begin{aligned} u_{H,\eta}^{\sigma_{\mathfrak{b}}} &= \mathrm{N}_{K_{\mathfrak{m}}/H}(u_{K_{\mathfrak{m}},\eta}(\mathfrak{b})) \\ &= \prod_{x \in T} u_{K_{\mathfrak{m}},\eta}(x\mathfrak{b}). \end{aligned} \tag{29}$$

By the definition of T and the injectivity of ι on $E_{\mathfrak{p}}(\mathfrak{f})$, any $\alpha \in F^{\times}$ can be written uniquely as

$$\alpha = x_{\alpha} \cdot \pi^{-d_{\alpha}} \cdot \epsilon_{\alpha}^{-1} \cdot \beta_{\alpha}, \tag{30}$$

where $x_{\alpha} \in T$, $d_{\alpha} \in \mathbf{Z}$, $\epsilon_{\alpha} \in E(\mathfrak{f})$, and $\beta_{\alpha} \in \ker \iota$. Define an equivalence relation $\sim$ on $F^{\times}$ by $\alpha \sim \alpha'$ if $x_{\alpha} = x_{\alpha'}$. The following lemma is easy to verify. We write simply $\Sigma_N(x\mathfrak{b})$ for $\Sigma_N(x\mathfrak{b}A, \mathfrak{f}\infty)$, where $x\mathfrak{b}A$ denotes the ideal of A corresponding to $x\mathfrak{b}$.

Lemma 7.1. *Fix $x \in T$. For any ideal $\mathfrak{b} \subset \mathcal{O}$ relatively prime to $\mathfrak{p}\mathfrak{f}\infty$, the map*

$$\alpha \mapsto \beta_{\alpha}$$

induces a bijection between $\{\alpha \in S_N(\mathfrak{b}) \cap \mathbf{O} \mid x_{\alpha} = x\}$ and $\Sigma_N(x\mathfrak{b})$.

Lemma 7.2. *We have*

$$u_{H,\eta}^{\sigma_{\mathfrak{b}}} = \lim_{N \to \infty} \left(\prod_{\alpha \in S_N(\mathfrak{b}\eta) \cap \mathbf{O}} \alpha \pi^{d_{\alpha}} \epsilon_{\alpha} \Big/ \prod_{\alpha \in S_N(\mathfrak{b}) \cap \mathbf{O}} (\alpha \pi^{d_{\alpha}} \epsilon_{\alpha})^{\mathrm{N}\eta} \right).$$

Proof. For each $x \in T$, the change of variables in Lemma 7.1 shows that the limit

$$\lim_{N \to \infty} \left(\prod_{\substack{\alpha \in S_N(\mathfrak{b}\eta) \cap \mathbf{O} \\ \alpha \sim x}} \alpha \pi^{d_{\alpha}} \epsilon_{\alpha} \Big/ \prod_{\substack{\alpha \in S_N(\mathfrak{b}) \cap \mathbf{O} \\ \alpha \sim x}} (\alpha \pi^{d_{\alpha}} \epsilon_{\alpha})^{\mathrm{N}\eta} \right) \tag{31}$$

is equal to

$$\lim_{N \to \infty} \left(\prod_{\beta \in \Sigma_N(x\mathfrak{b}\eta)} x\beta \Big/ \prod_{\beta \in \Sigma_N(x\mathfrak{b})} (x\beta)^{\mathrm{N}\eta} \right). \tag{32}$$

The exponent of x inside the limit of (32) is equal to

$$\#\Sigma_N(x\mathfrak{b}\eta) - \mathrm{N}\eta \cdot \#\Sigma_N(x\mathfrak{b}). \tag{33}$$

By Corollary 4.15 of [7], or by using an argument similar to the derivation of Corollary 5.4 with $\mathcal{O}$ replaced by A, we see that for N large enough the value of (33) is equal to $\zeta_{S,\eta}(K_{\mathfrak{m}}/F, \sigma_{x\mathfrak{b}}, 0)$. This zeta value is 0 by (10), so the x terms may be removed in (32) and the expression simplifies to the formula for $u_{K_{\mathfrak{m}}}(x\mathfrak{b})$ given in Lemma 6.2. Taking the product of (31) over all x and applying (29) gives the desired result. ☐

8. Proof of the Shintani-type conjecture over function fields

Theorem 5.2 follows by combining Lemma 7.2 with the following three formulae for the terms appearing in the definition of $u_\eta(\mathfrak{b},\mathcal{D})$.

Proposition 8.1. *We have*

$$\tag{34} ⨰_{\mathbf{O}} x\, d\mu_{\mathfrak{b}} = \lim_{N\to\infty}\left(\prod_{\alpha\in S_N(\mathfrak{b}\eta)\cap\mathbf{O}}\alpha \;\Big/ \prod_{\alpha\in S_N(\mathfrak{b})\cap\mathbf{O}}\alpha^{\mathrm{N}\eta}\right).$$

Proposition 8.2. *For N large enough, we have*

$$\zeta_{R,\eta}(H_{\mathfrak{f}}/F,\sigma_{\mathfrak{b}},0) = \sum_{\alpha\in S_N(\mathfrak{b}\eta)\cap\mathbf{O}} d_\alpha \;-\; \mathrm{N}\eta\sum_{\alpha\in S_N(\mathfrak{b})\cap\mathbf{O}} d_\alpha.$$

Proposition 8.3. *For N large enough, we have*

$$\tag{35} \epsilon(\mathfrak{b},\mathcal{D},\pi) = \prod_{\alpha\in S_N(\mathfrak{b}\eta)\cap\mathbf{O}}\epsilon_\alpha \;\Big/ \prod_{\alpha\in S_N(\mathfrak{b})\cap\mathbf{O}}\epsilon_\alpha^{\mathrm{N}\eta}.$$

From Proposition 8.2 we obtain

$$\tag{36} \pi^{\zeta_{R,\eta}(H_{\mathfrak{f}}/F,\sigma_{\mathfrak{b}},0)} = \lim_{N\to\infty}\pi^{\sum_{\alpha\in S_N(\mathfrak{b}\eta)\cap\mathbf{O}} d_\alpha - \mathrm{N}\eta\sum_{\alpha\in S_N(\mathfrak{b})\cap\mathbf{O}} d_\alpha}.$$

Multiplying (34), (35), and (36), and combining with Lemma 7.2, we obtain

$$u_\eta(\mathfrak{b},\mathcal{D}) = u_{H,\eta}^{\sigma_{\mathfrak{b}}},$$

thereby completing the proof of Theorem 5.2 (with the Shimura reciprocity law following from the corresponding law in Theorem 6.1).

Proof of Proposition 8.1. Recall that we have chosen a uniformizer ϖ for the local field $F_{\mathfrak{p}}$. Any element of $\mathbf{O}$ can be written uniquely as $\varpi^i x$ with $0\le i\le e-1$ and $x\in\mathcal{O}_{\mathfrak{p}}^\times$. We convert our multiplicative integral to a limit.

$$\tag{37} ⨰_{\mathbf{O}} x\, d\mu_{\mathfrak{b}} = \lim_{r\to\infty}\prod_{i=0}^{e-1}\prod_{x\in(\mathcal{O}/\mathfrak{p}^r)^\times}(\varpi^i x)^{\nu(\mathfrak{b},\mathcal{D},B(\varpi^i x,i+r))}.$$

By Corollary 5.4, the product in (37) can be written

$$\tag{38} \prod_{i=0}^{e-1}\prod_{x\in(\mathcal{O}/\mathfrak{p}^r)^\times}(\varpi^i x)^{s_N(\mathfrak{b}\eta,B(\varpi^i x,i+r))-\mathrm{N}\eta\cdot s_N(\mathfrak{b},B(\varpi^i x,i+r))}$$

for N large enough, depending on r. If we collect the powers of ϖ in this product, the resulting exponent is

$$\tag{39} C := \sum_{i=0}^{e-1} i\cdot\left(s_N(\mathfrak{b}\eta,\varpi^i\mathcal{O}_{\mathfrak{p}}^\times) - \mathrm{N}\eta\cdot s_N(\mathfrak{b},\varpi^i\mathcal{O}_{\mathfrak{p}}^\times)\right)$$

for N large enough, not depending on r.

For any $\alpha \in B(\varpi^i x, i+r)$, the value $\varpi^{-i}\alpha$ is a $\mathfrak{p}$-adic unit, and is congruent to x modulo $\mathfrak{p}^r$. This yields the following congruence:

$$\begin{aligned}
\prod_{i=0}^{e-1} \prod_{x \in (\mathcal{O}/\mathfrak{p}^r)^\times} x^{s_N(\mathfrak{b}, B(\varpi^i x, i+r))} &= \prod_{i=0}^{e-1} \prod_{x \in (\mathcal{O}/\mathfrak{p}^r)^\times} \prod_{\alpha \in S_N(\mathfrak{b}) \cap B(\varpi^i x, i+r)} x \\
&\equiv \prod_{i=0}^{e-1} \prod_{x \in (\mathcal{O}/\mathfrak{p}^r)^\times} \prod_{\alpha \in S_N(\mathfrak{b}) \cap B(\varpi^i x, i+r)} \varpi^{-i}\alpha \pmod{\mathfrak{p}^r} \\
&\equiv \prod_{i=0}^{e-1} \prod_{\alpha \in S_N(\mathfrak{b}) \cap \varpi^i \mathcal{O}_\mathfrak{p}^\times} \varpi^{-i}\alpha \pmod{\mathfrak{p}^r}, \qquad (40)
\end{aligned}$$

since every $\alpha \in S_N(\mathfrak{b}) \cap \varpi^i \mathcal{O}_\mathfrak{p}^\times$ belongs to exactly one of the congruence classes $B(\varpi^i x, i+r)$.

Applying (40) twice in the expression (38), we obtain

$$\prod_{i=0}^{e-1} \prod_{x \in (\mathcal{O}/\mathfrak{p}^r)^\times} (\varpi^i x)^{\nu(\mathfrak{b}, \mathcal{D}, B(\varpi^i x, i+r))} \equiv \frac{\prod_{\alpha \in S_N(\mathfrak{b}\eta) \cap \mathbf{O}} \alpha}{\prod_{\alpha \in S_N(\mathfrak{b}) \cap \mathbf{O}} \alpha^{\mathrm{N}\eta}} \pmod{\mathfrak{p}^{r+C}}. \qquad (41)$$

Taking the limit first as $N \to \infty$ and then as $r \to \infty$ and using (37) yields the desired result. $\square$

Before proceeding to the proof of Proposition 8.2, we record the following convenient formula, obtained from (30):

$$d_\alpha = \frac{\deg_\infty(x) - \deg_\infty(\alpha)}{\deg_\infty(\pi)}. \qquad (42)$$

Proof of Proposition 8.2. We have

$$\zeta_R(H_\mathfrak{f}/F, \sigma_\mathfrak{b}, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O} \\ (\mathfrak{a}, R)=1 \\ \sigma_\mathfrak{a} = \sigma_\mathfrak{b} \text{ on } H_\mathfrak{f}}} \frac{1}{\mathrm{N}\mathfrak{a}^s} = \frac{1}{\mathrm{N}\mathfrak{b}^s} \sum_{\substack{\alpha \in \mathfrak{b}^{-1}/E(\mathfrak{f}) \\ \alpha \gg 0, (\alpha, R)=1 \\ \alpha \equiv 1 \pmod{\mathfrak{f}}}} \frac{1}{\mathrm{N}\alpha^s}, \qquad (43)$$

where the second equality uses the change of variables $\mathfrak{a} = (\alpha)\mathfrak{b}$, and α is well-defined modulo $E(\mathfrak{f})$. Any element of the quotient $\mathfrak{b}^{-1}/E(\mathfrak{f})$ can be represented by an element of the form $\alpha\pi^m$ where $\alpha \in \mathfrak{b}^{-1} \cap \mathbf{O}$ and m is a non-negative integer. Furthermore, α is well-defined modulo $E(\mathfrak{f})$, and is specified uniquely after imposing the condition $\iota(\alpha) \in \mathcal{D}$. This allows us to rewrite (43) by applying the formula for the sum of a geometric series, as follows:

$$\zeta_R(H_\mathfrak{f}/F, \sigma_\mathfrak{b}, s) = \frac{\mathrm{N}\mathfrak{b}^{-s}}{1 - \mathrm{N}\pi^{-s}} \sum_{\substack{\alpha \in \mathfrak{b}^{-1} \cap \mathbf{O},\ \iota(\alpha) \in \mathcal{D} \\ \alpha \gg 0, (\alpha, R)=1 \\ \alpha \equiv 1 \pmod{\mathfrak{f}}}} \frac{1}{\mathrm{N}\alpha^s}. \qquad (44)$$

Equation (44) yields the following expression for the shifted zeta function:

$$(45)\qquad \zeta_{R,\eta}(H_{\mathfrak{f}}/F,\sigma_{\mathfrak{b}},s)=\frac{\mathrm{N}(\mathfrak{b}\eta)^{-s}}{1-\mathrm{N}\pi^{-s}}\left(\sum_{\substack{\alpha\in\mathfrak{b}^{-1}\eta^{-1}\cap\mathbf{O},\ \iota(\alpha)\in\mathcal{D}\\ \alpha\gg 0,(\alpha,R)=1\\ \alpha\equiv 1\ (\mathrm{mod}\ \mathfrak{f})}}\frac{1}{\mathrm{N}\alpha^s}-\mathrm{N}\eta\sum_{\substack{\alpha\in\mathfrak{b}^{-1}\cap\mathbf{O},\ \iota(\alpha)\in\mathcal{D}\\ \alpha\gg 0,(\alpha,R)=1\\ \alpha\equiv 1\ (\mathrm{mod}\ \mathfrak{f})}}\frac{1}{\mathrm{N}\alpha^s}\right).$$

By Proposition 5.1, this is a finite Dirichlet series equal to

$$(46)\qquad \frac{\mathrm{N}(\mathfrak{b}\eta)^{-s}}{1-\mathrm{N}\pi^{-s}}\left(\sum_{\alpha\in S_N(\mathfrak{b}\eta)\cap\mathbf{O}}\frac{1}{\mathrm{N}\alpha^s}-\mathrm{N}\eta\sum_{\alpha\in S_N(\mathfrak{b})\cap\mathbf{O}}\frac{1}{\mathrm{N}\alpha^s}\right)$$

for N large enough. The term in parenthesis in (46) and the denominator $1-\mathrm{N}\pi^{-s}$ both vanish at $s=0$, so we apply L'Hôpital's rule to obtain

$$(47)\qquad \zeta_{R,\eta}(H_{\mathfrak{f}}/F,\sigma_{\mathfrak{b}},0)=-\sum_{\alpha\in S_N(\mathfrak{b}\eta)\cap\mathbf{O}}\frac{\deg_\infty(\alpha)}{\deg_\infty(\pi)}+\mathrm{N}\eta\sum_{\alpha\in S_N(\mathfrak{b})\cap\mathbf{O}}\frac{\deg_\infty(\alpha)}{\deg_\infty(\pi)}.$$

Equation (42) implies that $\frac{\deg_\infty(\alpha)}{\deg_\infty(\pi)}+d_\alpha$ depends only on the equivalence class of α with respect to $\sim$. Furthermore, for a given $x\in T$, we saw in the proof of Lemma 7.2 that

$$(48)\qquad \#\{\alpha\in S_N(\mathfrak{b}\eta)\cap\mathbf{O}\mid\alpha\sim x\}-\mathrm{N}\eta\cdot\#\{\alpha\in S_N(\mathfrak{b})\cap\mathbf{O}\mid\alpha\sim x\}=0$$

for N large enough. This implies that, after possibly increasing N, we can change all the $-\deg_\infty(\alpha)/\deg_\infty(\pi)$ terms to d_α in (47) without affecting the sum. Therefore, for N large enough, we have

$$\zeta_{R,\eta}(H_{\mathfrak{f}}/F,\sigma_{\mathfrak{b}},0)=\sum_{\alpha\in S_N(\mathfrak{b}\eta)\cap\mathbf{O}}d_\alpha\ -\ \mathrm{N}\eta\sum_{\alpha\in S_N(\mathfrak{b})\cap\mathbf{O}}d_\alpha$$

as desired. □

Proof of Proposition 8.3. Recall from the proof of Corollary 5.5 that there exists an M not depending on ϵ such that equation (24) holds for all $N\geq M$ and all $\epsilon\in E(\mathfrak{f})$. Fix such an N. For each totally positive $\beta\in F$ define $\upsilon(\beta)$ to be the unique $\epsilon\in E(\mathfrak{f})$ such that $\iota(\beta)\in\epsilon\mathcal{D}$. Equations (18) and (24) imply

$$(49)\qquad \epsilon(\mathfrak{b},\mathcal{D},\pi)=\prod_{\beta\in S_N(\mathfrak{b}\eta,\pi^{-1}\mathcal{D})}\upsilon(\beta)\ \Big/\ \prod_{\beta\in S_N(\mathfrak{b},\pi^{-1}\mathcal{D})}\upsilon(\beta)^{\mathrm{N}\eta}.$$

For each $x\in T$, let r_x be the unique integer such that

$$(50)\qquad N<\deg_\infty(x\pi^{r_x})\leq N+\deg_\infty\pi,$$

and let $\upsilon_x=\upsilon(x\pi^{r_x})$. From (42) and (50) we have for each totally positive $\alpha\in F$ with $x_\alpha=x$ the equality

$$(51)\qquad \left\lfloor\frac{N-\deg_\infty\alpha}{\deg_\infty\pi}\right\rfloor=d_\alpha+r_x-1.$$

For each $\beta\in S_N(\mathfrak{b},\pi^{-1}\mathcal{D})$, let i denote the unique exponent such that $\beta\pi^{-i}\in\mathbf{O}$. Let $\epsilon=\upsilon(\beta\pi^{-i})$. Equation (51) implies that the map $\phi(\beta)=(\beta\pi^{-i}\epsilon^{-1},i)$ induces a bijection between

$S_N(\mathfrak{b}, \pi^{-1}\mathcal{D})$ and the set of pairs (α, i) such that $\alpha \in S_N(\mathfrak{b}, \mathcal{D}) \cap \mathbf{O}$ and $0 \leq i \leq d_\alpha + r_x - 1$. Indeed, if for each $\alpha \in S_N(\mathfrak{b}, \mathcal{D}) \cap \mathbf{O}$ we define $\epsilon_i = \upsilon(\pi^i \alpha)^{-1}$, then the inverse map to ϕ is given by $(\alpha, i) \mapsto \beta = \epsilon_{i+1} \pi^i \alpha$. Furthermore, we have $\upsilon(\beta) = \epsilon_{i+1}\epsilon_i^{-1}$ for $0 \leq i < d_\alpha + r_x - 1$. In particular,

$$\epsilon_{d_\alpha + r_x} = \prod_{i=0}^{d_\alpha + r_x - 1} \upsilon(\phi^{-1}(\alpha, i)). \tag{52}$$

Since it follows from (30) that $\epsilon_\alpha = \epsilon_{d_\alpha + r_x} \upsilon_x$, equation (52) states that $\upsilon_x^{-1}\epsilon_\alpha$ is equal to the product of $\upsilon(\beta)$ as β ranges over the $d_\alpha + r_x$ elements to which α correponds under the bijection ϕ. Therefore equation (49) yields

$$\epsilon(\mathfrak{b}, \mathcal{D}, \pi) = \prod_{\alpha \in S_N(\mathfrak{b}\eta) \cap \mathbf{O}} (\upsilon_x^{-1}\epsilon_\alpha) \ \Big/ \prod_{\alpha \in S_N(\mathfrak{b}) \cap \mathbf{O}} (\upsilon_x^{-1}\epsilon_\alpha)^{\mathrm{N}\eta}. \tag{53}$$

We saw in equation (48) that the exponent of each $x \in T$ in (53) is zero; this gives the desired result. □

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