The Eisenstein Cocycle and Gross's Tower of Fields Conjecture

Samit Dasgupta and Michael Spieß

Dedicated to Glenn Stevens on the occasion of his 60th birthday

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Abstract

This paper is an announcement of the following result, whose proof will be forthcoming. Let F be a totally real number field, and let $F \subset K \subset L$ be a tower of fields with L/F a finite abelian extension. Let I denote the kernel of the natural projection from $\mathbf{Z}[\mathrm{Gal}(L/F)]$ to $\mathbf{Z}[\mathrm{Gal}(K/F)]$. Let $\Theta \in \mathbf{Z}[\mathrm{Gal}(L/F)]$ denote the Stickelberger element encoding the special values at zero of the partial zeta functions of L/F, taken relative to sets S and T in the usual way. Let r denote the number of places in S that split completely in K. We show that $\Theta \in I^r$, unless K is totally real in which case we obtain $\Theta \in I^{r-1}$ and $2\Theta \in I^r$. This proves a conjecture of Gross up to the factor of 2 in the case that K is totally real and $\#S \neq r$. In this article we sketch the proof in the case that K is totally complex.

Résumé

Ce papier est une annonce du résultat suivant, dont la preuve est imminente. Soit F un corps de nombres totalement réel, et soit $F \subset K \subset L$ une tour d'extensions, où l'extension L/F est abélienne finie. Soit I le noyau de la projection naturelle de $\mathbf{Z}[\mathrm{Gal}(L/F)]$ vers $\mathbf{Z}[\mathrm{Gal}(K/F)]$. Soit $\Theta \in \mathbf{Z}[\mathrm{Gal}(L/F)]$ l'élément de Stickelberger qui encode les valeurs spéciales en zéro des fonctions zêta partielles de L/F, prise par rapport à des ensembles S et T de places de F de la manière usuelle. Soit r le nombre de places dans S qui sont totalement déployées dans K. Nous démontrons que $\Theta \in I^r$, à moins que K ne soit totalement réel auquel cas nous obtenons $\Theta \in I^{r-1}$ et $2\Theta \in I^r$. Ceci démontre une conjecture de Gross, à un facteur de 2 près dans le cas où K est totalement réel et $\#S \neq r$. Dans cet article, nous esquissons une preuve dans le cas où l'extension K est totalement complexe.

Contents

1 Introduction

This paper serves as an announcement for Theorem 1 stated below, which implies a conjecture of Gross (up to a small issue at 2 in an extremal case). In this paper we only sketch the arguments in a special setting, leaving the general results and full details of proofs to the article [4]. These results were presented at a conference in July 2014 in celebration of the 60th birthday of Glenn Stevens.

Let F be a totally real number field, and let $F \subset K \subset L$ be a tower of fields with L/F a finite abelian extension. Let $\tilde{G} = \mathrm{Gal}(L/F), H = \mathrm{Gal}(L/K)$, and $G = \mathrm{Gal}(K/F) \cong \tilde{G}/H$. Let I denote the relative augmentation ideal:

$$I = \ker(\mathbf{Z}[\tilde{G}] \longrightarrow \mathbf{Z}[G]).$$

Let S be a finite set of places of F containing all the archimedean places and all places ramified in L. Let T be a disjoint finite set of places of F satisfying a certain technical condition.[1]

[1]The conditions on T necessary for the Cassou-Noguès/Deligne–Ribet integrality statement below is that T contains two primes of different residue characteristic, or one place with residue characteristic p that is large enough. Here "large enough" depends on the value k at which Θ will be evaluated; for $k = 0$, $p \geq n+2$ is large enough. We will impose additional conditions on T for the use of Shintani's method in the next section.

For $\sigma \in \tilde{G}$, define the partial zeta function

$$\zeta_S(\sigma, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F, (\mathfrak{a}, S)=1 \\ \mathrm{Frob}(L/F, \mathfrak{a})=\sigma}} \frac{1}{\mathrm{N}\mathfrak{a}^s}, \qquad \mathrm{Re}(s) > 1.$$

These can be combined into group-ring valued Stickelberger elements:

$$\Theta_S^{L/F}(s) := \sum_{\sigma \in \tilde{G}} \zeta_S(\sigma, s)[\sigma^{-1}] \in \mathbf{C}[\tilde{G}] \tag{1}$$

$$\Theta_{S,T}^{L/F}(s) := \Theta_S^{L/F}(s) \prod_{\mathfrak{p} \in T} (1 - \mathrm{N}\mathfrak{p}^{1-s}[\mathrm{Frob}(L/F, \mathfrak{p})^{-1}]) \in \mathbf{C}[\tilde{G}]. \tag{2}$$

Theorem (Cassou-Noguès [1], Deligne–Ribet [5])**.** *Let $k \in \mathbf{Z}$, $k \leq 0$. Under the appropriate conditions on T, we have $\Theta_{S,T}^{L/F}(k) \in \mathbf{Z}[\tilde{G}]$.*

Gross conjectured the following concerning the "order of vanishing" of the Stickelberger element.

Conjecture (Gross [8])**.** *Let r be the number places in S that split completely in K. Then*

$$\Theta_{S,T}^{L/F}(0) \in I^r$$

unless $r = \#S$, in which case we have $\Theta_{S,T}^{L/F}(0) \in I^{r-1}$.

Our main result is:

Theorem 1 ([4], Corollary 5.10)**.** *Under appropriate conditions on T, we have $\Theta_{S,T}^{L/F}(0) \in I^r$ unless K is totally real, in which case we have $\Theta_{S,T}^{L/F}(0) \in I^{r-1}$ and $2\Theta_{S,T}^{L/F}(0) \in I^r$.*

Our result implies Gross's conjecture for $r = \#S$. It implies Gross's conjecture for $r < \#S$ as well, except for the 2-part when K is totally real. In [4], we prove a stronger result, where I^r is replaced by the product of the relative augmentation ideals associated to the decomposition groups of each $v \in S$ (see [4, Theorem 5.9]).[2] Furthermore, we prove a similar order of vanishing result for $\Theta_{S,T}^{L/F}(k)$ for $k \in \mathbf{Z}, k < 0$.

In this paper, we make the simplifying assumption that K is totally complex and we only consider $k = 0$. We sketch the arguments, which are greatly simplified under these assumptions.

Remark 2. Prior to our work, the following result was known due to Popescu and Greither [7, Theorem 6.5]. Let K be totally complex. Let p be prime, $p \neq 2$. Assume that S contains all the primes of F above p. Then the "p-part" of Gross's conjecture is true, i.e. $\Theta_{S,T}^{L/F}(0) \in I^r \otimes \mathbf{Z}_p$.

[2] We are grateful to C. Popescu for suggesting that we strengthen our result in this form at the conference.

Our method of proof involves realizing the group-ring element $\Theta_{S,T}^{L/F}(0)$ as the cap product between a cohomology class for the group of units in F with a homology class encoding a basis for this group of units and the reciprocity map of class field theory for the extension L/F. One can view this realization as separating the analytic component—namely the cohomology class, defined using Shintani's method and involving zeta functions—and the algebraic component, namely the homology class defined in terms of algebraic information about the field extension L/F. We prove that the image of the homology class in question vanishes in $\mathbf{Z}[\tilde{G}]/I^r$, and hence *a fortiori* the cap product with the cohomology class (namely, the image of $\Theta_{S,T}^{L/F}(0)$ modulo I^r) vanishes as well.

The vanishing of the homology class follows from an explicit cap product formula, proven in Theorem 8. The proof of this theorem is the technical heart of this paper (and of [4]). In this paper we given an explicit combinatorial and computational proof Theorem 8, in contrast to the more conceptual and formal proof given in [4, Theorem 3.6]. The inclusion of this new proof is one motivation for the writing of this paper; we hope that the methods described here may have use in other contexts.

The cohomology class involved is the so-called Eisenstein cocycle. Perhaps the first appearance of this cocycle in the literature was the article [16] by Glenn Stevens, where a cocycle for $\mathbf{GL}_2(\mathbf{Q})$ was defined using integration along horocycles in the Borel–Serre compactification of the modular curve. Later, alternate methods were introduced to define the cocycle for $\mathbf{GL}_n(\mathbf{Q})$, notably by Sczech [11] using principles of conditional convergence and by Solomon, Hu, and Hill ([10], [9]) using Shintani's method. These methods were generalized with integral refinements in [2], [3], [14], [15] and in the current project [4] to deduce results about classical and p-adic L-functions. It is a true pleasure to acknowledge the great debt this field pays to the seminal work of Glenn Stevens.

2 Stickelberger Elements and the Eisenstein cocycle

2.1 Shintani's Method

In this paper, we follow [3] in defining the Eisenstein cocycle using Shintani's method. We first recall Shintani's approach. Consider the sum defining Θ:

$$\Theta_S^{L/F}(s) = \sum_{\substack{\mathfrak{a}\in\mathcal{O}_F \\ (\mathfrak{a},S)=1}} \frac{[\mathrm{Frob}(L/F,\mathfrak{a})^{-1}]}{\mathrm{N}\mathfrak{a}^s}, \qquad \mathrm{Re}(s) > 1.$$

Let $\mathfrak{b}_1, \dots, \mathfrak{b}_h$ denote a set of integral ideals representing the narrow ideal class group of F. Each $\mathfrak{a}$ is equivalent to a unique $\mathfrak{b}_i^{-1}$ in the narrow ray class group, and using the change of variables $\mathfrak{a}\mathfrak{b}_i = (\alpha), \alpha \gg 0$, we have

$$\Theta_S^{L/F}(s) = \sum_{i=1}^{h} \mathrm{N}\mathfrak{b}_i^s[\mathrm{Frob}(L/F,\mathfrak{b}_i)] \sum_{\substack{\alpha\in\mathfrak{b}_i/E_+,\alpha\gg 0 \\ (\alpha,S)=1}} \frac{[\mathrm{Frob}(L/F,(\alpha))^{-1}]}{\mathrm{N}\alpha^s}. \tag{3}$$

Here E_+ denotes the group of totally positive units of $\mathcal{O}_F$.

Shintani's strategy is to view the inner sum in (3) geometrically. Embed $F \subset \mathbf{R}^n$ via the n archimedean embeddings of F. The ideal $\mathfrak{b}_i$ is a lattice in $\mathbf{R}^n$. Now F^* acts on $\mathbf{R}^n$, where $x \in F^*$ acts by multiplication by the ith archimedean embedding of x on the ith coordinate. Shintani proved that there is a fundamental domain $\mathcal{D}$ for the action of E_+ on the totally positive orthant $(\mathbf{R}^{>0})^n \subset \mathbf{R}^n$ consisting of a disjoint union of simplicial cones generated by elements of F^*. Such a set $\mathcal{D}$ is called a Shintani domain. Here, a simplicial cone generated by linearly independent vectors $x_1, \ldots, x_m \in (\mathbf{R}^{>0})^n$ refers to the set

$$C(x_1, \ldots, x_m) = \{t_1 x_1 + \cdots + t_m x_m : t_i \in \mathbf{R}^{>0}\} \subset (\mathbf{R}^{>0})^n.$$

For a fixed $\sigma \in \tilde{G}$, the coefficient of $[\sigma]$ in the inner sum of (3) can be written simply $\sum_\alpha \frac{1}{\mathrm{N}\alpha^s}$, where α ranges over the elements of $\mathfrak{b} \cap \mathcal{D}$ such that

$$(\alpha, S) = 1 \quad \text{and} \quad \mathrm{Frob}(L/F, (\alpha))^{-1} = \sigma.$$

The set of such α can be expressed as the disjoint union of lattices in $\mathbf{R}^n$ intersected with simplicial cones generated by elements of F^*. Shintani proved the analytic continuation of such sums—called Shintani zeta functions—and gave their evaluation at nonpositive integers in terms of Bernoulli numbers.

We conclude this section by giving an explicit construction of a Shintani domain $\mathcal{D}$. In fact, we will describe only a "signed Shintani domain", which is a formal linear combination $\mathcal{D} = \sum_{i=1}^m n_i C_i$ where C_i is a simplicial cone and $n_i \in \mathbf{Z}$. We call $\mathbf{1}_\mathcal{D} := \sum_{i=1}^m n_i \mathbf{1}_{C_i}$ the characteristic function of $\mathcal{D}$, where $\mathbf{1}_{C_i}$ denotes the characteristic function of the cone C_i. The property of being a fundamental domain for the action of E_+ is then replaced by the condition

$$\sum_{\epsilon \in E_+} \mathbf{1}_\mathcal{D}(\epsilon * x) = 1 \qquad \text{for all } x \in (\mathbf{R}^{>0})^n.$$

Let $\epsilon_1, \ldots, \epsilon_{n-1}$ denote an ordered basis for the free abelian group E_+. Define the orientation

$$w := \operatorname{sign} \det(\log(\epsilon_{ij}))_{i,j=1}^{n-1}) = \pm 1, \tag{4}$$

where ϵ_{ij} denotes the jth coordinate of ϵ_i as an element of $\mathbf{R}^n$. For each permutation $\sigma \in S_{n-1}$ let

$$v_{i,\sigma} = \epsilon_{\sigma(1)} \cdots \epsilon_{\sigma(i-1)} \in E_+, \qquad i = 1, \ldots, n$$

(so by convention $v_{1,\sigma} = 1$ for all σ). Define

$$w_\sigma = (-1)^{n-1} w \cdot \operatorname{sign}(\sigma) \operatorname{sign}(\det(v_{i,\sigma})_{i=1}^n) \in \{0, \pm 1\}.$$

Now, given linearly independent elements $x_1, \ldots, x_n \in (\mathbf{R}^{>0})^n$, one can associate to the open cone $C(x_1, \ldots, x_n)$ its union with certain boundary faces via the process of "Colmez perturbation" as follows. Choose the auxiliary vector $Q = (1, 0, 0, \ldots, 0)$, which has the property that its ray (i.e. its set of $\mathbf{R}^{>0}$ multiples) is preserved by the action of E_+. Then define $C^*(x_1, \ldots, x_n)$ to be the union of $C(x_1, \ldots, x_n)$ with the boundary faces that are brought into the interior of the cone by a small perturbation by Q, i.e.

$$\mathbf{1}_{C^*(x_1,\ldots,x_n)}(x) = \lim_{h \to 0^+} 1_{C(x_1,\ldots,x_n)}(x + hQ).$$

We then have the following theorem, which was proved independently by Diaz y Diaz–Friedman [6] and Charollois–Dasgupta–Greenberg [3]:

Theorem 3. *The formal linear combination*

$$\sum_{\sigma \in S_{n-1}} w_\sigma C^*(v_{1,\sigma}, \ldots, v_{n,\sigma})$$

is a signed fundamental domain for the action of E_+ on $(\mathbf{R}^{>0})^n$, i.e. we have

$$\sum_{\epsilon \in E_+} \sum_{\sigma \in S_{n-1}} w_\sigma \mathbf{1}_{C^*(v_{1,\sigma},\ldots,v_{n,\sigma})}(\epsilon * x) = 1 \qquad \text{for all } x \in (\mathbf{R}^{>0})^n.$$

2.2 Eisenstein Cocycle on F_+^*

We now define the Eisenstein cocycle using Shintani's method, following [3]. Whereas the cocycle may be defined on the larger group $\mathbf{GL}_n(\mathbf{Q})$, for simplicity here we only describe the restriction to the group of totally positive elements of F, denoted $F_+^* \subset F^*$ (here "restriction" refers to an embedding $F^* \subset \mathbf{GL}_n(\mathbf{Q})$ given by choosing a $\mathbf{Q}$-basis for F).

Let $\mathcal{N}$ denote the abelian group of functions $(\mathbf{R}^{>0})^n \longrightarrow \mathbf{Z}$, which is endowed with a natural action of $(\mathbf{R}^{>0})^n \supset F_+^*$ given by $(t \cdot f)(x) := f(t^{-1}x)$. We then have:

Theorem 4 ([3], Theorem 1.6)**.** *Given $x_1, \ldots, x_n \in (\mathbf{R}^{>0})^n$, let $x = (x_{ij})$ denote the $n \times n$ matrix whose columns are the vectors x_i. Define*

$$\tilde{\Phi}(x_1, \ldots, x_n) = \operatorname{sgn}\det(x)\mathbf{1}_{C^*(x_1,\ldots,x_n)}.$$

Then $\tilde{\Phi}$ is a homogenous $(n-1)$-cocycle for the group F_+^* acting on $\mathcal{N}$.

The cocycle $\tilde{\Phi}$ is our first manifestation of the Eisenstein cocycle. We now use Shintani zeta functions to turn $\tilde{\Phi}$ into a measure valued cocycle. For notational simplicity, in this paper we let $\mathbf{A}_F$ denote the ring of finite adeles. This is contrary to standard notation and in particular to the notation of [4], where $\mathbf{A}_F$ denotes the usual full adele ring.

For an abelian group A, we denote by $\operatorname{Dist}(\mathbf{A}_F, A)$ the A-module of functions μ that assign to each compact open subset $U \subset \mathbf{A}_F$ an element $\mu(U) \in A$, such that for disjoint open compacts U and V we have $\mu(U \cup V) = \mu(U) + \mu(V)$. The A-module $\operatorname{Dist}(\mathbf{A}_F, A)$ is endowed with an F^* action by $(t \cdot \mu)(U) = \mu(t^{-1}U)$.

Let $U \subset \mathbf{A}_F$ be an open compact subset. For linearly independent elements $x_1, \ldots, x_n \in F_+^* \subset (\mathbf{R}^{>0})^n$, define

$$\zeta(x_1, \ldots, x_n; U, s) = \sum_{\substack{\alpha \in C^*(x_1,\ldots,x_n) \\ \alpha \in F \cap U}} \frac{1}{\mathrm{N}\alpha^s}, \qquad \operatorname{Re}(s) > 1.$$

As mentioned above, Shintani proved that these zeta functions have analytic continuation to $\mathbf{C}$, and that the values at non-positive integers are rational [12].

The following result follows directly from Proposition 4; it gives our next manifestation of the Eisenstein cocycle.

Proposition 5. *Define*

$$\Phi(x_1, \ldots, x_n)(U) := \operatorname{sgn}\det(x)\zeta(x_1, \ldots, x_n; U, 0) \in \mathbf{Q}.$$

Then Φ is a homogenous $(n-1)$-cocycle for the group F_+^* acting on the space $\operatorname{Dist}(\mathbf{A}_F, \mathbf{Q})$ *of $\mathbf{Q}$-valued distributions on $\mathbf{A}_F$.*

Finally we introduce a smoothing of the Eisenstein cocycle using primes in T, employing the "Cassou-Noguès trick". For this, we must introduce extra conditions on the set T further than those listed in the introduction. In the paper [4], these conditions are given in (A1)–(A3) in §5.2. Here for simplicity, we assume that $T = \{\mathfrak{q}\}$ consists of exactly one element $\mathfrak{q}$ such that $\mathrm{N}\mathfrak{q} = \ell$ is a prime larger than $n+1$. We furthermore assume that S contains no primes lying above ℓ.

Given such a set T, we write $\overline{T}$ for the set of all places of F above ℓ. Let $F_{\overline{T}}^* \subset F_+^*$ be subgroup of elements with valuation 0 at every prime in $\overline{T}$. We denote by

$$\mathbf{A}_F^{\overline{T}} = \prod_{v \nmid \ell}{}' F_v$$

the ring of finite adeles away from $\overline{T}$. Given a compact open subset $U \subset \mathbf{A}_F^{\overline{T}}$ we define two associated compact open subsets $U_0, U_1 \subset \mathbf{A}_F$ by

$$U_0 = U \times \prod_{v \mid \ell} \mathcal{O}_v, \qquad U_1 = U \times \prod_{v \mid \ell, v \neq \mathfrak{q}} \mathcal{O}_v \times \mathfrak{q}\mathcal{O}_{\mathfrak{q}}.$$

The following result gives the form of the Eisenstein cocycle that will be most useful in this paper.

Proposition 6. *For $x_1, \ldots, x_n \in F_{\overline{T}}^*$, and $U \subset \mathbf{A}_F^{\overline{T}}$, define*

$$\Phi_i(x_1, \ldots, x_n)(U) := \Phi(x_1, \ldots, x_n)(U_i) \qquad i = 0, 1,$$

and define

$$\Phi_T := \Phi_0 - \ell \cdot \Phi_1.$$

Then Φ_0, Φ_1, and Φ_T are homogenous $(n-1)$-cocycles for the group $F_{\overline{T}}^*$ acting on the space $\operatorname{Dist}(\mathbf{A}_F^{\overline{T}}, \mathbf{Q})$*, and $\Phi_T(x_1, \ldots, x_n)(U) \in \mathbf{Z}$.*

The fact that Φ_T takes on integer values is proven in [4, Prop. 4.1]. The cocycle property for the Φ_i (and hence for Φ_T) follows formally from Prop. 5.

For the next few sections, we will ignore the set of smoothing primes T. This will ensure some notational simplicity. In §3.3 we will note the minor changes that must be made to achieve full integral results.

2.3 Pairing Cohomology and Homology

Let R be a commutative ring endowed with the discrete topology. Let $C_c(\mathbf{A}_F, R)$ denote the R-module of compactly supported continuous (i.e. locally constant) functions on $\mathbf{A}_F$ valued in R. There is a canonical pairing

$$\mathrm{Dist}(\mathbf{A}_F, \mathbf{Z}) \times C_c(\mathbf{A}_F, R) \longrightarrow R \tag{5}$$
$$\mu \times f \mapsto \int_{\mathbf{A}_F} f d\mu$$

given by

$$\int_{\mathbf{A}_F} f d\mu := \sum_{r \in R} r \cdot \mu(f^{-1}(r)).$$

Note that since f is compactly supported, the sum is finite. Of course, if μ is only $\mathbf{Q}$-valued (rather than $\mathbf{Z}$-valued), then the integral will take values in $R \otimes \mathbf{Q}$. The pairing (5) induces via cap product a pairing

$$H^{n-1}(F_+^*, \mathrm{Dist}(\mathbf{A}_F, \mathbf{Z})) \times H_{n-1}(F_+^*, C_c(\mathbf{A}_F, R)) \longrightarrow R$$
$$(\Phi_T, \rho) \mapsto \Phi_T \cap \rho.$$

Let $R = \mathbf{Z}[\tilde{G}]$. In §2.4, we will define a class

$$\rho^{L/F} \in H_{n-1}(F_+^*, C_c(\mathbf{A}_F, R))$$

corresponding to taking a signed fundamental domain $\mathcal{D}$ for the action of E_+ on $(\mathbf{R}^{>0})^n$ and to the restriction of the Artin reciprocity map $\mathrm{rec}_{L/F} : \mathbf{A}_F^* \to \tilde{G} \subset R^*$. After an unwinding of definitions, the following theorem (which will be proved in §2.5) will follow from Shintani's realization of the element $\Theta_S^{L/F}(0)$ described in §2.1.

Theorem 7. $\Theta_S^{L/F}(0) = \Phi \cap \rho^{L/F} \in \mathbf{Q}[\tilde{G}]$.

Given Theorem 7, the idea behind the proof of Gross's conjecture is that the image of the homology class $\rho^{L/F}$ in $H_{n-1}(F_+^*, C_c(\mathbf{A}_F, R/I^{r+1}))$ vanishes.

2.4 Definition of $\rho^{L/F}$

Let $\eta \in H_{n-1}(E_+, \mathbf{Z}) \cong \mathbf{Z}$ be the generator represented by the homogenous $(n-1)$-cycle

$$\sum_{\sigma \in S_{n-1}} (-1)^{n-1} w\, \mathrm{sign}(\sigma)[(v_{1,\sigma}, v_{2,\sigma}, \ldots, v_{n,\sigma})] \in \mathbf{Z}[E_+^n]$$

where the notation is as in §2.1.

Let $U = \prod_v \mathcal{O}_v^*$ and $U^S = \prod_{v \notin S} \mathcal{O}_v^*$. Let $\mathcal{R}$ be a fundamental domain for action of F_+^*/E_+ on $\mathbf{A}_F^*/U$, i.e. a set of representatives for the narrow class group. We assume that these representative ideals are relatively prime to S. We can view the constant function 1 on $\mathcal{R}$ as an element $\mathbf{1}_{\mathcal{R}} \in H^0(E_+, C(\mathcal{R}, \mathbf{Z}))$ and hence consider

$$\mathbf{1}_{\mathcal{R}} \cap \eta \in H_{n-1}(E_+, C(\mathcal{R}, \mathbf{Z})) \cong H_{n-1}(F_+^*, C_c(\mathbf{A}_F^*/U, \mathbf{Z})),$$

where the last isomorphism is by Shapiro's Lemma since $C_c(\mathbf{A}_F^*/U, \mathbf{Z}) \cong \mathrm{Ind}_{E_+}^{F_+^*} C(\mathcal{R}, \mathbf{Z})$.

Now consider the Artin reciprocity map $\mathrm{rec}_{L/F}\colon \mathbf{A}_F^* \to \tilde{G} \subset R$ as an element

$$\mathrm{rec}_{L/F} \in H^0(F_+^*, C(\mathbf{A}_F^*/U^S, R))$$

and define

$$\overline{\rho}^{L/F} = \mathrm{rec}_{L/F} \cap (\mathbf{1}_{\mathcal{R}} \cap \eta) \in H_{n-1}(F_+^*, C_c(\mathbf{A}_F^*/U^S, R)).$$

Finally, write $\mathbf{A}_{F,S}^* := \prod_{v \in S} F_v^*$. Consider the map

$$C_c(\mathbf{A}_F^*/U^S, R) \longrightarrow C_c(\mathbf{A}_F, R) \tag{6}$$

induced by $f \cdot \mathbf{1}_{xU^S} \mapsto f_! \cdot \mathbf{1}_{\mathrm{ideal}(x) \otimes \hat{\mathbf{Z}}}$ for $f \in C_c(\mathbf{A}_{F,S}^*, R), x \in (\mathbf{A}_F^S)^*/U^S$. Here $f_!$ denotes the extension by zero of f to a function on $\mathbf{A}_{F,S}$. The map (6) is $\mathbf{A}_F^*$-equivariant and in particular F_+^*-equivariant and hence induces a map on homology. We define

$$\rho^{L/F} \in H_{n-1}(F_+^*, C_c(\mathbf{A}_F, R))$$

to be the image of $\overline{\rho}^{L/F}$ under this map.

2.5 Proof of Theorem 7

This involves simply tracing through the definitions. Theorem 3 precisely states that

$$\tilde{\Phi} \cap \eta \in H_0(E_+, \mathcal{N})$$

is the image in $H_0(E_+, \mathcal{N})$ of the characteristic function of a signed fundamental domain $\mathcal{D}$ for the action of E_+ on $(\mathbf{R}^{>0})^n$. In what follows, the notation $\sum_{\alpha \in \mathcal{D}}$ will mean the sum $\sum_i n_i \sum_{\alpha \in C_i}$ if $\mathcal{D} = \sum_i n_i \mathbf{1}_{C_i}$ for simplicial cones C_i.

Now let $\mathcal{R}$ as in §2.4 be a fundamental domain for the action of F_+^*/E_+ on $\mathbf{A}_F^*/U$, so $\mathcal{R}$ corresponds to a set $\mathfrak{b}_1, \ldots, \mathfrak{b}_h$ of representatives for the narrow class group of F (this correspondence is given by $\mathcal{R} = \{\alpha \in \mathbf{A}_F^*/U : \mathrm{ideal}(\alpha) = \mathfrak{b}_i \text{ for some } i\}$). As above we assume that each $\mathfrak{b}_i$ is relatively prime to S. We now calculate the image of

$$\mathbf{1}_{\mathcal{R}} \cdot \mathrm{rec}_{L/F} \in C_c(\mathbf{A}_F^*/U^S, R)$$

under (6). Write $\mathrm{rec}_{L/F}$ as a product $\mathrm{rec}_{L/F}^S \cdot \mathrm{rec}_{L/F,S}$, where these functions are the composition of $\mathrm{rec}_{L/F}$ with projection onto $(\mathbf{A}_F^S)^*$ and $\mathbf{A}_{F,S}^* = \prod_{v \in S} F_v^*$, respectively. If x_i is an element of $\mathbf{A}_F^*$ such that $\mathrm{ideal}(x_i) = \mathfrak{b}_i$, then we find

$$\begin{aligned}
\mathbf{1}_{\mathcal{R}} \cdot \mathrm{rec}_{L/F} &= \sum_{i=1}^{h} \mathbf{1}_{x_i U} \mathrm{rec}_{L/F}^{S} \mathrm{rec}_{L/F,S} \\
&= \sum_{i=1}^{h} [\mathrm{Frob}(L/F, \mathfrak{b}_i)] \mathrm{rec}_{L/F,S} \cdot \mathbf{1}_{x_i U} \\
&= \sum_{i=1}^{h} [\mathrm{Frob}(L/F, \mathfrak{b}_i)] (\mathrm{rec}_{L/F,S} \cdot \mathbf{1}_{U_S}) \cdot \mathbf{1}_{x_i^S U^S},
\end{aligned}$$

where the second equation follows from the fact that for $y \in x_i U^S$, we have $\mathrm{rec}_{L/F}^S(y) = \mathrm{Frob}(L/F, \mathfrak{b}_i)$. In the third equation, x_i^S denotes the projection of x_i onto $(\mathbf{A}_F^S)^*$, and the equation follows since $x_{i,S} U_S = U_S$, as $\mathrm{ideal}(x_i) = \mathfrak{b}_i$ is relatively prime to S (here $x_i = x_i^S x_{i,S}$).

Therefore, by definition of the map (6), we see that $\mathbf{1}_{\mathcal{R}} \cdot \mathrm{rec}_{L/F}$ maps to

$$\sum_{i=1}^{h} [\mathrm{Frob}(L/F, \mathfrak{b}_i)] (\mathrm{rec}_{L/F,S} \cdot \mathbf{1}_{U_S})_! \mathbf{1}_{\mathfrak{b}_i \otimes \hat{\mathbf{Z}}}.$$

In conclusion, we see that

$$\Phi \cap \rho^{L/F} = \left(\sum_{i=1}^{h} \sum_{\substack{\alpha \in \mathfrak{b}_i, (\alpha, S) = 1 \\ \alpha \in \mathcal{D}}} \frac{[\mathrm{Frob}(L/F, \mathfrak{b}_i) \cdot \mathrm{rec}_{L/F,S}(\alpha)]}{\mathrm{N}\alpha^s} \right) \Big|_{s=0}. \tag{7}$$

Now for $\alpha \gg 0$ with $(\alpha, S) = 1$, we have $\mathrm{rec}_{L/F}(\alpha) = 1$ and $\mathrm{rec}_{L/F}^S(\alpha) = \mathrm{Frob}(L/F, (\alpha))$; therefore we have $\mathrm{rec}_{L/F,S}(\alpha) = \mathrm{Frob}(L/F, (\alpha))^{-1}$. In view of this equality, it follows from (3) and (7) that $\Phi \cap \rho^{L/F} = \Theta_S^{L/F}(0)$ as desired.

3 The cap product formula

So far the intermediate field K has not played a role in our discussion. In this section we introduce another cohomology class

$$\rho^{K/F,X} \in H_{n+r-1}(F_+^*, C_c(\mathbf{A}_F^X, \mathbf{Z}[G]))$$

associated to the extension K/F and the set

$$X = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \subset S$$

of places in S that split completely in K. We then describe a formula relating $\rho^{L/F}$ to a cap product between certain 1-cocycles associated to the splitting primes X and the class $\rho^{K/F,X}$.

3.1 Definition of $\rho^{K/F,X}$

The definition of $\rho^{K/F,X}$ is very similar to that of $\rho^{L/F}$. However, we use the group of totally positive X-units of F, denoted $E_{X,+}$, which is a free abelian group of rank $n+r-1$. Let $\eta_X \in H_{n+r-1}(E_{X,+}, \mathbf{Z})$ be a generator corresponding to the same ordering of the basis of E_+ used to define the generator $\eta \in H_{n-1}(E_+, \mathbf{Z})$ and a fixed ordering $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of the primes in X.

Let $\mathcal{R}$ be a fundamental domain for action of $F_+^*/E_{X,+}$ on $(\mathbf{A}_F^X)^*/U^X$. We can view $\mathbf{1}_{\mathcal{R}} \in H^0(E_{X,+}, C(\mathcal{R}, \mathbf{Z}))$ and hence consider

$$\mathbf{1}_{\mathcal{R}} \cap \eta_X \in H_{n+r-1}(E_{X,+}, C(\mathcal{R}, \mathbf{Z})) \cong H_{n+r-1}(F_+^*, C_c((\mathbf{A}_F^X)^*/U^X, \mathbf{Z})).$$

Now consider the restriction of the Artin reciprocity map as an element

$$\mathrm{rec}_{K/F}^X \in H^0(F_+^*, C((\mathbf{A}_F^X)^*/U^S, \mathbf{Z}[G])).$$

Note here the fact that $\mathrm{rec}_{K/F}^X$ is invariant under F_+^* follows from the fact that each prime in X splits completely in K, so $\mathrm{rec}_{K/F,X} = 1$. Define

$$\overline{\rho}^{K/F,X} = \mathrm{rec}_{K/F}^X \cap (\mathbf{1}_{\mathcal{R}} \cap \eta_X) \in H_{n+r-1}(F_+^*, C_c((\mathbf{A}_F^X)^*/U^S, \mathbf{Z}[G])).$$

Then as in (6), we have a map

$$C_c((\mathbf{A}_F^X)^*/U^S, \mathbf{Z}[G]) \longrightarrow C_c(\mathbf{A}_F^X, \mathbf{Z}[G]) \tag{8}$$

induced by $f \cdot \mathbf{1}_{xU^S} \mapsto f_! \cdot \mathbf{1}_{\mathrm{ideal}(x) \otimes \hat{\mathbf{Z}}}$ for $f \in C_c(\mathbf{A}_{F,S-X}^*, \mathbf{Z}[G]), x \in (\mathbf{A}_F^S)^*/U^S$. We define

$$\rho^{K/F,X} \in H_{n+r-1}(F_+^*, C_c(\mathbf{A}_F^X, \mathbf{Z}[G]))$$

to be the image of $\overline{\rho}^{K/F,X}$ under the map on homology induced by (8).

3.2 1-cocycles attached to splitting primes

Recall that $H = \mathrm{Gal}(L/K)$, so $G \cong \tilde{G}/H$, and that $X \subset S$ denotes the set of places in S that split completely in K. Since K is assumed to be totally complex, each place $\mathfrak{p} \in X$ is finite. Let $\mathrm{rec}_{\mathfrak{p}} \colon F_{\mathfrak{p}}^* \to \mathbf{A}_F^* \to \tilde{G}$ denote the local component at $\mathfrak{p}$ of the Artin reciprocity map for L/F. Note that since $\mathfrak{p}$ splits in K, it follows that the image of $\mathrm{rec}_{\mathfrak{p}}$ actually lies in H.

For each $\mathfrak{p} \in X$ we now define a class $c_{\mathfrak{p}} \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, H))$. Intuitively, $c_{\mathfrak{p}}$ is the coboundary attached to the function $1_{\mathcal{O}_{\mathfrak{p}}} \cdot \mathrm{rec}_{\mathfrak{p}}$, i.e.

$$\text{“ } c_{\mathfrak{p}}(x) = (1-x)(1_{\mathcal{O}_{\mathfrak{p}}} \cdot \mathrm{rec}_{\mathfrak{p}}) \quad \text{for } x \in F_{\mathfrak{p}}^*. \text{ ”} \tag{9}$$

Now, the function $1_{\mathcal{O}_{\mathfrak{p}}} \cdot \mathrm{rec}_{\mathfrak{p}}$ on $F_{\mathfrak{p}}^*$ does not extend to an element of $C_c(F_{\mathfrak{p}}, H)$. For this reason (9) is not well-defined as written (and, being a coboundary, if it were well-defined it would represent the trivial class); nevertheless a formal manipulation of the right side of (9) yields the following definition, which *does* make sense:

$$c_{\mathfrak{p}}(x)(y) := 1_{x\mathcal{O}_{\mathfrak{p}}}(y) \cdot \mathrm{rec}_{\mathfrak{p}}(x) + (1_{\mathcal{O}_p} - 1_{x\mathcal{O}_p})(y) \cdot \mathrm{rec}_{\mathfrak{p}}(y) \tag{10}$$

for $x \in F_{\mathfrak{p}}^*$ and $y \in F_{\mathfrak{p}}$. Here we view H as a $\mathbf{Z}$-module, so its group law is written additively and the products in (10) should be understood accordingly (note that $\mathrm{rec}_{\mathfrak{p}}(y)$ is not defined if $y = 0$, but the coefficient of $\mathrm{rec}_{\mathfrak{p}}(y)$ in this case is 0). One easily checks that the definition (10) yields a 1-cocycle, whose cohomology class $H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, H))$ we again (by abuse of notation) denote $c_{\mathfrak{p}} \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, H))$. The cohomology class $c_{\mathfrak{p}}$ is nontrivial if and only if $\mathrm{rec}_{\mathfrak{p}}$ is nontrivial, i.e. if and only if $\mathfrak{p}$ is not split in L.

3.3 Smoothing

Let $T = \{\mathfrak{q}\}$ be a set as §2.2, so $\mathrm{N}\mathfrak{q} = \ell$ is a prime integer. We state without giving details the modifications of the constructions in the previous sections that are necessary. First, the definition of the cohomology classes $\rho^{L/F}$ and $\rho^{K/F,X}$ are modified to yield classes

$$\rho_T^{L/F} \in H_{n-1}(F_{\overline{T}}^*, C_c(\mathbf{A}_F^{\overline{T}}, R)), \qquad \rho_T^{K/F,X} \in H_{n+r-1}(F_{\overline{T}}^*, C_c(\mathbf{A}_F^{X\cup\overline{T}}, \mathbf{Z}[G])).$$

Then, the proof of Theorem 7 is modified to yield

$$\Phi_T \cap \rho_T^{L/K} = \Theta_{S,T}^{L/F}(0) \in R.$$

For the details, we refer the reader to [4]. In the following however we will drop T, $\overline{T}$ again from the notation. However the reader should be aware that for the application to the Gross conjecture in Theorem 8 below, we really need the variant with $\rho_T^{L/K}$ and $\rho_T^{K/F,X}$ instead of $\rho^{L/K}$ and $\rho^{K/F,X}$.

3.4 Gross's conjecture

The following is the key technical result leading to the proof of Gross's conjecture.

Theorem 8. *Let $X = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ denote the set of places in S splitting completely in K and write $R = \mathbf{Z}[\tilde{G}]$. We have*

$$\rho_T^{L/F} \equiv (c_{\mathfrak{p}_1} \cup c_{\mathfrak{p}_2} \cup \cdots \cup c_{\mathfrak{p}_r}) \cap \rho_T^{K/F,X} \text{ in } H^{n-1}(F_{\overline{T}}^*, C_c(\mathbf{A}_F^{\overline{T}}, R/I^{r+1})). \tag{11}$$

Remark 9. To view the cup and cap products on the right side of (11) as elements of R/I^{r+1}, note that $H \subset H \otimes \mathbf{Z}[G] \cong I/I^2$ and apply

$$H \times \cdots \times H \times \mathbf{Z}[G] \longrightarrow I/I^2 \times \cdots \times I/I^2 \times R/I \longrightarrow I^r/I^{r+1} \subset R/I^{r+1}. \tag{12}$$

Here there are r factors of H on the left and of I/I^2 in the middle. The second arrow is multiplication.

Gross's conjecture follows easily from Theorem 8. Indeed, the map (12) lands in I^r/I^{r+1}, so we obtain that the image of $\rho_T^{L/F}$ in $H^{n-1}(F_{\overline{T}}^*, C_c(\mathbf{A}_F, R/I^r))$ is trivial. Since

$$\Theta_{S,T}^{L/K}(0) = \Phi_T \cap \rho_T^{L/F},$$

it follows that $\Theta_{S,T}^{L/K}(0) \equiv 0 \pmod{I^r}$ as desired.

4 Proof of Theorem 8

In the remainder of the paper we sketch a proof of Theorem 8. For simplicty in the following however we will drop T, $\overline{T}$ from the notation, i.e. we prove only the variant of (11) with $\rho_T^{L/F}$ and $\rho_T^{K/F,X}$ replaced by $\rho^{L/F}$ and $\rho^{K/F,X}$. The proof we give here is an explicit computation, in contrast to the more conceptual argument described in [4].

For the rest of the paper, let $R = \mathbf{Z}[\tilde{G}]/I^{r+1}$. Write $X = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, and $F_i := F_{\mathfrak{p}_i}$, $\mathcal{O}_i = \mathcal{O}_{\mathfrak{p}_i}$. We denote

$$\mathrm{rec}_{L/F,X} \colon \prod_{i=1}^{r} F_{\mathfrak{p}_i}^* \to \tilde{G} \to R$$

by the character χ, and let $\chi_i : F_i^* \to \tilde{G} \to R$ be the local component at $\mathfrak{p}_i$. Thus $\chi = \prod \chi_i$. Here the map $\tilde{G} \to R$ is induced by the natural inclusion $\tilde{G} \to \mathbf{Z}[\tilde{G}], g \mapsto [g]$.

Note that we have $\chi_i \equiv 1 \pmod{I}$, i.e. we can write $\chi_i = 1 + \psi_i$ for a function $\psi_i : F_{\mathfrak{p}}^* \to I$. The image of ψ_i mod I^2 is a group homomorphism $\overline{\psi}_i : F_{\mathfrak{p}}^* \to I/I^2$.

For each $\mathfrak{p}_i$, choose a power $\mathfrak{p}_i^{n_i} = (t_i)$ that is principal, generated by an element t_i that is totally positive and such that $\chi(t_i) = 1$. For each i, let $\mathcal{F}_i = \mathcal{O}_i - t_i \mathcal{O}_i$. Let T be the subgroup of $E_{+,X}$ generated by the t_i, let E' be the intersection of the kernel of χ with E_+, and let $\Gamma = T \times E_+$, which has finite index in $E_{+,X}$. The following is the first major step in our proof of Theorem 8.

Proposition 10. *Let $[\chi \cdot \mathbf{1}_{\mathcal{F}_1 \times \cdots \times \mathcal{F}_r}]$ be the class of the function $\chi \cdot \mathbf{1}_{\mathcal{F}_1 \times \cdots \times \mathcal{F}_r}$ in $H_0(T, C_c(F_1 \times \cdots \times F_r, R))$. We have*

$$[\chi \cdot \mathbf{1}_{\mathcal{F}_1 \times \cdots \times \mathcal{F}_r}] = (c_{\overline{\psi}_1} \cup \cdots \cup c_{\overline{\psi}_r}) \cap \eta_T$$

where $c_{\overline{\psi}_i} \in H^1(F_i, C_c(F_i, I/I^2))$ is the 1-cocycle associated to the homomorphism $\overline{\psi}_i$ as in §3.2 (i.e. replace $\mathrm{rec}_{\mathfrak{p}}$ by $\overline{\psi}_i$ in (10)) and $\eta_T \in H_{r-1}(T, \mathbf{Z})$ is the generator associated to $t_1, \ldots, t_r$.

Here the coefficients for the cup/cap products are viewed as taking values in $I^r/I^{r+1} \subset R$ as in Remark 9.

Proposition 10 implies the following stronger version. Note that by Shapiro's Lemma, we have an isomorphism

$$H_{n-1}(\Gamma, C_c(F_1^* \times \cdots \times F_r^*, \mathbf{Z})) \cong H_{n-1}(E', C_c(\mathcal{F}_1 \times \cdots \times \mathcal{F}_r, \mathbf{Z})). \tag{13}$$

Let $\epsilon_1, \ldots, \epsilon_{n-1}$ be an ordered basis of E'. For the group on the right, we get by taking cap product with the corresponding generator $\eta_{E'} \in H_{n-1}(E', \mathbf{Z}) \cong \mathbf{Z}$ an isomorphism

$$H^0(E', C_c(\mathcal{F}_1 \times \cdots \times \mathcal{F}_r, \mathbf{Z})) \cong H_{n-1}(E', C_c(\mathcal{F}_1 \times \cdots \times \mathcal{F}_r, \mathbf{Z})). \tag{14}$$

Denote by $\vartheta \in H_{n-1}(\Gamma, C_c(F_1^* \times \cdots \times F_r^*, \mathbf{Z}))$ the image of $\mathbf{1}_{\mathcal{F}_1 \times \cdots \times \mathcal{F}_r}$ under the composition of (14) with the inverse of the isomorphism (13). Taking the cap product with $\chi \in H^0(\Gamma, C_c(F_1^* \times \cdots \times F_r^*, R))$ gives a class

$$\chi \cap \vartheta \in H_{n-1}(\Gamma, C_c(F_1^* \times \cdots \times F_r^*, R)),$$

which by "extension by zero" yields a class (also denoted by $\chi \cap \vartheta$ by abuse of notation)

$$\chi \cap \vartheta \in H_{n-1}(\Gamma, C_c(F_1 \times \cdots \times F_r, R)).$$

The following is a strengthening of Proposition 10.

Proposition 11. *Let $\eta_\Gamma \in H_{n+r-1}(\Gamma, \mathbf{Z}) \cong \mathbf{Z}$ be the fundamental class corresponding to $\eta_T \otimes \eta_{E'}$ under the canonical isomorphism $H_r(T, \mathbf{Z}) \otimes H_{n-1}(E', \mathbf{Z}) \cong H_{n+r-1}(\Gamma, \mathbf{Z})$. Then*

$$\chi \cap \vartheta = (c_{\overline{\psi}_1} \cup \cdots \cup c_{\overline{\psi}_r}) \cap \eta_\Gamma$$

in $H_{n-1}(\Gamma, C_c(F_1 \times \cdots \times F_r, R))$.

To deduce Proposition 11 from Proposition 10, we show that the canonical map

$$H_{n-1}(\Gamma, C_c(F_1 \times \cdots \times F_r, R)) \to H_{n-1}(E_+, H_0(T, C_c(F_1 \times \cdots \times F_r, R))) \tag{15}$$

is an isomorphism; this will suffice since

$$H_{n-1}(E_+, H_0(T, C_c(F_1 \times \cdots \times F_r), R)) \cong H^0(E_+, H_0(T, C_c(F_1 \times \cdots \times F_r, R)))$$

and the right side is a subgroup of $H_0(T, C_c(F_1 \times \cdots \times F_r, R))$. By the Hochschild–Serre spectral sequence, the following proposition implies that (15) is an isomorphism.

Proposition 12. *The T-module $C_c(F_1 \times \cdots \times F_r, R)$ is homologically trivial, i.e. we have*

$$H_i(T, C_c(F_1 \times \cdots \times F_r, R)) = 0, \qquad i \geq 1.$$

For the remainder of this section we explain how Proposition 11 implies Theorem 8. Then in §5 and §6, we prove Propositions 10 and 12, thereby completing the proof.

Note that for Theorem 8 it suffices to show that

$$\overline{\rho}^{L/F} \equiv (c_{\mathfrak{p}_1} \cup c_{\mathfrak{p}_2} \cup \cdots \cup c_{\mathfrak{p}_r}) \cap \overline{\rho}^{K/F,X} \text{ in } H_{n-1}(F_+^*, C_c(\mathbf{A}_{F,X} \times (\mathbf{A}_F^X)^*/U^S, R)). \tag{16}$$

We denote the prime-to-X-part of the reciprocity map $\mathrm{rec}_{L/F}$ by

$$\xi = \mathrm{rec}_{L/F}^X : (\mathbf{A}_F^X)^*/U^S \longrightarrow \tilde{G} \subset R$$

so that $\mathrm{rec}_{L/F}(x) = \chi(x')\xi(x'')$ for

$$x = (x', x'') \in \left(\prod_{i=1}^r F_{\mathfrak{p}_i}^*\right) \times (\mathbf{A}_F^X)^*/U^S.$$

Note that $\xi(x) = \chi(x)\xi(x) = 1$ for all $x \in \Gamma$. There exists a subgroup Δ of F_+^* such that (i) $\Delta \cap E_{+,X} = 1$, (ii) $E_{+,X} \times \Delta$ has finite index in F^* and (iii) $\xi(a) = 1 = \chi(a)$ for all $a \in \Delta$. Indeed, since $F_+^* U^X/U^X$ is (as a subgroup of the free-abelian group $(\mathbf{A}_F^X)^*/U^X$) free-abelian, the projection $F_+^* \to F_+^* U^X/U^X$ has a section $s : F_+^* U^X/U^X \to F_+^*$. The group $\Delta' = s(F_+^*)$ satisfies conditions (i), (ii) and the subgroup $\Delta = \Delta' \cap \ker(\xi)$ conditions (i)-(iii).

Let $\mathcal{F}'$ be a fundamental domain for the action of Δ on $(\mathbf{A}_F^X)^*/U^X$ and let $\mathcal{F} \subseteq (\mathbf{A}_F^X)^*/U^S$ be its preimage under the projection $(\mathbf{A}_F^X)^*/U^S \to (\mathbf{A}_F^X)^*/U^X$. Note that $\mathcal{F}$ is compact and Γ-stable. Since $\Gamma \times \Delta$ has finite index in F_+^* there exists a restriction map

$$H_{n-1}(F_+^*, C_c(\mathbf{A}_{F,X} \times (\mathbf{A}_F^X)^*/U^S, R)) \longrightarrow H_{n-1}(\Gamma \times \Delta, C_c(\mathbf{A}_{F,X} \times (\mathbf{A}_F^X)^*/U^S, R)). \quad (17)$$

By Shapiro's Lemma we have an isomorphism

$$H_{n-1}(\Gamma \times \Delta, C_c(\mathbf{A}_{F,X} \times (\mathbf{A}_F^X)^*/U^S, R)) \cong H_{n-1}(\Gamma, C_c(\mathbf{A}_{F,X} \times \mathcal{F}, R)). \quad (18)$$

Furthermore we have an obvious cap-product pairing

$$\cap : H^0(\Gamma, C(\mathcal{F}, R)) \times H_{n-1}(\Gamma, C_c(F_1 \times \cdots \times F_r, R)) \to H_{n-1}(\Gamma, C_c(\mathbf{A}_{F,X} \times \mathcal{F}, R)).$$

Since Γ lies in the kernel of ξ we can view the restriction $\xi|_{\mathcal{F}}$ of the character ξ to $\mathcal{F}$ as an element in $H^0(\Gamma, C(\mathcal{F}, R))$. It is easy to see that the image of $\overline{\rho}^{L/F}$ under the composite of (17) and (18) is the class

$$\xi|_{\mathcal{F}} \cap (\chi \cap \vartheta)$$

and the image of $(c_{\mathfrak{p}_1} \cup c_{\mathfrak{p}_2} \cup \cdots \cup c_{\mathfrak{p}_r}) \cap \overline{\rho}^{K/F,X}$ is the class

$$\xi|_{\mathcal{F}} \cap (c_{\overline{\psi}_1} \cup \cdots \cup c_{\overline{\psi}_r}) \cap \eta_\Gamma.$$

We will show in §6 that (17) is injective. Thus (16) follows from Proposition 11.

5 Proof of Proposition 10

Let R be a commutative ring and $I \subset R$ an ideal such that $I^{r+1} = 0$. Let $T = \langle t_1, \ldots, t_r \rangle$, a free abelian group of rank r, and consider the R-module $M = H_0(T, C_c(F_1 \times \ldots \times F_r, R))$. Let $\eta_T = \sum_{\sigma \in S_r} \operatorname{sign}(\sigma)[t_{\sigma_1}| \cdots |t_{\sigma(r)}]$ be a generator of $H_r(T, \mathbf{Z}) \cong \mathbf{Z}$. Let

$$\chi = (\chi_1, \ldots, \chi_r) : F_1^* \times \cdots \times F_r^* \to R^*$$

be a character such that $\chi(T) = 1$ and $\chi_i \equiv 1 \pmod{I}$. Write $\psi_i = \chi_i - 1$ and $\mathcal{F}_i := \mathcal{O}_i - t_i\mathcal{O}_i$. Our goal is to prove that in M, we have

$$\chi \cdot \mathbf{1}_{\mathcal{F}_1 \times \cdots \times \mathcal{F}_r} \equiv (c_{\overline{\psi}_1} \cup \cdots \cup c_{\overline{\psi}_r}) \cap \rho, \quad (19)$$

where $c_{\overline{\psi}_i} \in H^1(F_i, C_c(F_i, I/I^2))$ as in (10).

To continue we introduce some notation, similar to [13]. Let A and B be disjoint subsets of $V = \{1, \ldots, r\}$. Define

$$\Lambda(A, B) := \prod_{a \in A} \psi_a \mathbf{1}_{\mathcal{F}_a} \prod_{b \in B} \mathbf{1}_{\mathcal{F}_b} \prod_{c \in V-A-B} \mathbf{1}_{\mathcal{O}_c}.$$

Writing $\chi = \prod \chi_i = \prod (1 + \psi_i)$ and multiplying out, we see that the left hand side of (19) is equal to

$$\chi \cdot \mathbf{1}_{\mathcal{F}_1 \times \cdots \times \mathcal{F}_r} = \sum_{A \subset V} \Lambda(A, V - A). \quad (20)$$

Let $J \subset V$. We say that a function $f : J \to V$ is *non-self including* if it satisfies the property

$$f(S) \not\subset S \text{ for all } S \subset J, S \neq \phi.$$

We let $N(J,V)$ be the set of non-self including maps $f\colon J \to V$. For any i,j, write

$$x(i,j) := \psi_i(t_j^{-1}) \in I.$$

We will prove that both sides of (19) are equal to

$$\sum_{J\subset V}\sum_{f\in N(J,V)}\prod_{j\in J} x(f(j),j)\Lambda(V-J,\phi).$$

5.1 LHS

We begin by noting that for any i and j, we have:

$$\begin{aligned}(1-t_j)(\psi_i)(x) &= (\psi_i(x)-\psi_i(t_j^{-1}x))\\ &= (\chi_i(x)-\chi_i(t_j^{-1}x))\\ &= \chi_i(x)(1-\chi_i(t_j^{-1}))\\ &= -(1+\psi_i(x))\psi_i(t_j^{-1}). \end{aligned} \tag{21}$$

If $i \neq j$, we have $(1-t_j)\mathbf{1}_{\mathcal{F}_i} = 0$ and hence in this case (21) yields

$$(1-t_j)(\psi_i\mathbf{1}_{\mathcal{F}_i}) = -x(i,j)(1+\psi_i)\mathbf{1}_{\mathcal{F}_i}. \tag{22}$$

This will enable us to deduce an inductive formula for the $\Lambda(A,B)$. For notational simplicity, if $b \in B$ we simply write $B-b$ for $B-\{b\}$ and similarly for $B\cup a$, etc.

Lemma 13. *Let A,B be disjoint subsets of V and let $b\in B$. Then*

$$\sum_{J\subset A, J\neq\phi}\left(\prod_{j\in J}x(j,b)\right)\sum_{J'\subset J}\Lambda(A-J',J'\cup B-b) = \sum_{J\subset A}\left(\prod_{j\in J}x(j,b)\right)\sum_{J'\subset J}\Lambda(A-J',J'\cup B). \tag{23}$$

Proof. We evaluate $(1-t_b)\Lambda(A,B-\{b\})$, which is 0 in M by definition. In general note that we have

$$(1-t)(f_1\dots f_r) = \sum_{J\subset V, J\neq\phi}(-1)^{|J|+1}\prod_{j\in J}(1-t)f_j\prod_{j\notin J}f_j.$$

In the present application, $(1-t_b)\mathbf{1}_{\mathcal{O}_c} = 0 = (1-t_b)\mathbf{1}_{\mathcal{F}_c}$ for $c\neq b$, and $(1-t_b)\mathbf{1}_{\mathcal{O}_b} = \mathbf{1}_{\mathcal{F}_b}$. Hence we need only consider sets J contained in $A\cup b$. Equation (22) shows that if $J\subset A$, then

$$\begin{aligned}\prod_{j\in J}(1-t_b)(\psi_j\mathbf{1}_{\mathcal{F}_j}) &= \prod_{j\in J}(-x(j,b))(1+\psi_j)\mathbf{1}_{\mathcal{F}_j}\\ &= \prod_{j\in J}(-x(j,b))\cdot\sum_{J'\subset J}\left(\prod_{i\in J-J'}\psi_i\mathbf{1}_{\mathcal{F}_i}\prod_{i\in J'}\mathbf{1}_{\mathcal{F}_i}\right).\end{aligned}$$

Combining these facts, we see that the contribution to $(1-t_b)\Lambda(A, B-b)$ from terms J not containing b (i.e. $J \subset A$) is the negative of the left side of (23), whereas the contribution from the terms $J \subset A \cup b$ containing b is the right side of (23), (with $J-b$ replacing J in the notation). $\square$

For any nonnegative integer i, let $M_i \subset M$ be the kernel of the natural map

$$M \to H_0(T, C_c(F_1 \times \cdots \times F_r, R/I^i)).$$

Note that

$$\Lambda(A, B) \in M_{|A|}, \quad I^i M_j \subset M_{i+j}, \quad M_0 = M, \quad \text{and } M_{r+1} = 0. \tag{24}$$

We can now evaluate the LHS of (19) without any more consideration of function spaces; in other words, the following result follows purely combinatorially from Lemma 13 and (24).

Lemma 14. *We have*

$$\sum_{A \subset V} \Lambda(A, V-A) = \sum_{J \subset V} \sum_{f \in N(J,V)} \left(\prod_{j \in J} x(f(j), j) \right) \Lambda(V-J, \phi).$$

To prove Lemma 14, we must first prove a general formula for $\Lambda(A, B)$. Given disjoint sets $A, B \subset V$ and a subset $C \subset A$, let $N'(A, B, C)$ denote the set of maps $f \in N(B \cup C, A)$ such that $C \subset \mathrm{Im}(f)$. By convention, we let $N'(A, \phi, \phi)$ have one element for any set A.

Lemma 15. *Let A and B be disjoint subsets of V. Let $n = |A| + |B|$. Then $\Lambda(A, B) \in M_n$ and*

$$\Lambda(A, B) \equiv \sum_{\substack{C \subset A \\ f \in N'(A,B,C)}} \Lambda(A-C, \phi) \prod_{j \in B \cup C} x(f(j), j) \pmod{M_{n+1}}. \tag{25}$$

Before proving the lemma, we first show that it completes our evaluation of the LHS.

Proof of Lemma 14. When $B = V - A$, $n = |A| + |B| = r$, and $M_{r+1} = 0$. Therefore Lemma 15 gives us a true equality in M in this case. Summing over all $A \subset V$, we obtain

$$\sum_{A \subset V} \Lambda(A, V-A) = \sum_{A \subset V} \sum_{\substack{C \subset A \\ f \in N'(A,V-A,C)}} \Lambda(A-C, \phi) \prod_{j \in (V-A) \cup C} x(f(j), j).$$

Change variables on the right side by letting $J = (V-A) \cup C$. We then obtain

$$\sum_{J \subset V} \sum_{\substack{C \subset J \\ f \in N'((V-J) \cup C, J-C, C)}} \prod_{j \in J} x(f(j), j) \Lambda(V-J, \phi).$$

To complete the proof, it remains to show that for a fixed $J \subset V$, there is a bijection between $N(J, V)$ and the set of pairs (C, f) with $C \subset J$ and $f \in N'((V-J) \cup C, J-C, C)$. This bijection is simply $f \mapsto (\mathrm{Im}(f) \cap J, f)$. $\square$

Proof of Lemma 15. We use induction. To this end, define a partial order on the set of pairs (A, B) of disjoint subsets of V lexicographically on the sizes of A and B; in other words,

$$(A, B) < (A', B') \text{ if } |A| < |A'| \text{ or } (|A| = |A'| \text{ and } |B| < |B'|).$$

For the base case of the induction, consider the case $B = \phi$ for any set A. $\Lambda(A, \phi) \in M_{|A|}$ is given. Furthermore, $N'(A, \phi, C)$ is empty unless $C = \phi$ in which case $N'(A, \phi, \phi)$ has one element by our convention; our congruence then simply reads $\Lambda(A, \phi) \equiv \Lambda(A, \phi)$.

For the inductive step we have $B \neq \phi$. Let $b \in B$ and apply Lemma 13. On the right side of (23), consider the terms where $J' = \phi$. When $J = \phi$ as well, we obtain $\Lambda(A, B)$. When $J' = \phi$ but $J \neq \phi$, we obtain terms of the form $x\Lambda(A, B)$ with $x \in I$. Therefore the total contribution of terms on the right side with $J' = \phi$ has the form $(1 + x)\Lambda(A, B)$ with $x \in I$. For terms on the right side with $J' \neq \phi$, we have $(A - J', B \cup J') < (A, B)$ with respect to our ordering, hence we may use the induction hypothesis to conclude that $\Lambda(A - J', B \cup J') \in M_n$. Furthermore $J' \neq \phi$ implies $J \neq \phi$ and therefore there is at least one factor $x(f(j), j) \in I$ in each such term; since $IM_n \subset M_{n+1}$, none of these terms contribute modulo M_{n+1}.

Now we consider the left side of (23). Note that all of the pairs appearing are less than (A, B) in our ordering: if $J' \neq \phi$, then $|A - J'| < |A|$, and if $J' = \phi$ then $|J' \cup B - b| < |B|$. We may use the induction hypothesis on all these terms, therefore, and conclude that $\Lambda(A - J', J' \cup B - b) \in M_{n-1}$. Furthermore, in this sum $J \neq \phi$ so there is at least one factor $x(f(j), j) \in I$ in each term. This implies all the following:

- each term on the left lies in M_n;
- the class modulo M_{n+1} of each term on the left depends only on $\Lambda(A - J', J' \cup B - b)$ modulo M_n;
- only terms where $|J| = 1$ contribute on the left modulo M_{n+1}, since other J give at least two factors $x(f(j), j) \in I$.

Combining all of these observations, we therefore deduce that

$$\sum_{a \in A} x(a, b)(\Lambda(A, B - b) + \Lambda(A - a, (B \cup a) - b)) \equiv (1 + x)\Lambda(A, B) \pmod{M_{n+1}}, \qquad (26)$$

with the left side (and hence the right) lying in M_n. Since $x \in I$ and $I^{r+1} = 0$, we have $(1 + x) \in R^*$ and $(1 + x)^{-1} \equiv 1 \pmod{I}$. Since the left hand side of (26) is in M_n, its class modulo M_{n+1} is unaffected by multiplication by elements in $1 + I$; therefore we obtain the same congruence with the $(1 + x)$ eliminated. Using the induction hypothesis for the two Λ

terms on the left, we obtain

$$\Lambda(A,B) \equiv \sum_{a\in A} x(a,b) \sum_{\substack{C\subset A \\ f\in N'(A,B-b,C)}} \Lambda(A-C,\phi) \prod_{j\in(B-b)\cup C} x(f(j),j)$$
$$+ \sum_{a\in A} x(a,b) \sum_{\substack{C\subset A-a \\ f\in N'(A-a,(B\cup a)-b,C)}} \Lambda(A-a-C,\phi) \prod_{j\in(B-b)\cup a\cup C} x(f(j),j) \pmod{M_{n+1}}.$$

$$\equiv \sum_{C\subset A} \Lambda(A-C,\phi) \left[\sum_{\substack{a\in A \\ f\in N'(A,B-b,C)}} x(a,b) \prod_{j\in B-b\cup C} x(f(j),j) + \right. \tag{27}$$

$$\left. \sum_{\substack{a\in C \\ f\in N'(A-a,(B\cup a)-b,C-a)}} x(a,b) \prod_{j\in B-b\cup C} x(f(j),j) \right] \pmod{M_{n+1}}. \tag{28}$$

Therefore, to conclude the proof, we must show that for A, B, C fixed (with A, B disjoint, $C\subset A$ and $b\in B$), that there is a bijection between $N'(A,B,C)$ and the disjoint union of the sets D_1 and D_2, where D_1 is the set of pairs (a,f) with $a\in A$ and $f\in N'(A,B-b,C)$, and D_2 is the set of pairs (a,f) with $a\in C$ and $f\in N'(A-a,(B\cup a)-b,C-a)$. Furthermore under this bijection, the product on the right side of (25) should match up with those in (27) and (28).

This bijection is easy to write down. Given $f\in N'(A,B,C)$, let $a=f(b)$. If $a\notin \mathrm{Im}(f|_{(B-b)\cup C})$ and $a\in C$, then we map f to $(a, f|_{(B-b)\cup C})$ in D_2. Otherwise, map f to $(a, f|_{(B-b)\cup C})$ in D_1. One checks that this is a bijection and that the corresponding products match up. (When $B=\{b\}$ and $C=\phi$, special care must be taken to note that one must adopt our convention that $N'(A,\phi,\phi)$ has one element to ensure that we retain a bijection in this case; this justifies our convention and our verification of the base case above.) This concludes the proof. □

5.2 RHS

The calculation of the right hand side of (19) follows exactly along the analogous part of [13], see e.g. §4.2 of *loc. cit.* For concreteness we write out some details.

The right side of (19) is by definition the image in M of

$$\sum_{\sigma\in S_r} \mathrm{sign}(\sigma) \prod_{i=1}^{r} c_{\overline{\psi}_i}(t_{\sigma(i)}) = \det(c_{\overline{\psi}_i}(t_j))_{i,j=1\ldots,r}. \tag{29}$$

Note that by definition we have

$$c_{\overline{\psi}_i}(t_j) = \begin{cases} ((1-t_i)\psi_i)(\mathbf{1}_{\mathcal{O}_i} - \mathbf{1}_{\mathcal{F}_i}) + \psi_i \mathbf{1}_{\mathcal{F}_i} & i=j \\ ((1-t_j)\psi_i)\mathbf{1}_{\mathcal{O}_i} & i\neq j. \end{cases}$$

In particular, all of these functions take values in I. Since (29) is a linear combination of products of r such terms, the result lies in M_r and furthermore depends only on the value of each $c_{\overline{\psi}_i}(t_j)$ mod I^2. In particular it follows from (21) that modulo I^2 we have

$$c_{\overline{\psi}_i}(t_j) \equiv \begin{cases} -x(i,i)(\mathbf{1}_{\mathcal{O}_i} - \mathbf{1}_{\mathcal{F}_i}) + \psi_i \mathbf{1}_{\mathcal{F}_i} & i = j \\ -x(i,j)\mathbf{1}_{\mathcal{O}_i} & i \neq j. \end{cases} \tag{30}$$

Lemma 16. *The value of* $(c_{\overline{\psi}_1} \cup \cdots \cup c_{\overline{\psi}_r}) \cap \rho$ *depends only on each* $c_{\overline{\psi}_i}(t_j)$ *modulo* $I\mathbf{1}_{\mathcal{F}_i}$.

Proof. If we multiply either of the two expressions on the right of (30) for $i = 1, \ldots, r$ and expand, then we obtain a linear combination of terms of the form $\Lambda(A,B)$ for disjoint subsets $A, B \subset V$. Furthermore, the coefficient of $\Lambda(A,B)$ arising from the various $x(i,j)$ terms lies in $I^{r-|A|}$. Since we proved in Lemma 15 that $\Lambda(A,B) \in M_{|A|+|B|}$, the product will lie in $M_{r+|B|}$. Therefore, this will be zero unless $B = \phi$. In other words, adding an element of $I\mathbf{1}_{\mathcal{F}_i}$ to $c_{\overline{\psi}_i}(t_j)$ does not alter the value of the cup product, as claimed. □

Thus (30) can be written more succinctly as

$$c_{\overline{\psi}_i}(t_j) \equiv -x(i,j)\mathbf{1}_{\mathcal{O}_i} + \delta_{ij}\psi_i \mathbf{1}_{\mathcal{F}_i} \pmod{(I^2, I\mathbf{1}_{\mathcal{F}_i})},$$

and we thus obtain:

Lemma 17. *We have*

$$(c_{\overline{\psi}_1} \cup \cdots \cup c_{\overline{\psi}_r}) \cap \rho = \det(-x(i,j)\mathbf{1}_{\mathcal{O}_i} + \delta_{ij}\psi_i \mathbf{1}_{\mathcal{F}_i})_{i,j=1,\ldots,r}.$$

We now conclude exactly as in [13, pp. 105-106]. Namely, expanding this determinant, we obtain

$$(c_{\overline{\psi}_1} \cup \cdots \cup c_{\overline{\psi}_r}) \cap \rho = \sum_{J \subset V} a(J)\Lambda(V - J, \phi) \qquad \text{where } a(J) = \det(-x(i,j))_{i,j \in J}.$$

Now since $1 = \chi(t_j^{-1}) = \prod_{i=1}^r (1 + \psi_i(t_j^{-1}))$, it follows that $\sum_{i=1}^r x(i,j) \equiv 0 \pmod{I^2}$. Lemma 4.9 of *loc. cit.* therefore implies that

$$a(J) \equiv \sum_{f \in N(J,V)} \prod_{i \in J} x(f(i), i) \pmod{I^{|J|+1}}.$$

In conclusion, we obtain the desired result:

Lemma 18. *We have in* M*:*

$$(c_{\overline{\psi}_1} \cup \cdots \cup c_{\overline{\psi}_r}) \cap \rho = \sum_{J \subset V} \sum_{f \in N(J,V)} \left(\prod_{i \in J} x(f(i), i) \right) \Lambda(V - J, \phi).$$

Combining (20), Lemma 14 and Lemma 18, we obtain Proposition 10.

6 Proof of Proposition 12 and the injectivity of (17)

Let R be a commutative ring endowed with the discrete topology and let $v_i : F \to \mathbf{Z} \cup \{+\infty\}$ be the normalized additive valuation associated to $\mathfrak{p}_i$ for $i = 1, \ldots, r$. Here we consider (slightly more generally than in §4) a subgroup T of F^* such that $T \mapsto \mathbf{Z}^r, t \mapsto (v_i(t))_{i=1,\ldots,r}$ is injective. Consider the $R[T]$-module $M = C_c(F_1 \times F_2 \times \cdots \times F_r, R)$. In this section we show that that M has a decreasing filtration of length 2^r

$$M = M^0 \supseteq M^1 \supseteq \ldots \supseteq M^{2^r} = 0 \tag{31}$$

by $R[T]$-submodules such that the following holds

(i) Each quotient M^m/M^{m+1} is an induced $R[T]$-module.

(ii) As an R-module the filtration (31) splits, i.e. the sequence $0 \to M^{n+1} \to M^n \to M^n/M^{n+1} \to 0$ splits as a sequence of R-modules (but not necessarily as $R[T]$-modules).

Since extensions of homologically trivial modules are again homologically trivial (i) implies in particular that M is homologically trivial, i.e. $H_i(T, M) = 0$ for $i \geq 1$.

The proof of the existence of (31) follows [13, §4.1] closely. We will use induction over r. Property (ii) is a technical condition needed in the induction step. It is equivalent to the existence of a sequence of R-modules $N_1, \ldots, N_{2^r}$ of M such that $M^n = \bigoplus_{j=n+1}^{2^r} N_j$ for $n = 0, \ldots, 2^r - 1$.

6.1 Construction of the filtration

Note that $M \cong M_1 \otimes_R M_2$ with $M_1 = C_c(F_1 \times \cdots \times F_{r-1}, R)$ and $M_2 = C_c(F_r, R)$. Let $T_1 = \{t \in T : v_r(t) = 0\}$ and choose $t_r \in T$ with $v_r(t_r) > 0$ such that T is generated by T_1 and t_r.

Let $\mathcal{F}$ denote a T_1-stable fundamental domain for the action of t_r on F_r^*, e.g. $\mathcal{F} = \mathcal{O}_r - t_r\mathcal{O}_r$. We define a filtration on M_2 by $M_2^0 = M_2$,

$$M_2^1 = \{f \in M_2 : f \text{ is constant on each } t_r^n\mathcal{F}, n \in \mathbf{Z}\},$$

and $M_2^2 = 0$. Note that each M_2^j is T-stable. Furthermore it is easy to see that the sequence

$$0 \to M_2^1 \to M_2 \to M_2/M_2^1 \to 0 \tag{32}$$

splits as a sequence of R-modules.

Proposition 19. *As a T-module, we have*

$$\operatorname{gr}^1 M_2 := M_2^1/M_2^2 = M_2^1 \cong \operatorname{Ind}_{T_1}^T R.$$

Proof. First we note that there is an isomorphism

$$M_2^1 \to M_2^1 \cap C_c(F_r^*, R) = \{g \in M_2^1 : g(0) = 0\}$$

given by $f \mapsto (1 - t_r)f$. Note that $(1 - t_r)f$ vanishes at 0 and hence defines an element of $C_c(F_r^*, R)$. To see this is an isomorphism, one may define an inverse by sending $g \in M_2^1 \cap C_c(F_r^*, R)$ to the function

$$((1 - t_r)^{-1}g)(x) := \sum_{n=0}^{\infty}(t_r^n g)(x) := \sum_{n=0}^{\infty} g(x/t_r^n), \qquad x \in F_i^*.$$

For a given $x \in F_r^*$, this is a finite sum, and one checks that it provides an inverse to the map $(1 - t_r)$. The key fact here is that $(1 - t_r)^{-1}g$ extends to a continuous function at 0. Indeed, the value at 0 is given by

$$\sum_{n=-\infty}^{\infty} g(x/t_r^n)$$

for any $x \in F_r^*$. (This is sum is actually finite on both ends.) The independence of this value on x follows from the fact that g is constant on $t_r^n \mathcal{F}_r$ for each n.

To conclude the proof, it is easy to see that

$$M_2^1 \cap C_c(F_r^*, R) \cong \operatorname{Ind}_{T_1}^T R.$$

Indeed, T_1 acts trivially on M_2^1, and the module R corresponds to the inclusion $R \cdot \mathbf{1}_{\mathcal{F}_1} \subset M_2^1$. □

Proposition 20. *With notation as above, we have an isomorphism of T-modules*

$$\operatorname{gr}^0 M_2 := M_2^0/M_2^1 \cong \operatorname{Ind}_{T_1}^T(C_c(\mathcal{F}_r, R)/R).$$

Proof. First note that there is an isomorphism of T-modules

$$C_c(F_r^*, R)/M_2^1 \cap C_c(F_r^*, R)) \to C_c(F_r, R)/M_2^1 = \operatorname{gr}^0 M_2.$$

Indeed, this map is simply induced by the canonical inclusion; it is surjective since every $f \in C_c(F_i, R)$ is equivalent modulo M_2^1 to $f - f(0)\mathbf{1}_{\mathcal{O}_r} \in C_c(F_r^*, R)$.

Now we note that $C_c(\mathcal{F}_r, R)/R \subset C_c(F_r^*, R)/(M_2^1 \cap C_c(F_r^*, R))$ is a T_1-stable subspace, and the isomorphism

$$C_c(F_r^*, R)/(M_2^1 \cap C_c(F_r^*, R)) \cong \operatorname{Ind}_{T_1}^T(C_c(\mathcal{F}_r, R)/R)$$

is easily seen. □

Tensoring (32) with M_1 and using the fact that for any $R[T_1]$-module N we have

$$M_1 \otimes_R \operatorname{Ind}_{T_1}^T N \cong \operatorname{Ind}_{T_1}^T(\widetilde{M}_1 \otimes_R N)$$

(where $\widetilde{M}_1$ is the group M_1 considered only as an $R[T_1]$-module) we obtain an exact sequence

$$0 \to \mathrm{Ind}_{T_1}^{T}(\widetilde{M}_1) \to M \to \mathrm{Ind}_{T_1}^{T}(\widetilde{M}_1 \otimes_R N) \to 0 \tag{33}$$

with $N = C_c(\mathcal{F}_r, R)/R$.

By the induction hypotheses the $R[T_1]$-module $\widetilde{M}_1$ has a filtration $\widetilde{M}_1 = \widetilde{M}_1^0 \supseteq \widetilde{M}_1^1 \supseteq \ldots \supseteq \widetilde{M}_1^{2^{r-1}} = 0$ by $R[T_1]$-submodules satisfying (i) and (ii). The latter implies that tensoring with N also yields a filtration on $\widetilde{M}_1 \otimes_R N$ of length 2^{r-1} with properties (i) and (ii). Applying $\mathrm{Ind}_{T_1}^{T}$, we see that the first and third $R[T]$-module in (33) have a filtration of length 2^{r-1} of the desired form. Hence we obtain a decreasing filtration $M^\bullet$ of length 2^r on the M, namely $M^{2^{r-1}} \supseteq M^{2^{r-1}+1} \supseteq \ldots \supseteq M^{2^r} = 0$ is the image of the filtration on $\mathrm{Ind}_{T_1}^{T}(\widetilde{M}_1)$ and the image of $M^0 \supseteq M^1 \supseteq \ldots \supseteq M^{2^{r-1}}$ under the second map in (33) is the filtration on $\mathrm{Ind}_{T_1}^{T}(\widetilde{M}_1 \otimes_R N)$. That (i) holds for $M^\bullet$ is obvious and (ii) is easily seen using the fact that (32) splits.

6.2 Injectivity of (17)

We show here more generally that the restriction

$$H_{n-1}(H, C_c(\mathbf{A}_{F,X} \times (\mathbf{A}_F^X)^*/U^S, R)) \to H_{n-1}(H', C_c(\mathbf{A}_{F,X} \times (\mathbf{A}_F^X)^*/U^S, R)) \tag{34}$$

is injective for a pair $H' \subseteq H$ of subgroups of F_+^* of finite index. For that we can decompose H in the form $H = \Gamma \times \Delta$ where $\Gamma = H \cap E_{+,X}$. It is clearly enough to consider the case $H' = \Gamma' \times \Delta'$ with either (1) $\Gamma' = \Gamma$ or (2) $\Delta' = \Delta$.

Put $M = C_c(\mathbf{A}_{F,X}, R)$, $M' = C_c((\mathbf{A}_F^X)^*/U^S, R)$ so that we have

$$C_c(\mathbf{A}_{F,X} \times (\mathbf{A}_F^X)^*/U^S, R) = M \otimes M'.$$

There exists a $R[\Gamma]$-module N such that $M' = \mathrm{Ind}_\Gamma^H N$. In fact we can choose $N = C(\mathcal{F}, R)$ where $\mathcal{F}$ is the preimage under the projection $(\mathbf{A}_F^X)^*/U^S \to (\mathbf{A}_F^X)^*/U^X$ of a fundamental domain for the action of Δ on $(\mathbf{A}_F^X)^*/U^X$. It follows that

$$C_c(\mathbf{A}_{F,X} \times (\mathbf{A}_F^X)^*/U^S, R) = \mathrm{Ind}_\Gamma^H(M \otimes N)$$

and (34) can be identified in case (1) with the diagonal embedding

$$H_{n-1}(\Gamma, M \otimes N) \to \bigoplus_{j=1}^{[\Delta:\Delta']} H_{n-1}(\Gamma, M \otimes N).$$

In case (2) the map (34) can be identified with the restriction

$$H_{n-1}(\Gamma, M \otimes N) \to H_{n-1}(\Gamma', M \otimes N) \tag{35}$$

To proceed further we decompose Γ as $E \times T$ with $E = H \cap E_+$. Again, we may assume $\Gamma' = E' \times T'$.

Using the filtration (31) above it is easy to see that $Q := M \otimes N$ as well has a decreasing filtration by $R[\Gamma]$-modules such that the graded quotients are induced as T-modules. In particular Q is homologically trivial as a T- as well as T'-module and $\mathrm{res} : H_0(T, Q) \to H_0(T', Q)$ is injective. Using the Hochschild-Serre spectral sequence we may identify (35) with the upper horizontal arrow of

$$\begin{array}{ccc} H_{n-1}(E, H_0(T,Q)) & \longrightarrow & H_{n-1}(E', H_0(T',Q)) \\ \Big\uparrow{\scriptstyle \cdot\cap\eta_E} & & \Big\uparrow{\scriptstyle \cap\cdot\eta_{E'}} \\ H^0(E, H_0(T,Q)) & \longrightarrow & H^0(E', H_0(T',Q)). \end{array}$$

Note that the vertical maps are isomorphisms. Since the lower horizontal map is injective we conclude that (35) is injective.

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