A Conjectural Product Formula for Brumer–Stark Units over Real Quadratic Fields

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Dedicated to the memory of David Hayes

Abstract

Following methods of Hayes, we state a conjectural product formula for ratios of Brumer–Stark units over real quadratic fields.

Contents

Introduction

Let F be a real quadratic field with ring of integers $\mathcal{O}$, and let $\mathfrak{f}$ be an integral ideal of F. Let f denote the positive generator of $\mathfrak{f} \cap \mathbf{Z}$. For each fractional ideal $\mathfrak{b}$ of F relatively prime to $\mathfrak{f}$ we associate the partial zeta function

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O} \\ (\mathfrak{a}, \mathfrak{f})=1 \\ \mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}}} \frac{1}{\mathrm{N}\mathfrak{a}^s}, \qquad s \in \mathbf{C},\ \mathrm{Re}(s) > 1.$$

Here we write $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ to denote that $\mathfrak{a}$ and $\mathfrak{b}$ have the same image in $\mathrm{Cl}^+(\mathfrak{f})$, the narrow ray class group of F associated to the conductor $\mathfrak{f}$.

Let $\mathfrak{c}$ denote a prime ideal of F not dividing $\mathfrak{f}$. The $\mathfrak{c}$-smoothed partial zeta functions of F are defined as follows:

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, s) = \zeta_{\mathfrak{f}}(\mathfrak{b}, s) - \mathrm{N}\mathfrak{c}^{1-s}\zeta_{\mathfrak{f}}(\mathfrak{b}\mathfrak{c}^{-1}, s). \tag{1}$$

The functions $\zeta_{\mathfrak{f}}$ and $\zeta_{\mathfrak{f},\mathfrak{c}}$ extend to meromorphic functions on the complex plane. The function $\zeta_{\mathfrak{f}}$ is analytic except for a simple pole at $s = 1$, and $\zeta_{\mathfrak{f},\mathfrak{c}}$ is analytic everywhere. The values of $\zeta_{\mathfrak{f}}$ at nonnegative integers s are rational numbers, and the values of $\zeta_{\mathfrak{f},\mathfrak{c}}$ at nonnegative integers s lie in $\mathbf{Z}[1/\ell]$, where ℓ is the prime of $\mathbf{Z}$ below $\mathfrak{c}$. In fact, if $\ell \geq 5$, then $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, 0)$ is an integer. Let us assume that this condition holds.

Let $H_{\mathfrak{f}}$ denote the narrow ray class field of F. Class field theory provides a canonical isomorphism

$$\mathrm{rec}\colon \mathrm{Cl}^+(\mathfrak{f}) \longrightarrow \mathrm{Gal}(H_{\mathfrak{f}}/F), \tag{2}$$

denoted $\mathfrak{a} \mapsto \sigma_{\mathfrak{a}}$. Let p be a prime number that is inert in F, and let H denote the maximal subfield of $H_{\mathfrak{f}}$ in which p splits completely. The reciprocity map induces an isomorphism between the quotient of $\mathrm{Cl}^+(\mathfrak{f})$ by the subgroup generated by the image of (p) and $\mathrm{Gal}(H/F)$. The partial zeta functions associated to the extension H/K are given by

$$\zeta_{H/K}(\mathfrak{b}, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O} \\ (\mathfrak{a},\mathfrak{f})=1 \\ \mathfrak{a} \sim_{\mathfrak{f},p} \mathfrak{b}}} \frac{1}{\mathrm{N}\mathfrak{a}^s} = \sum_{\substack{\mathfrak{a} \in \mathrm{Cl}^+(\mathfrak{f}) \\ \mathfrak{a} \sim_{\mathfrak{f},p} \mathfrak{b}}} \zeta_{\mathfrak{f}}(\mathfrak{a}, s).$$

Here $\mathfrak{a} \sim_{\mathfrak{f},p} \mathfrak{b}$ denotes equivalence in the quotient $\mathrm{Cl}^+(\mathfrak{f})/\langle (p) \rangle$. We assume that $\mathfrak{c}$ does not divide p, and define $\zeta_{H/K,\mathfrak{c}}(\mathfrak{b}, s)$ from $\zeta_{H/K}(\mathfrak{b}, s)$ as in (1).

As we now recall, the Brumer–Stark conjecture purports the existence of certain p-units in H whose valuations at the places above p are related to zeta values.

Conjecture 1 (Brumer–Stark). *Fix a prime $\mathfrak{P}$ of H above p. For every fractional ideal $\mathfrak{b}$ relatively prime to $\mathfrak{f}$, there exists a p-unit $u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}) \in H^\times$ such that:*

- $|u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})| = 1$ *at every archimedean place of H,*
- $\mathrm{ord}_{\mathfrak{P}}(\sigma_{\mathfrak{a}}(u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}))) = \zeta_{H/K,\mathfrak{c}}(\mathfrak{a}\mathfrak{b}, 0)$ *for every fractional ideal $\mathfrak{a}$ prime to $\mathfrak{f}$, and*
- $u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}) \equiv 1 \pmod{\mathfrak{c}\mathcal{O}_H}$.

If the Brumer–Stark units $u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})$ exist, then they are unique, and they satisfy

$$\sigma_{\mathfrak{a}}(u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})) = u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}\mathfrak{b}).$$

The goal of this paper is to give a conjectural infinite product formula for the elements $u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})$ in the p-adic field $H_{\mathfrak{P}} \cong F_p$. In fact, we can only give a product formula for the *ratio* of the p-unit $u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})$ with a certain power of the p-unit $u_{\mathfrak{e}}(\mathfrak{b}\mathfrak{f}^{-1})$ arising from the narrow Hilbert class field $H_{\mathfrak{e}}$ of F (here $\mathfrak{e} = (1)$ is the unit ideal). In the case where F has narrow class number 1, the p-units $u_{\mathfrak{e}}(\mathfrak{b})$ are equal to 1, so we do obtain a product formula for $u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})$ in this case.

In earlier work, we stated a conjectural formula for $u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}) \in H_{\mathfrak{P}}$ in terms of a certain p-adic multiplicative integral [D]. In that paper, we showed that our conjectural formula implied the refinements of the Brumer–Stark conjecture stated by Gross in [G1] and [G2]. In this paper, we manipulate our earlier integral formula following ideas of Hayes gleaned from [H1] and [H2] , and arrive at the desired product formula. The resulting infinite product formula shares a strong formal resemblance to the product defining the exponential functions of Drinfield modules. Indeed, Hayes's proof of both the Brumer–Stark conjecture and Gross's refinements in the function field setting using these exponential functions is the motivation for our present work.

The product formula

We assume that the prime ℓ splits in F, i.e. $(\ell) = \mathfrak{c}\bar{\mathfrak{c}}$ with $\mathfrak{c} \neq \bar{\mathfrak{c}}$, and that ℓ is relatively prime to $\mathfrak{f}$. For simplicity, we assume that $p \equiv 1 \pmod{f}$ so in particular $H = H_{\mathfrak{f}}$, though it should be possible to give a similar formula without this last assumption. Fix a positive integer ℓ' such that $\ell\ell' \equiv 1 \pmod{f}$.

Denote by $e\colon F \hookrightarrow \mathbf{R}^2$ the standard embedding $e(x) = (x^{(1)}, x^{(2)})$ given by the two real places of F. Define $\mathrm{Tr}, \mathrm{N} : \mathbf{R}^2 \to \mathbf{R}$ by

$$\mathrm{Tr}(x_1, x_2) = x_1 + x_2, \quad \mathrm{N}(x_1, x_2) = x_1 x_2,$$

so that under the composition with e these maps have the expected interpretation as trace and norm.

The group $F^\times$ acts on $\mathbf{R}^2$ by componentwise multiplication with $e(x)$, and under this action the group of totally positive elements of $F^\times$ preserves the positive quadrant $Q = (\mathbf{R}^{>0})^2$. We denote by $E(\mathfrak{f}) = \langle \epsilon_{\mathfrak{f}} \rangle$ the group of totally positive units of F congruent to 1 modulo $\mathfrak{f}$. Choose $\epsilon_{\mathfrak{f}}$ such that $\epsilon_{\mathfrak{f}}^{(2)} > \epsilon_{\mathfrak{f}}^{(1)}$. Let

$$m(\mathfrak{f}) = [E(\mathfrak{e}) : E(\mathfrak{f})], \quad \text{i.e. } \epsilon_{\mathfrak{f}} = \epsilon_{\mathfrak{e}}^{m(\mathfrak{f})}.$$

Let $\mathcal{D}$ denote the *Shintani domain* given by the sector

$$\mathcal{D} = \{x \cdot e(1) + y \cdot e(\epsilon_{\mathfrak{f}})\colon 0 < x,\ 0 \leq y\} \subset Q, \tag{3}$$

which is a fundamental domain for the action of $E(\mathfrak{f})$ on Q. For any positive integer n, let $N = N(n) = \ell\ell' p^n$. Let $\mathcal{D}_n$ denote the triangular region consisting of the intersection of $\mathcal{D}$ with the open half-plane bounded above by the line through $e(N)$ and $e(N\epsilon_{\mathfrak{f}})$:

$$\mathcal{D}_n = \{x \cdot e(1) + y \cdot e(\epsilon_{\mathfrak{f}})\colon 0 < x,\ 0 \leq y,\ x + y < N\}.$$

Let $\mathfrak{b}$ be an integral ideal of F relatively prime to $\ell p \mathfrak{f}$. Let $\mathscr{S}_n(\mathfrak{b})$ denote the set of elements of $\mathfrak{b}^{-1}$ that are congruent to 1 modulo $\mathfrak{f}$, relatively prime to p, and with image under e lying in $\mathcal{D}_n$. (Note that here and throughout this paper, the condition $\alpha \equiv 1 \pmod{\mathfrak{f}}$ for an arbitrary $\alpha \in F$ means that $\alpha - 1 \in \mathfrak{f}\mathcal{O}_{\mathfrak{q}}$ for each prime $\mathfrak{q}$ dividing $\mathfrak{f}$; here $\mathcal{O}_{\mathfrak{q}}$ is the completion of $\mathcal{O}$ at $\mathfrak{q}$.) Define $\mathscr{P}_n(\mathfrak{b})$ to be the product of the elements in $\mathscr{S}_n(\mathfrak{b})$, viewed as an element of $\mathcal{O}_p^\times$. We then propose:

Conjecture 2. *Fix a prime $\mathfrak{P}$ of H above p, giving an embedding $H \subset H_{\mathfrak{P}} \cong F_p$. The Brumer–Stark units satisfy*

$$\frac{u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})}{u_{\mathfrak{e},\mathfrak{c}}(\mathfrak{b}\mathfrak{f}^{-1})^{m(\mathfrak{f})}} = p^{\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b},0) - m(\mathfrak{f})\zeta_{\mathfrak{e},\mathfrak{c}}(\mathfrak{b}\mathfrak{f}^{-1},0)} \cdot \lim_{n\to\infty} \frac{\mathscr{P}_n(\mathfrak{b})}{\mathscr{P}_n(\mathfrak{b}\mathfrak{c}^{-1})^{\ell}} \in F_p^\times. \tag{4}$$

The integral formula

In earlier work [D], we presented a conjectural formula for the image of the Brumer–Stark unit $u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})$ in $F_p^\times$. This formula has the shape of a multiplicative integral rather than an infinite product. We now recall this formula, applied to our current setting.

Define for each $x \in \mathcal{O}/p^n$ the *Shintani zeta function*

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, \mathcal{D}, x, s) = \mathrm{N}\mathfrak{b}^{-s} \sum \frac{1}{\mathrm{N}\alpha^s}, \quad \mathrm{Re}(s) \gg 0,$$

where the sum runs over all $\alpha \in \mathfrak{b}^{-1}$ such that $\alpha \equiv 1 \pmod{\mathfrak{f}}$, $\alpha \equiv x \pmod{p^n}$, and $e(\alpha) \in \mathcal{D}$. Shintani proved that this zeta function extends to a meromorphic function on $\mathbf{C}$ and assumes rational values at nonpositive integers s. Define

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, \mathcal{D}, x, s) = \zeta_{\mathfrak{f}}(\mathfrak{b}, \mathcal{D}, x, s) - \ell^{1-s}\zeta_{\mathfrak{f}}(\mathfrak{b}\mathfrak{c}^{-1}, \mathcal{D}, x, s).$$

We showed in [D] (though this was already present in the earlier work of Cassou-Nogues [CN]) that the values $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, \mathcal{D}, x, 0)$ are integral. The following conjectural formula was presented in [D].

Conjecture 3. *Fix a prime $\mathfrak{P}$ of H above p, giving an embedding $H \subset H_{\mathfrak{P}} \cong F_p$. The Brumer–Stark units are given by the formula*

$$u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}) := p^{\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b},0)} \cdot \lim_{n\to\infty} \prod_{x \in (\mathcal{O}/p^n)^\times} x^{\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b},\mathcal{D},x,0)}. \tag{5}$$

Despite the fact that the formulae (4) and (5) both take the shape of p-adic limits of finite products, we would like to stress the qualitative difference between these two expressions. Formula (4) is truly an "infinite product" in that it is a limit of products of elements in a growing nested sequence of sets. Formula (5) does not have this shape — the terms in the product for a given n do not appear in the product for $n+1$. Instead, the limit in formula (5) should be interpreted as a multiplicative integral over the space $\mathcal{O}_p^\times$.

It is clear that the right side of (5) satisfies the second condition of Conjecture 1 for $\mathfrak{a} = 1$, i.e. its $\mathfrak{P}$-adic valuation is $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, 0)$. It is proven in [D] that this element is independent of the Shintani domain $\mathcal{D}$ chosen, and that it satisfies the analytic properties of the refinements of Conjecture 1 given by Gross in [G1] and [G2]. (See [D], Theorem 5.3 and the paragraph following its proof, Proposition 3.5, and Theorem 3.22.) Of course, the algebraicity of this element is left open.

The main theorem of this paper is that formula (5) implies (4).

Theorem 4. *Conjecture 3 implies Conjecture 2.*

As mentioned earlier, our interest in formula (4) lies with the connection to the function field setting of Stark's conjectures. In [H1], Hayes proved the Brumer–Stark conjecture and Gross's refinements over function fields using the theory of Drinfeld modules. Using Hayes's methods, we proved the analogs of Conjectures 2 and 3 in the function field setting in [DM].

As part of that study, we proved the analog of Theorem 4 in the function field setting (see [DM, Proposition 8.1]).

We conclude this introduction by noting the strong influence of the work of David Hayes on our current investigations. Our paper [DM] was built upon Hayes's paper [H1], and the methods of the present paper are drawn entirely from Hayes's paper [H2] . We are grateful also to Benedict Gross and Paul Gunnels for making us aware of this latter paper.

1 Strategy of the proof

Let $\mathcal{D}_{\mathfrak{e}}$ be the Shintani domain for $E(\mathfrak{e})$ given by (3), with $\epsilon_{\mathfrak{f}}$ replaced by $\epsilon_{\mathfrak{e}}$. Then the Shintani domain $\mathcal{D}$ for $E(\mathfrak{f})$ can be written as a disjoint union

$$\mathcal{D} = \bigcup_{i=0}^{m(\mathfrak{f})-1} \epsilon_{\mathfrak{e}}^i \mathcal{D}_{\mathfrak{e}}.$$

Since formula (5) is independent of domain (see [D, Theorem 5.3]), Conjecture 3 yields

$$\frac{u_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b})}{u_{\mathfrak{e}}(\mathfrak{b}\mathfrak{f}^{-1})^{m(\mathfrak{f})}} = p^{\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b},0)-m(\mathfrak{f})\zeta_{\mathfrak{e},\mathfrak{c}}(\mathfrak{b}\mathfrak{f}^{-1},0)} \cdot \lim_{n\to\infty} \prod_{x\in(\mathcal{O}/p^n)^\times} x^{Z_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b},\mathcal{D},x,0)} \tag{6}$$

where

$$Z_{\mathfrak{f}}(\mathfrak{b},\mathcal{D},x,s) = \zeta_{\mathfrak{f}}(\mathfrak{b},\mathcal{D},x,s) - \mathrm{N}\mathfrak{f}^{-s}\zeta_{\mathfrak{e}}(\mathfrak{b}\mathfrak{f}^{-1},\mathcal{D},x,s)$$

and

$$Z_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b},\mathcal{D},x,s) = Z_{\mathfrak{f}}(\mathfrak{b},\mathcal{D},x,s) - \ell^{1-s}\cdot Z_{\mathfrak{f}}(\mathfrak{b}\mathfrak{c}^{-1},\mathcal{D},x,s). \tag{7}$$

Now

$$\mathrm{N}\mathfrak{b}^s \cdot Z_{\mathfrak{f}}(\mathfrak{b},\mathcal{D},x,s) = \sum_{\substack{\alpha\in\mathfrak{b}^{-1},\, e(\alpha)\in\mathcal{D} \\ \alpha\equiv 1\,(\mathrm{mod}\ \mathfrak{f}) \\ \alpha\equiv x\,(\mathrm{mod}\ p^n)}} \frac{1}{\mathrm{N}\alpha^s} - \sum_{\substack{\alpha\in\mathfrak{b}^{-1}\mathfrak{f},\, e(\alpha)\in\mathcal{D} \\ \alpha\equiv x\,(\mathrm{mod}\ p^n)}} \frac{1}{\mathrm{N}\alpha^s}. \tag{8}$$

Let $N = \ell\ell' p^n$, so in particular $N \in \mathfrak{b}^{-1}\mathfrak{c}$, $N \equiv 0 \pmod{p^n}$, and $N \equiv 1 \pmod{\mathfrak{f}}$. Let

$$M = M(\mathfrak{b},x) = \{\alpha\in\mathfrak{b}^{-1}\colon \alpha\equiv x\,(\mathrm{mod}\ p^n),\ \alpha\equiv 1\,(\mathrm{mod}\ \mathfrak{f})\}.$$

Following Hayes, we decompose the first sum in (8) as $\Sigma_1 + \Sigma_2$, where

$$\Sigma_1 := \sum_{\alpha} \frac{1}{\mathrm{N}\alpha^s} \qquad (\alpha\in M;\quad e(\alpha)\in\mathcal{D}-(e(N)+\mathcal{D}))$$

and

$$\Sigma_2 := \sum_{\alpha} \frac{1}{\mathrm{N}\alpha^s} \qquad (\alpha\in M;\quad e(\alpha)\in e(N)+\mathcal{D}).$$

Letting

$$L = \{\alpha\in\mathfrak{b}^{-1}\mathfrak{f}\colon \alpha\equiv x\,(\mathrm{mod}\ p^n)\}$$

(so $M = L + N$), we see from (8) that

$$Z_{\mathfrak{f}}(\mathfrak{b}, \mathcal{D}, x, s) = \Sigma_1 + \Sigma_2 - \Sigma_3 \tag{9}$$

where

$$\Sigma_2 - \Sigma_3 = \sum_{\alpha} \left(\frac{1}{\mathrm{N}(\alpha + N)^s} - \frac{1}{\mathrm{N}\alpha^s} \right), \qquad (\alpha \in L; \quad e(\alpha) \in \mathcal{D}).$$

In section 3 we prove that the contribution of the $\Sigma_2 - \Sigma_3$ terms to $Z_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, x, \mathcal{D}, 0)$ is 0:

$$\big((\Sigma_2 - \Sigma_3)(\mathfrak{b}) - \ell^{1-s}(\Sigma_2 - \Sigma_3)(\mathfrak{b}\mathfrak{c}^{-1})\big)\,|_{s=0} = 0. \tag{10}$$

To study the contribution of Σ_1, we decompose $\mathcal{D} - (e(N) + \mathcal{D})$ as a disjoint union $\mathcal{D}_n \sqcup T$, where T is the half-open strip

$$T = T(n) = \{x \cdot e(1) + y \cdot e(\epsilon_{\mathfrak{f}} 1) : \quad 0 < x \leq N, \quad N \leq x + y\}.$$

Now

$$\sum_{\gamma} \frac{1}{\mathrm{N}\gamma^s}, \qquad (\gamma \in M; \quad e(\gamma) \in \mathcal{D}_n)$$

is a finite sum since $\mathcal{D}_n$ is bounded and M is discrete; therefore its value at $s = 0$ is simply $\#(\mathcal{D}_n \cap e(M))$. If we write Σ_T for the value of the analytic continuation of the series

$$\sum_{\gamma} \frac{1}{\mathrm{N}\gamma^s}, \qquad (\gamma \in M; \quad e(\gamma) \in T)$$

at $s = 0$, we thus have

$$(\Sigma_1)|_{s=0} = \#(\mathcal{D}_n \cap e(M)) + \Sigma_T. \tag{11}$$

We will show in section 4 that the contribution of Σ_T to $Z_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, x, \mathcal{D}, 0)$ is equal to zero:

$$\Sigma_T(\mathfrak{b}) - \ell \cdot \Sigma_T(\mathfrak{b}\mathfrak{c}^{-1}) = 0. \tag{12}$$

Combining (7), (9), (10), (11), and (12), we therefore have

$$Z_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, x, \mathcal{D}, 0) = \#(\mathcal{D}_n \cap e(M(\mathfrak{b}, x))) - \ell \cdot \#(\mathcal{D}_n \cap e(M(\mathfrak{b}\mathfrak{c}^{-1}, x))).$$

We use this expression to evaluate the limit in (6). Note that

$$\prod_{x \in (\mathcal{O}/p^n)^\times} x^{\#(\mathcal{D}_n \cap e(M(\mathfrak{b}, x)))} \equiv \prod_{\alpha \in \mathscr{S}_n(\mathfrak{b})} \alpha = \mathscr{P}_n(\mathfrak{b}) \pmod{p^n}$$

and similarly

$$\prod_{x \in (\mathcal{O}/p^n)^\times} x^{\#(\mathcal{D}_n \cap e(M(\mathfrak{b}\mathfrak{c}^{-1}, x)))} \equiv \mathscr{P}_n(\mathfrak{b}\mathfrak{c}^{-1}) \pmod{p^n}.$$

It follows that the right side of (6) is equal to the right side of (4). This completes the proof of Theorem 4, up to the demonstration of (10) and (12).

2 Preliminaries

We recall the notation and some basic results from [H2]. For a fractional ideal $\mathfrak{a}$ of F, let $\mu(\mathfrak{a})$ denote the positive generator of the cyclic group $\mathfrak{a}\cap\mathbf{Q}$. Define $\nu(\mathfrak{a})$ by $\mathrm{N}\mathfrak{a} = \mu(\mathfrak{a})\cdot\nu(\mathfrak{a})$. Now let $\mathfrak{b}$ and $\mathfrak{f}$ be relatively prime integral ideals as in the introduction, and let $f = \mu(\mathfrak{f})$. We write $q = \mu(\mathfrak{b}^{-1}\mathfrak{f})$. By [H2, Lemma 3.8], we have

$$q = f/g \text{ for some positive integer } g \text{ relatively prime to } f. \tag{13}$$

Write $\mathcal{O} = \mathbf{Z}+\mathbf{Z}\omega$. Recall that $(\ell) = \mathfrak{c}\bar{\mathfrak{c}}$ is a prime that splits in F, and that is ℓ relatively prime to $\mathfrak{b}$ and $\mathfrak{f}$. Note that

$$\mu(\mathfrak{b}^{-1}\mathfrak{f}\mathfrak{c}) = q\ell \quad \text{and} \quad \nu(\mathfrak{b}^{-1}\mathfrak{f}) = \nu(\mathfrak{b}^{-1}\mathfrak{f}\mathfrak{c}).$$

By [H2, Lemma 3.6], there exists $y \in \mathbf{Q}$ such that $\sigma := y + \nu(\mathfrak{b}^{-1}\mathfrak{f})\omega \in \mathfrak{b}^{-1}\mathfrak{f}\mathfrak{c}$, and then we have

$$\mathfrak{b}^{-1}\mathfrak{f} = \mathbf{Z}q \oplus \mathbf{Z}\sigma, \quad \mathfrak{b}^{-1}\mathfrak{c}\mathfrak{f} = \mathbf{Z}q\ell \oplus \mathbf{Z}\sigma. \tag{14}$$

Define a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$ by

$$(q \ \ \sigma)\,\epsilon_{\mathfrak{f}} = (q \ \ \sigma)\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{15}$$

Then in fact $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(f)$, i.e. we have

$$c \equiv a-1 \equiv d-1 \equiv 0 \pmod{f}, \tag{16}$$

by [H2, Lemma 3.10]. We require one additional result before proceeding with the proof of (10) and (12).

Lemma 2.1. *Define $\delta \in \mathbf{Z}$ by $f\delta = \gcd(c, d-1)$. Then $\ell \nmid \frac{d-1}{f\delta}$.*

Proof. From (15) we find

$$\begin{aligned} \epsilon_{\mathfrak{f}}^{-1} - 1 &= (d-1) - \frac{c}{q}\sigma \\ &= \delta g\left(\frac{d-1}{f\delta}q - \frac{c}{f\delta}\sigma\right). \end{aligned} \tag{17}$$

From (14) we see that $\ell \mid \frac{d-1}{f\delta}$ implies that

$$\epsilon_{\mathfrak{f}}^{-1} - 1 \in \delta g \cdot \mathfrak{b}^{-1}\mathfrak{f}\mathfrak{c}. \tag{18}$$

Conjugating (18) and multiplying by $-\epsilon_{\mathfrak{f}}$, we obtain

$$\epsilon_{\mathfrak{f}}^{-1} - 1 \in \delta g \cdot \overline{\mathfrak{b}^{-1}\mathfrak{f}\mathfrak{c}}. \tag{19}$$

Since $\mathfrak{b}$ and $\mathfrak{f}$ are relatively prime to $(\ell) = \mathfrak{c}\bar{\mathfrak{c}}$ by assumption, (18) and (19) imply that

$$\epsilon_{\mathfrak{f}}^{-1} - 1 \in \delta g\ell \cdot \mathfrak{b}^{-1}\mathfrak{f},$$

which in conjunction with (17) implies that $\ell \mid \frac{c}{f\delta}$. This divisibility along with $\ell \mid \frac{d-1}{f\delta}$ contradicts the definition of δ. $\square$

3 The contribution of $\Sigma_2 - \Sigma_3$

In this section, we use a formula of Shintani (again following Hayes) to evaluate the analytic continuation of $\Sigma_2 - \Sigma_3$ at $s = 0$.

If ρ, τ are totally positive basis vectors for $\mathbf{R}^2$ and $\gamma \in Q$, the function

$$\sum_{m,n=0}^{\infty} \frac{1}{\mathrm{N}(\gamma + m\rho + n\tau)^s}$$

converges absolutely for $\mathrm{Re}(s) > 1$. This function can be analytically continued to the entire complex plane and at $s = 0$ has the value

$$B_1(w)B_1(z) + \frac{1}{4}\left(B_2(w)\,\mathrm{Tr}(\rho/\tau) + B_2(z)\,\mathrm{Tr}(\tau/\rho)\right), \tag{20}$$

where $w, z \in \mathbf{R}$ are given by $\gamma = w\rho + z\tau$. Here $B_1(x) = x - 1/2$ and $B_2(x) = x^2 - x + 1/6$ are the usual Bernoulli polynomials. We will apply formula (20) with $\rho = e(Nq)$ and $\tau = e(Nq\epsilon_{\mathfrak{f}})$, in which case

$$\mathrm{Tr}(\rho/\tau) = \mathrm{Tr}(\tau/\rho) = a + d.$$

Let us write $x \equiv uq\ell + v\sigma \pmod{p^n}$ where u and v are integers that are uniquely well-defined modulo p^n. Every $\alpha \in L$ with $e(\alpha) \in \mathcal{D}$ can be written uniquely as

$$\alpha = \gamma + mp^n q + np^n q\epsilon_{\mathfrak{f}},$$

where $m, n \geq 0$ are integers and

$$\gamma = x_1 p^n q + x_2 p^n q \epsilon_{\mathfrak{f}}, \qquad 0 < x_1 \leq 1,\ 0 \leq x_2 < 1,$$

such that $\gamma \in L$. In view of (14) and the equation $q\epsilon_{\mathfrak{f}} = qa + \sigma c$ (from (15)), the condition $\gamma \in L$ is equivalent to

$$\begin{aligned} x_1 + x_2 a &\in \frac{u\ell}{p^n} + \mathbf{Z} \\ x_2 &\in \frac{1}{c}\left(\frac{v}{p^n} + \mathbf{Z}\right). \end{aligned}$$

Therefore, writing $\langle x \rangle = x - \lfloor x \rfloor$ and

$$\{x\} = \begin{cases} x - \lfloor x \rfloor & \text{if } x \notin \mathbf{Z} \\ 1 & \text{if } x \in \mathbf{Z}, \end{cases}$$

we have

$$\Sigma_3|_{s=0} = \sum_{h=0}^{c-1} \left(B_1(x_1)B_1(x_2) + \frac{a+d}{4}(B_2(x_1) + B_2(x_2)) \right)$$

where

$$x_2 = \left\langle \frac{1}{c}\left(\frac{v}{p^n}+h\right)\right\rangle \qquad \text{and} \qquad x_1 = \left\{\frac{u\ell}{p^n} - x_2 a\right\}.$$

Now $\Sigma_2|_{s=0}$ is given by the exact same expression with x_1 replaced by $x_1 + \ell\ell'/q$, and hence we obtain

$$(\Sigma_2 - \Sigma_3)|_{s=0} = \sum_{h=0}^{c-1}\left(\frac{\ell\ell'}{q}B_1(x_2) + \frac{a+d}{4}\left(B_2\left(x_1+\frac{\ell\ell'}{q}\right) - B_2(x_1)\right)\right).$$

The first term in this expression is easily calculated to equal $(\langle v/p^n\rangle - 1/2)\ell\ell'/q$ using the distribution relation for B_1. The second term is calculated using the formula

$$B_2(x+y) - B_2(x) = 2yB_1(x) + y^2$$

and the distribution relation for B_1 to equal

$$\frac{a+d}{4}\cdot\frac{\ell\ell'}{q}\left(2\left\{\frac{uc\ell - va}{p^n}\right\} - 1 + \frac{c\ell\ell'}{q}\right).$$

Therefore our final expression is

$$(\Sigma_2 - \Sigma_3)|_{s=0} = \frac{\ell\ell'}{q}\left[\left\langle\frac{v}{p^n}\right\rangle - \frac{1}{2} + \frac{a+d}{4}\left(2\left\{\frac{u\ell c - va}{p^n}\right\} + \frac{c\ell\ell'}{q} - 1\right)\right].$$

The corresponding expression for $\mathfrak{b}$ replaced by $\mathfrak{b}\mathfrak{c}^{-1}$ is obtained by replacing q by $q\ell$, $u\ell$ by u, and c by $c\ell$ — we see by inspection that we obtain exactly the same value divided by ℓ. In other words, the contribution of the $\Sigma_2 - \Sigma_3$ terms to $Z_{\mathfrak{f},\mathfrak{c}}(\mathfrak{b}, x, \mathcal{D}, 0)$ is 0:

$$\left((\Sigma_2 - \Sigma_3)(\mathfrak{b}) - \ell^{1-s}(\Sigma_2 - \Sigma_3)(\mathfrak{b}\mathfrak{c}^{-1})\right)|_{s=0} = 0.$$

4 The strip T

It remains to prove (12). We once again write $x \equiv uq\ell + v\sigma \pmod{p^n}$ with $u, v \in \mathbf{Z}$, so that by (14) we have $M = uq\ell + v\sigma + N + \mathbf{Z}qp^n \oplus \mathbf{Z}\sigma p^n$. Therefore we can write any $\gamma \in M$ in the form

$$\begin{aligned}\gamma &= uq\ell + v\sigma + N + qp^n h + \sigma p^n j \qquad (h, j \in \mathbf{Z})\\ &= x + y\epsilon_{\mathfrak{f}}\end{aligned}$$

where

$$\begin{aligned}x &= uq\ell + N + qp^n h - \frac{aq}{c}(v + p^n j),\\ y &= \frac{q}{c}(v + p^n j).\end{aligned}$$

These latter equalities follow from $\sigma = \frac{q}{c}(\epsilon_{\mathfrak{f}} - a)$, which is deduced from (15).

Let us enact the change of variables $(w \;\; z) = (j \;\; h)\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Then

$$\begin{aligned} x = x(w) &= N + \frac{q}{c}(u\ell c - av) - \frac{qp^n}{c}w \\ y = y(w,z) &= \frac{q}{c}(v + p^n dw + p^n cz). \end{aligned}$$

The inequalities $0 < x \leq N$ and $N \leq x + y$ defining the condition $e(\gamma) \in T$ simplify to

$$\begin{aligned} \frac{u\ell c - av}{p^n} \leq w < \frac{u\ell c - av + Nc/q}{p^n}, \\ z \geq \frac{-u\ell c + (a-1)v + (1-d)wp^n}{cp^n}. \end{aligned}$$

Therefore, if we let

$$\lambda := \left\lceil \frac{u\ell c - av}{p^n} \right\rceil \quad \text{and} \quad \beta_w := \left\lceil \frac{-u\ell c + (a-1)v + (1-d)wp^n}{cp^n} \right\rceil,$$

then

$$\sum_{\substack{\gamma \in M \\ e(\gamma) \in T}} \frac{1}{\mathrm{N}\gamma^s} = \sum_{w=\lambda}^{\lambda + \ell\ell' c/q - 1} \sum_{k=0}^{\infty} \frac{1}{\mathrm{N}(x(w) + y(w, \beta_w)\epsilon_{\mathfrak{f}} + kqp^n\epsilon_{\mathfrak{f}})^s}. \tag{21}$$

We now invoke another formula of Shintani: for $\beta \in Q$, the function

$$\sum_{k=0}^{\infty} \frac{1}{\mathrm{N}(\beta + e(k))^s}$$

extends to a meromorphic function in $s \in \mathbf{C}$, and obtains the value $\frac{1}{2}(1 - \mathrm{Tr}(\beta))$ at $s = 0$. Applying this formula with $\beta = e(\beta')$, where

$$\begin{aligned} \beta' &= \frac{x(w) + y(w, \beta_w)\epsilon_{\mathfrak{f}}}{qp^n\epsilon_{\mathfrak{f}}} \\ &= \left(\frac{N}{qp^n} + \frac{u\ell c - av}{cp^n} - \frac{w}{c} \right) \epsilon_{\mathfrak{f}}^{-1} + \frac{1}{cp^n}\left(v + p^n dw + p^n c\beta_w\right), \end{aligned}$$

we see from (21) that

$$\Sigma_T = \sum_{w=\lambda}^{\lambda + \ell\ell' c/q - 1} \left[\frac{1}{2} - \left(\frac{N}{qp^n} + \frac{u\ell c - av}{cp^n} - \frac{w}{c} \right) \frac{a+d}{2} - \frac{1}{cp^n}\left(v + p^n dw + p^n c\beta_w\right) \right]. \tag{22}$$

As in section 3, the analogous expression for $\mathfrak{b}$ replaced by $\mathfrak{b}\mathfrak{c}^{-1}$ is obtained by replacing c by $c\ell$, q by $q\ell$, and $u\ell$ by u. One sees by inspection that in the difference $\Sigma_T(\mathfrak{b}) - \ell\Sigma_T(\mathfrak{b}\mathfrak{c}^{-1})$, all the terms in (22) cancel except the first and last, i.e. the $1/2$ and the $-\beta_w$. In other words, if we write

$$\alpha = \frac{-u\ell c + (a-1)v}{cp^n} \quad \text{and} \quad \alpha_w = \alpha + \frac{1-d}{c} w$$

so that $\beta_w = \lceil \alpha_w \rceil$, then

$$\begin{aligned} \Sigma_T(\mathfrak{b}) - \ell \cdot \Sigma_T(\mathfrak{b}\mathfrak{c}^{-1}) &= \sum_{w=\lambda}^{\lambda+\ell\ell' c/q-1} \left(\left(\frac{1}{2} - \lceil \alpha_w \rceil \right) - \ell \left(\frac{1}{2} - \left\lceil \frac{\alpha_w}{\ell} \right\rceil \right) \right) \\ &= \sum_{w=\lambda}^{\lambda+\ell\ell' c/q-1} \left(\tilde{B}_1(\alpha_w) - \ell \tilde{B}_1(\alpha_w/\ell) \right), \end{aligned} \tag{23}$$

where $\tilde{B}_1(x) = B_1(\{x\}) = \{x\} - 1/2$. Since $\tilde{B}_1(x)$ depends only on x mod $\mathbf{Z}$, and since

$$\frac{1-d}{q} = g \cdot \frac{1-d}{f} \in \mathbf{Z}$$

(see (13) and (16)), it follows that the terms $\tilde{B}_1(\alpha_w)$ and $\tilde{B}_1(\alpha_w/\ell)$ depend only the value of w modulo $\ell\ell' c/q$; in other words, the value of the integer λ does not affect the sum (23). Furthermore, if we recall that δ is the integer such that $f\delta = \gcd(c, d-1)$, then the distribution relation for $\tilde{B}_1(x)$ implies that

$$\sum_{w \bmod \ell\ell' c/q} \tilde{B}_1(\alpha_w) = \ell\ell' g\delta \cdot \tilde{B}_1\left(\frac{\alpha c}{f\delta}\right).$$

Lemma 2.1 implies that $f\delta = \gcd(c\ell, d-1)$, so

$$\sum_{w \bmod \ell\ell' c/q} \tilde{B}_1(\alpha_w/\ell) = \ell' g\delta \cdot \tilde{B}_1\left(\frac{\alpha c}{f\delta}\right).$$

It follows that $\Sigma_T(\mathfrak{b}) - \ell \cdot \Sigma_T(\mathfrak{b}\mathfrak{c}^{-1}) = 0$ as desired.

References

[CN] Pi. Cassou-Noguès, *Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques.* Invent. Math. **51** (1979), no. 1, 29–59.

[D] S. Dasgupta, *Shintani zeta functions and Gross–Stark units for totally real fields.* Duke Math. J. **143** (2008), no. 2, 225–279.

[DM] S. Dasgupta and A. Miller, *A Shintani-type formula for Gross–Stark units over function fields.* J. Math. Sci. Univ. Tokyo **16** (2009), no. 3, 415–440.

[G1] B. Gross, *p-adic L-series at $s = 0$.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 979–994 (1982).

[G2] B. Gross, *On the values of abelian L-functions at $s = 0$.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 35 (1988), no. 1, 177–197.

[H1] D. Hayes, *The refined $\mathfrak{p}$-adic abelian Stark conjecture in function fields.* Invent. Math. **94** (1988), no. 3, 505–527.

[H2] D. Hayes, *The partial zeta functions of a real quadratic number field evaluated at* $s = 0$, in "Number theory (Banff, AB, 1988)," 207–226, de Gruyter, Berlin, 1990.