Problem 1 (p.9 #2). Suppose a word is picked at random from this sentence. Find:

a) the chance the word has at least 4 letters;

SOLUTION: All words are equally likely to be chosen. The sentence has 10 words; 7 are at least 4 letters long. We have \( P(\text{at least 4 letters}) = \frac{7}{10} \).

b) the chance that the word contains at least 2 vowels (a,e,i,o,u);

SOLUTION: Assuming that the vowels need not be distinct (i.e. the three instances of “e” in “sentence” count as three vowels in the word), then exactly 4 words contain at least two vowels, so \( P(\text{at least 2 vowels}) = \frac{4}{10} \).

c) the chance that the word contains at least 4 letters and at least 2 vowels.

SOLUTION: Exactly four words contain at least 4 letters and at least two vowels, so \( P(\text{at least 4 letters and at least 2 vowels}) = \frac{4}{10} \).

d) What is the distribution of the length of the word picked?

SOLUTION: Let \( i \) be the length of the word. Then the values of \( i \) which occur with nonzero probability are \( i = 1, 2, 4, 6, 7, \) and 8. We have \( P(1) = \frac{1}{10}, P(2) = \frac{2}{10}, P(4) = \frac{3}{10}, P(6) = \frac{2}{10}, P(7) = \frac{1}{10}, P(8) = \frac{1}{10} \).

e) What is the distribution of the number of vowels in the word?

SOLUTION: Let \( j \) be the number of vowels in the word. Then the values of \( j \) which occur with nonzero probability are \( j = 1, 2, 3 \). We have \( P(1) = \frac{6}{10}, P(2) = \frac{2}{10}, P(3) = \frac{2}{10} \).

Problem 2 (p.9 #7). Suppose two 4-sided dice are rolled. Find the probabilities of the following events:

a) the maximum of the two numbers rolled is less than or equal to 2;

SOLUTION: The sample space is the set of ordered pairs \((i,j)\), where \(i\) and \(j\) are integers between 1 and 4, and \(i\) represents the number on the upper face of the first die and \(j\) the number on the upper face of the second die. All outcomes are equally likely. There are 16 outcomes. The outcomes \((1, 1), (1, 2), (2, 1),\) and \((2, 2)\) are the outcomes whose maximum is less than or equal to 2. Hence \( P(\text{maximum is less than or equal to 2}) = \frac{1}{4} \).

b) the maximum of the two numbers rolled is less than or equal to 3;

SOLUTION: In order for the maximum of the two numbers to be less than or equal to 3,
both numbers must be less than or equal to 3. There are $3 \times 3 = 9$ ways for this to happen, so $P(\text{maximum is less than or equal to } 3) = \frac{9}{16}$.

c) the maximum of the two numbers rolled is equal to 3:

SOLUTION: In order for the maximum to be equal to 3, at least one number must be 3, and the other number can be anything less than or equal to 3. There are two choices for which die results in a 3; once this is fixed, there are 2 choices for the outcome of second die to be strictly less than 3, yielding $2(3 - 1)$ outcomes in which the one number is 3 and the other is strictly less than 3. Finally there is the outcome in which both numbers are 3. So there are $2 \times 3 - 1$ outcomes in which the maximum is exactly 3, so the probability of this event is $\frac{5}{16}$.

d) Repeat part c for the maximum equal to 1, 2, and 4.

SOLUTION: The solution to this part is identical to the solutions above; $P(\text{max} = 1) = \frac{1}{16}$, $P(\text{max} = 2) = \frac{3}{16}$, $P(\text{max} = 4) = \frac{7}{16}$.

e) If $M$ is the maximum of the two numbers, then

$$P(M = 1) + P(M = 2) + P(M = 3) + P(M = 4) = 1,$$

check that your answers for c) and d) satisfy this relationship.

SOLUTION: Indeed, if the maximum is equal to $i$, then the upper face of at least one die must be $i$ (there are two possibilities for which die will have $i$ on its upper face) and the other die must result in a number strictly less than $i$ or exactly equal to $i$. Now, there are $(i-1)$ numbers strictly less than $i$, and 1 number exactly equal to $i$, so we get $2(i - 1) + i = 2(i) - 1$ possibilities for a maximum exactly equal to $i$, where $i$ ranges from 1 to 4. Hence $P(i) = \frac{2i-1}{16}$, and evaluating this at each of $i = 1, 2, 3$ and 4 and then summing gives the desired result.

Problem 3. Now suppose that the die has $n$ sides:

a) the maximum of the two numbers rolled is less than or equal to 2;

SOLUTION: Again, there are $n^2$ possible outcomes, all equally likely, and 4 outcomes in which the maximum of the two numbers is less than 2, so the desired probability is $\frac{4}{n^2}$.

b) the maximum of the two numbers rolled is less than or equal to $i \in \{1, \ldots, n\}$;

SOLUTION: As in the previous problem, for the maximum to be less than or equal to $i$, both die rolls must result in numbers less than or equal to $i$. There are $i$ choices for the outcome of the first die and $i$ choices for the second, so $i^2$ ways in total for the maximum of the two numbers to be less than $i$. The desired probability is $\frac{i^2}{n^2}$.

c) the maximum of the two numbers rolled is equal to $i \in \{1, \ldots, n\}$;
SOLUTION: As in the previous problem, one of the numbers must be equal to $i$ and the other must be less than or equal to $i$. If the first one is equal to $i$, the second number can take on $i$ values; if the second is equal to $i$, the first can take on $i$ values, and the ordered pair $(i,i)$ should not be counted twice. Hence there are $2i - 1$ ways for the maximum to be equal to $i$. The desired probability is therefore $\frac{2i-1}{n}$. 

**Problem 4.** Consider that the data 

$$\{1, 4, 1.2, 1.22, 1.75, 2.1, 2.5, 2.6, 2.9, 2.99, 3.1415, 3.5, 3.3, 3.7\}$$ 

and the cut points $\{0, 1, 2, 3, 4\}$. Draw the empirical distribution for this data. Label the height for each rectangle as in Section 1.3. What is the total area of all of the rectangles?

SOLUTION. We note that the difference between each consecutive pair of cut points $(b_j, b_{j+1})$ is 1, and there are 14 data points in total. For each pair $(b_j, b_{j+1})$ of cut points, where the height of the rectangle whose base is the interval between $(b_j, b_{j+1})$ is given by 

$$H_j = \frac{\# \text{ of data points } x_i \text{ for which } b_j < x_i < b_{j+1}}{\text{total number of data points}} \cdot [b_{j+1} - b_j] .$$

So, for instance, the height above the interval $(2,3)$ is $\frac{3}{14}$. The total area of all the rectangles sums to 1, because the height of each rectangle is the fraction of data points that lie in the given interval and the width of each rectangle is 1.

**Problem 5 (p.31 #10).** Events $A, B,$ and $C$ are defined on an outcome space. Find expressions for the following probabilities in terms of $P(A)$, $P(B)$, $P(C)$, $P(AB)$, $P(AC)$, $P(BC)$, and $P(ABC)$.

a) The probability that exactly two of the $A, B, C$ occur.

SOLUTION: We want to find the probability of the events $E_1 = A \cap B \cap C^c = ABC^c$, $E_2 = A \cap C \cap B^c$, and $E_3 = B \cap C \cap A^c$. Each of these events $E_i$ represents exactly two of $A, B,$ or $C$ occurring. Note that these events have no intersection. Therefore, 

$$P \left( \{A \cap B \cap C^c\} \cup \{A \cap C \cap B^c\} \cup \{B \cap C \cap A^c\} \right) = P(E_1) + P(E_2) + P(E_3).$$

Now, $P(E_1) = P(AB) - P(ABC)$, $P(E_2) = P(AC) - P(ABC)$, and $P(E_3) = P(BC) - P(ABC)$. Hence we get

$$P(\text{exactly two of the events occur}) = P(AB) + P(AC) + P(BC) - 3P(ABC).$$

b) The probability that exactly one of these events occurs.

SOLUTION: Here we want to find the probabilities of the events $F_1 = AB^cC^c$, $F_2 = BA^cC^c$, and $F_3 = CA^cB^c$. Again, the events $F_i$ are mutually exclusive and their union is event that exactly one of the events occurs. We find 

$$P(F_1) = P(A) - P(AB) - P(AC) + P(ABC);$$

$$P(F_2) = P(B) - P(AB) - P(BC) + P(ABC);$$

$$P(F_3) = P(C) - P(AC) - P(BC) + P(ABC)$$

which yields $P(A) + P(B) + (C) - 2P(AB) - 2P(BC) - 2(AC) + 3P(ABC)$ as the answer.
c) The probability that none of these events occur.

SOLUTION: The probability that no event occurs is the complement of the probability that at least one event occurs. The probability that at least one event occurs is \( P(A \cup B \cup C) \). By the inclusion-exclusion formula, we have

\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)
\]

so the probability that none of the events occur is \( 1 - P(A \cup B \cup C) \), which by inclusion-exclusion is

\[
1 - \left[ P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \right].
\]

Problem 6. How many children should a family plan to have so that the probability of having at least one child of each sex is at least 0.95? (Assume that both sexes are equally likely.)

SOLUTION: Let \( n \) be the number of children that the family will have, and let \( A_n \) be the event that there is at least one boy and one girl among the \( n \) children. It is easier to compute the probability of the event, \( A_n^c \), that all \( n \) children are of the same sex. We want to find \( n \) so that:

\[
P(A_n) = 1 - P(A_n^c) \geq 0.95
\]

\[
\Leftrightarrow P(A_n^c) \leq 0.05
\]

Observe that the events \( M_n = \text{"} n \text{ boys} \), and \( F_n = \text{"} n \text{ girls} \) are a partition of \( A_n^c \), and \( P(M_n) = P(F_n) = 1/2^n \). So we solve:

\[
P(A_n^c) = \frac{2}{2^n} \leq 0.05
\]

\[
\Leftrightarrow 40 \leq 2^n
\]

\[
\Leftrightarrow n \geq \log_2 40
\]

The smallest value which satisfies this inequality is \( n = 6 \).

Problem 7. Suppose we roll two fair six-sided dice. What is the distribution of the difference (in absolute value) between the two numbers on the top faces?

SOLUTION: The smallest possible difference is 0 and the largest is \( 6 - 1 = 5 \). When the difference is 0, the numbers must be equal – there are 6 ways for this to occur. When the difference is 1, there are 5 possibilities for the smaller number (6 cannot be the smaller of the two numbers, and the larger number is determined by the smaller by adding 1), and for each possibility, there are two outcomes (e.g. \( (1, 2) \) and \( (2, 1) \)), giving 10 ways for the difference to be 1.

By similar reasoning, there are \( 4 \times 2 = 8 \) outcomes with a difference of 2, \( 3 \times 2 = 6 \) outcomes with a difference of 3, and so on. The distribution is then

<table>
<thead>
<tr>
<th>difference, ( i )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(i) )</td>
<td>( \frac{6}{36} = \frac{1}{6} )</td>
<td>( \frac{5}{36} )</td>
<td>( \frac{2}{9} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{9} )</td>
<td>( \frac{1}{18} )</td>
</tr>
</tbody>
</table>

The next problem is a classic puzzle.
Problem 8. Three people are going to play a cooperative game. They are allowed to strategize before the game begins, but once it starts they cannot communicate with one another. The game goes as follows. A fair coin is tossed for each player to determine whether that player will receive a red hat or a blue hat, but the color of the hat (and result of the coin toss) is not revealed to the player. Then the three players are allowed to see one another, so each player sees the other two players’ hats, but not her own. Simultaneously, each player must either guess at the color of her own hat or ‘pass’. They win if nobody guesses incorrectly and at least one person guesses correctly (so they can’t all pass).

The players would like to maximize their probability of winning, so the question is what should their strategy be? A naive strategy is for them to agree in advance that two people will pass and one person (designated in advance) will guess either red or blue. This strategy gives them a 50% chance of winning, but it is not optimal. Devise a strategy that gives the players a greater probability of winning.

SOLUTION: First, write out the outcome space for the assignment of hats.

\[ \Omega = \{RRR, RRB, RBR, BRR, RBB, BRB, BBR, BBB\} \]

The goal is not to win for every outcome, but to win for more than half of the outcomes. Notice that in all but two of the outcomes, two people have the same color hat and one person has a different colored hat. In each of these scenarios, then, one person sees two like-colored hats, while the other two people see one hat of each color. For example, if the outcome is RRB, then the person with the blue hat sees two red hats on the other players’ heads, and each person with a red hat sees a red and a blue hat. Therefore, the following strategy will guarantee a win whenever two people have the same colored hat and one person has the other color: If a player sees two red hats on the other players’ heads, and each person with a red hat sees a red and a blue hat. Then, the following strategy will guarantee a win whenever two people have the same colored hat and one person has the other color: If a player sees two hats of the same color, then guess the opposite color; If a player sees two different colored hats, then pass. This strategy always fails when all three hats are the same color, since all three players will guess incorrectly. Therefore, the probability of winning with this strategy is \( \frac{6}{8} = 75\% \).

Now for a little review of calculus. These are things you should have learned in a class before this one. If you have problems with them, get help now to refresh your memory.

Problem 9. Do the following integrals:

\[ \int_0^1 x^3 \, dx \quad \int_0^0 x \exp(x) \, dx \quad \int_{-\infty}^{\infty} \exp(-x^2/2) \, dx \]

SOLUTION. For the first integral, by the fundamental theorem of calculus and the power rule,

\[ \int_0^1 x^3 \, dx = \left. \frac{x^4}{4} \right|_0^1 = \frac{1}{4} \]

For the second integral, note that the integrand is the product of two functions: \( x \) and \( \exp(x) \), one of which gets simpler with differentiation and the other of which is easy to integrate. Hence this type of integral can be addressed through integration-by-parts. Also, as \( x \to -\infty \), \( x \exp(x) \to 0 \) (write the expression as a quotient and use L’hôpital’s rule). We derive

\[ \int_{-\infty}^{0} x \exp(x) \, dx = [x \exp(x)]_{-\infty}^{0} - \int_{-\infty}^{0} \exp(x) \, dx = -1 \]
The last integral requires a bit of ingenuity. Let us first observe that
\[
\int_{-\infty}^{\infty} \exp(-x^2/2)\,dx = \int_{-\infty}^{\infty} \exp(-y^2/2)\,dy;
\]
that is, in a definite integral, the particular choice of letter to represent the variable of integration is irrelevant. Hence we get
\[
\left[ \int_{-\infty}^{\infty} \exp(-x^2/2)\,dx \right] \left[ \int_{-\infty}^{\infty} \exp(-y^2/2)\,dy \right] = \left[ \int_{-\infty}^{\infty} \exp(-x^2/2)\,dx \right]^2.
\]
Now, consider the double integral
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[ -\left( x^2 + y^2 \right)/2 \right] \,dxdy
\]
This is an *iterated integral*—first we integrate with respect to the variable \( x \), and then with respect to the variable \( y \). *A priori*, we cannot say that the iterated integral and the product of the two separate integrals are equal. But since \( \exp\left[ -\left( x^2 + y^2 \right)/2 \right] = \exp(-x^2/2) \cdot \exp(-y^2/2) \), the function \( f(x,y) = \exp\left[ -\left( x^2 + y^2 \right)/2 \right] \) can be decomposed into a product of functions \( f(x,y) = g(x)h(y) \), where \( g(x) = \exp(-x^2/2) \), and \( h(y) = \exp(-y^2/2) \). While integrating \( f(x,y) = g(x)h(y) \) first in the variable \( x \), we can treat \( h(y) \) as a constant. Namely, we have
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[ -\left( x^2 + y^2 \right)/2 \right] \,dxdy = \int_{-\infty}^{\infty} \exp(-y^2/2) \left[ \int_{-\infty}^{\infty} \exp(-x^2/2)\,dx \right] dy
\]
\[
= \left[ \int_{-\infty}^{\infty} \exp(-x^2/2)\,dx \right] \left[ \int_{-\infty}^{\infty} \exp(-y^2/2)\,dy \right]
\]
\[
= \left[ \int_{-\infty}^{\infty} \exp(-x^2/2)\,dx \right]^2.
\]
From the above chain of equalities, we see that if we can evaluate the double integral over the whole plane, then we can determine the value of the integral we want. The double integral over the whole plane
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[ -\left( x^2 + y^2 \right)/2 \right] \,dxdy
\]
can be evaluated by changing to polar coordinates: \( x = r\cos\theta \), \( y = r\sin\theta \), where \( 0 \leq r < \infty \), \( 0 \leq \theta \leq 2\pi \). By the change-of-variables formula for multiple integrals, in a change of variables from \((x,y)\) to \((r,\theta)\), the area differential \( dxdy \) is transformed to the differential \( rdrd\theta \) multiplied by the absolute value of the determinant of the Jacobian matrix of the change-of-coordinate map \( \frac{\partial(x,y)}{\partial(r,\theta)} \).
You can compute this matrix (in fact you should!), and you will find its determinant to be \( r \), which is always nonnegative. Hence the area differential \( dxdy \) is transformed under this change of variable to \( rdrd\theta \). The double integral in the new \((r,\theta)\) coordinates is
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[ -\left( x^2 + y^2 \right)/2 \right] \,dxdy = \int_{0}^{2\pi} \int_{0}^{\infty} \exp\left[ -r^2/2 \right] \,rdrd\theta
\]
Since the function being integrated does not depend on \( \theta \), this becomes
\[
2\pi \int_{0}^{\infty} \exp\left[ -r^2/2 \right] \,rdr
\]
which can be evaluated by simple substitution. The final answer for the double integral is $2\pi$, so the final answer for the original integral is $\sqrt{2\pi}$.

**Problem 10.** Perform the following differentiations:

$$\frac{d}{dx}(x^4) \quad \frac{d}{dx}(x^2 \exp(-x)) \quad \frac{d}{dx}(\ln(x^2))$$

**SOLUTION.** These are straightforward and we omit the details: the first differentiation is an application of the power rule; the second an application of the product rule; the third an application of the chain rule. We get

$$\frac{d}{dx}(x^4) = 4x^3 \quad \frac{d}{dx}(x^2 \exp(-x)) = (2x - x^2) \exp(-x) \quad \frac{d}{dx}(\ln(x^2)) = \frac{2}{x}$$

**Problem 11.** Evaluate the following infinite sums:

$$\sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k \quad \sum_{k=1}^{\infty} \frac{3^k}{k!}$$

**SOLUTION.** The first series is the sum of a geometric series whose first term is $\frac{1}{4}$ and whose ratio is $\frac{1}{4}$. Hence its sum is $\frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$. The second series is almost the series expansion for the exponential function $\exp(x)$, except the first term in that series, $\frac{3^0}{0!} = 1$, is not included. Hence the numerical value of the infinite sum is $\exp(3) - 1$.

**Problem 12.** Draw a picture of the region of the $xy$-plane were both $x$ and $y$ are between 0 and 1 and $y \geq x^2$. Find the area of this region.

**SOLUTION.** The area is represented by the integral

$$\int_{0}^{1} [1 - x^2] \, dx = [x - x^3/3]_{0}^{1} = 2/3$$

**Problem 13.** Evaluate the integral

$$\int_{0}^{4} \int_{\frac{y}{2}}^{2} e^{x^2} \, dx \, dy$$

**Hint:** Draw the region over which you are integrating and change the order of integration.

**SOLUTION.** The region is $\{y/2 \leq x \leq 2, 0 \leq y \leq 4\}$. In order to change the order of integration, we need to express the limits first in terms of $y$ (so the $y$ limits depend on $x$), and then in terms of $x$, so the $x$ limits are then two numbers. We see that if $y/2 \leq x \leq 2$ and $0 \leq y \leq 4$, then $0 \leq y \leq 2x$, and $0 \leq x \leq 2$. (Draw these two regions separately and check that they are the same region!) By interchanging the order of integration, we get

$$\int_{0}^{4} \int_{\frac{y}{2}}^{2} e^{x^2} \, dx \, dy = \int_{0}^{4} \int_{0}^{2x} e^{x^2} \, dy \, dx$$

$$= \int_{0}^{2} 2xe^{x^2} \, dx$$

$$= \exp(x^2)|_{0}^{2} = \exp(4) - 1$$