Squeezing and Non-Squeezing in Real and p-Adic Symplectic Geometry

Ezra Jordan Aylaian

December 9, 2025

This talk was given at the Grapevine Conference at Duke on December 9th, 2025.

1 Symplectic Vector Spaces

1.1 Linear Symplectic Forms

Definition 1. A bilinear antisymmetric linear 2-form (henceforth simply a linear 2-form) ω on a k-vector space V is *non-degenerate* if $\omega(v, w) = 0$ implies $v = 0$, or, equivalently, if $v \mapsto \omega(v, \cdot)$ is an injection $V \to V^*$. A *symplectic k-vector space* (V, ω) is a k-vector space V equipped with a non-degenerate linear 2-form ω, called the *linear symplectic form.*

Remark. One of the most natural symplectic vector spaces occurring in nature is the cotangent space of a smooth manifold. We will be better equipped to understand this motivating example when we talk about symplectic manifolds, which are logically situated after our discussion of symplectic vector spaces. I promise that if we are willing to go without a good discussion of this motivating example for now, we'll come back to it and understand it triumphantly later.

Remark. Alternating linear 2-forms are always antisymmetric. The converse is true given the assumption that $\operatorname{char} k \neq 2$, but it need not hold otherwise.

In the finite-dimensional case, where we will spend much of our energy, an injection $V \to V^*$ is automatically an isomorphism, since $\dim V = \dim V^*$, so a symplectic vector space is canonically isomorphic to its dual space via the isomorphism $v \mapsto \omega(v, \cdot)$. This is analogous to how an inner product space is canonically isomorphic to its dual space via the isomorphism $v \mapsto \langle v, \cdot \rangle$.

Examples.

- $(k^2, dx \wedge dy)$ is a symplectic vector space. Indeed, $dx \wedge dy$ is the only linear 2-form on k^2 up to a nonzero constant, so it is also the only symplectic form on k^2 up to a nonzero constant.
- The direct sum of the symplectic vector spaces

$$(V, \omega) \oplus (W, \eta) := (V \oplus W, \omega \oplus \eta := \pi_V^* \omega + \pi_W^* \eta)$$

is a symplectic vector space, where $\pi_V : V \oplus W \to V$ and $\pi_W : V \oplus W \to W$ are the projections out of a product.

- For $n \geq 0$, the n-fold direct sum of $(k^2, dx \wedge dy)$, namely

  $$(k^{2n}, \omega_{\text{std}} := dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n),$$

  where k^{2n} has coordinates $x^1, y^1, \ldots, x^n, y^n$, is a symplectic vector space.

Definition 2. Taking inspiration from inner product spaces, if W is a subspace of a symplectic vector space (V, ω), we define the *subspace perpendicular to* W

$$W^\perp := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}.$$

Exercise: just like for inner product spaces, if V is finite-dimensional, then $\dim W + \dim W^\perp = \dim V$ and $(W^\perp)^\perp = W$.

It is not automatic that a non-degenerate linear 2-form restricts to a non-degenerate linear 2-form on a subspace. If this does happen, we christen the subspace with the following appropriate name.

Definition 3. If W is a subspace of a symplectic vector space (V, ω), then W is a *symplectic subspace* if $(W, \omega|_W)$ is a symplectic vector space, that is, if $\omega|_W$ is non-degenerate, equivalently, if $W \cap W^\perp = 0$, equivalently, if $W^\perp$ is non-degenerate.

Example. The symplectic subspaces of $(k^{2n}, \omega_{\text{std}}$ are exactly the subspaces of the form

$$\text{span}\left(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial y^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_l}}, \frac{\partial}{\partial y^{i_l}}\right)$$

for some $0 \leq i_1, \ldots, i_l \leq n$.

Exercise: if W is a symplectic subspace of the finite-dimensional symplectic vector space (V, ω), then V splits as a direct sum of W and $W^\perp$:

$$(V, \omega) = (W, \omega|_W) \oplus (W^\perp, \omega|_{W^\perp}).$$

Compare to the case of inner product spaces.

1.2 Linear Symplectic Maps

Definition 4. A *linear symplectic map* $\varphi : (V, \omega) \to (W, \eta)$ is a linear map $\varphi : V \to W$ that preserves the symplectic form: $\varphi^*\eta = \omega$. A *linear symplectomorphism* is a linear symplectic map with a linear symplectic inverse.

Proposition 1. *All linear symplectic maps are injective.*

Proof. If $v \in \ker \varphi$, then

$$\omega(v, u) = \eta(\varphi(v), \varphi(u) = \eta(0, \varphi(u)) = 0$$

for all $u \in V$, so $v = 0$ by non-degeneracy. □

Corollary. *The category of symplectic vector spaces with linear symplectic maps between them sucks.*

For example, if $V, W \neq 0$, then the projection maps out of the product $V \oplus W$ are not injective, so the category doesn't have them as morphisms. Hence $V \oplus W$ with its projections to V and W is not the product of V and W in this category.

Every finite-dimensional vector space is (non-canonically) isomorphic to k^n for some $n \geq 0$. A similar statement holds for symplectic vector spaces.

Proposition 2. *If* $\operatorname{char} k \neq 2$*, every finite-dimensional symplectic* k*-vector space* (V, ω) *is (non-canonically) linear symplectomorphic to* (k^{2n}, ω_{std}) *for some* $n \geq 0$*.*

Thus, every finite-dimensional symplectic vector space is even-dimensional.

Proof. If $\dim V = 0, 1$, this is trivial, and if $\dim V = 2$, this is because there is a unique linear 2-form up to scaling, so suppose $\dim V > 2$. Pick a nonzero vector $e \in V$. Non-degeneracy implies there is $f \in V$ such that $\omega(e, f) = 1$. The assumption $\operatorname{char} k \neq 2$ implies ω alternating implies e, f are linearly independent, so $W = \operatorname{span}(e, f)$ is a 2-dimensional symplectic subspace of V. Thus $(V, \omega) = (W, \omega|_W) \oplus (W^\perp, \omega|_{W^\perp})$, and we are done by induction.[1] □

We can use this proposition to prove another characterization of non-degeneracy:

Proposition 3. *Let* $\dim V = 2n$ *and let* ω *be a linear* 2*-form on* V*. If* $\operatorname{char} k \neq 2$ *and* $n! \neq 0$ *in* k*, then* ω *is non-degenerate iff* $\omega^n \neq 0$*.*

Proof. Suppose ω is non-degenerate. By the preceding proposition, WLOG $(V, \omega) = (k^{2n}, \omega_{\text{std}})$. It is an exercise to check that

$$\omega_{\text{std}}^n = n!\, dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \neq 0.$$

Suppose ω is degenerate. Then there exists $e_1 \in V$ such that $\omega(e_1, \cdot) = 0$. Extend e_1 to a basis $e_1, \ldots, e_{2n}$ for V. Then $\omega^n(e_1, \ldots, e_{2n}) = 0$, so $\omega^n = 0$. □

1.3 The Symplectic and Heisenberg Groups

There are two groups we can associate to a symplectic vector space.

Definition 5. The *symplectic group* of (V, ω) is

$$\operatorname{Sp}(V, \omega) := \{\text{linear symplectomorphisms } (V, \omega) \to (V, \omega)\}.$$

The symplectic group of (V, ω), $\dim V = 2n$, is a subgroup of $\operatorname{SL}(V)$ because a map that preserves ω will also preserve the nonzero linear top form ω^n, and preserving a nonzero linear top form is equivalent to preserving the determinant. In the special case $\dim V = 2$, we have $n = 1$ and $\omega^n = \omega$ so $\operatorname{Sp}(V, \omega) = \operatorname{SL}(V)$. But in general, there are more elements in $\operatorname{SL}(V)$ than in $\operatorname{Sp}(V, \omega)$.

Example. If we pick a basis $e_1, \ldots, e_{2n}$ for V, then we can define the $2n$ by $2n$ matrix Ω representing the bilinear 2-form ω as the matrix with $\omega(e_i, e_j)$ as its ijth entry. Then a linear map $V \to V$ preserves ω if and only if its matrix representation A satisfies $A^T \Omega A = \Omega$, so

$$\operatorname{Sp}(V, \omega) = \{A : A^T \Omega A = \Omega\}.$$

[1] If one wishes to be all too careful, they can check that if a factor in a direct sum of symplectic vector spaces is replaced by a symplectomorphic symplectic vector space, the result of the direct sum is symplectomorphic.

If $(V, \omega) = (k^{2n}, \omega_{\text{std}})$, we order the coordinates on k^{2n} as $x^1, \ldots, x^n, y^1, \ldots, y^n$, and we take the standard basis on k^{2n}, then Ω may be written especially compactly:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

On the other hand, if we order the coordinates as $x^1, y^1, \ldots, x^n, y^n$, as we have been doing in this talk, the matrix is

$$\Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Definition 6. If $k = \mathbb{R}$, the *Heisenberg group* of (V, ω), which we denote $\text{Heis}(V, \omega)$, is the set $V \times \mathbb{R}$ with group multiplication given by

$$(v, t) \cdot (w, s) := (v + w, t + s + \omega(v, w)/2).$$

Example. Putting the coordinates x, y, t on $\mathbb{R}^3$, the group $\text{Heis}(\mathbb{R}^2, \omega_{\text{std}})$ is the set $\mathbb{R}^3$ with multiplication given by

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + (xy' - yx')/2).$$

This group, which you may have seen before, is called the 3-*dimensional Heisenberg group*. In general, for $n \geq 3$, $\text{Heis}(\mathbb{R}^{n-1}, \omega_{\text{std}})$ is called the n-*dimensional Heisenberg group*.

2 Real Symplectic Geometry

2.1 Symplectic Forms

Definition 7. A 2-form ω on a smooth boundaryless manifold M is *non-degenerate* if ω_p is non-degenerate for all $p \in M$ and *symplectic* if it is non-degenerate and closed. If ω is symplectic, the pair (M, ω) is a *symplectic manifold*.

Three easy observations are at hand. First, the tangent spaces to a symplectic manifold are symplectic vector spaces, so the dimension of a symplectic manifold is even. Second, just as a Riemannian manifold (M, g) has a musical isomorphism $TM \cong T^*M$ given by $(x, v) \mapsto (x, g(v, \cdot))$, a symplectic manifold (M, ω) has an isomorphism $TM \cong T^*M$ given by $(x, v) \mapsto (x, \omega(v, \cdot))$. Third, ω^n is a nonvanishing top form, a.k.a. a *volume form*. A volume form defines both an orientation on M and a Radon measure on M.

Examples.

- For $n \geq 0$, $(\mathbb{R}^{2n}, \omega_{\text{std}} := dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n)$ is a symplectic manifold.[2] Similarly, any finite-dimensional symplectic vector space over $\mathbb{R}$ is a symplectic manifold.
- Any nonvanishing 2-form on a surface is a symplectic form. Therefore, a surface admits a symplectic structure iff it is orientable.

[2]We are overloading notation by referring to both this differential form and the linear form from the previous section by the same notation ω_{std}.

- The product of two symplectic manifolds is symplectic.

- If N has a symplectic form η, the pullback $f^*\eta$ of η through an immersion $f : M \to N$ is a symplectic form on M. As a special case, an open submanifold of a symplectic submanifold is symplectic.

- If M is a smooth manifold without boundary, T^*M admits a natural structure as a symplectic manifold. This important example is what I referred to in the first remark of the talk as one of the most natural ways that symplectic vector spaces (indeed, symplectic manifolds) arise in nature. Let $\pi : T^*M \to M$ be the bundle projection. The map π induces, at the point $(p, \varphi) \in T^*M$, the pointwise pullback map

$$\pi^*_{(p,\varphi)} : T^*_p M \to T^*_{(p,\varphi)}(T^*M).$$

  The Liouville 1-form τ is the 1-form on T^*M defined by $\tau_{(p,\varphi)} := \pi^*_{(p,\varphi)}\varphi$. Exercise: in charts, $-d\tau = \omega_{\text{std}}$. Hence, not only is $-d\tau$ a natural symplectic form on T^*M, but also, if you do this exercise, you will see how the formula $dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$ might have arisen naturally to someone studying the cotangent bundle, even if they hadn't yet defined symplectic forms.

The following proposition is an obstruction to a manifold admitting a symplectic structure. For example, it establishes that S^0 and S^2 are the only spheres that admit symplectic structures.

Proposition 4. *If (M^{2n}, ω) is a closed symplectic manifold, ω^k is nontrivial in $H^{2k}_{\text{dR}}(M)$ for all $0 \leq k \leq n$.*

Proof. WLOG assume M is connected.

If $\omega^k = d\alpha$ is exact, then $\omega^{k+1} = d\alpha \wedge \omega = d(\alpha \wedge \omega)$ is too. Repeating this, ω^n is exact, so $\int_M \omega^n = 0$ since M is closed.

On the other hand, choose a finite oriented atlas $\{U_i\}$ for M. If U_i has coordinates $x^1, \ldots, x^{2n}$, then the coordinate expression of ω in U_i is $\omega = f_i \, dx^1 \wedge \cdots \wedge dx^{2n}$ for $f_i : U_i \to \mathbb{R}$ nonvanishing. But since the atlas is finite and oriented and M is connected, the signs of the f_is all agree. Therefore, if ψ_i is a partition of unity subordinate to $\{U_i\}$,

$$\int_M \omega^n = \sum_i \int_{U_i} \psi_i \omega^n$$

is a sum of terms all of the same sign, so it is $\neq 0$. □

2.2 Symplectic Maps and Gromov's Non-Squeezing Theorem

Definition 8. A *symplectic map* $f : (M, \omega) \to (N, \eta)$ is a smooth map $f : M \to N$ that preserves the symplectic form: $f^*\eta = \omega$. A *symplectomorphism* is a symplectic map with a symplectic inverse. A *symplectic embedding* is a symplectic map that is also an embedding. The *symplectomorphism group* of (M, ω) is the group

$$\text{Sympl}(M, \omega) := \{\text{symplectomorphisms } (M, \omega) \to (M, \omega)\}.$$

Our question in this talk is this: **how rigid are symplectic maps?** For example, we know smooth maps are extremely wobbly and flexible, and the space of diffeomorphisms on a smooth manifold is huge, and we know that contrastingly, Riemannian isometries are extremely rigid and inflexible, with the space of self-isometries on a generic Riemannian manifold trivial. We can build a sequence of inclusions

$$\{\text{smooth maps}\} \supset \{\text{symplectic maps}\} \supset \{\text{volume-preserving maps}\}.$$

It seems a priori likely to me that symplectic maps are more similar, whatever that means, to volume-preserving maps than they are to smooth maps. If we take this far enough, maybe we can guess that symplectic maps can be described as volume-preserving maps plus some small extra condition?

This guess is completely wrong, due to Gromov's famous theorem:

Theorem 1 (Gromov's Non-Squeezing Theorem)**.** *Let*

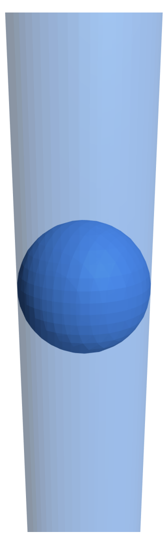
$$\begin{aligned} B^{2n}(r) &:= \{z = (x^1, y^1, \dots, x^n, y^n) \in \mathbb{R}^{2n} : |z| < r\} \\ &= \textit{ball of radius } r \end{aligned}$$

and

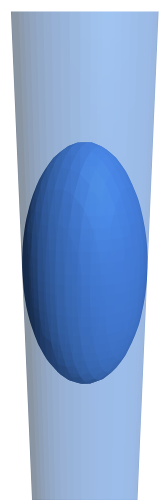
$$\begin{aligned} Z^{2n}(R) &:= B^2(R) \times \mathbb{R}^{2n-2} \\ &= \{(x^1, y^1, \dots, x^n, y^n) \in \mathbb{R}^{2n} : |(x^1, y^1)| < r\} \\ &= \textit{cylinder of radius } R \end{aligned}$$

be endowed with symplectic structures as open subsets of $(\mathbb{R}^{2n}, \omega_{std})$*. Then*

$$\exists \textit{ symplectic embedding } B^{2n}(r) \to Z^{2n}(R) \qquad \Longleftrightarrow \qquad r \le R.$$



(a) The ball of radius 1 sits symplectically inside the cylinder of radius 1 as a subset.



(b) The ball of radius $\sqrt{2}$ can be squeezed inside the cylinder of radius 1 by rescaling the coordinates. This map does not preserve the symplectic form, even though it preserves volume.

In other words, an oversized ball cannot be symplectically squeezed in to a small cylinder even though a volume-preserving embedding can easily do the trick! Explicitly, for example, for any $r > 0$, the linear map $A : \mathbb{R}^4 \to \mathbb{R}^4$ given by the diagonal matrix

$$A = \begin{pmatrix} 1/r & & & \\ & 1/r & & \\ & & r & \\ & & & r \end{pmatrix}$$

sends $B^4(r)$ into $Z^4(1)$ and is volume-preserving, but it is not symplectic as $A^T \Omega A \neq \Omega$ (using the appropriate matrix Ω from earlier).

The proof of Gromov's Non-Squeezing Theorem studies the moduli space of pseudo-holomorphic spheres inside the manifold $\mathbb{C}P^1 \times T^{2n-2}$. It is too difficult and too far afield for me to squeeze a proper discussion of it into this already lengthy talk. Instead, we will end the talk with a treat.

3 Naive p-Adic Symplectic Geometry

Fix a prime p. Naive p-adic analytic manifolds are like real manifolds, but with the p-adic field $\mathbb{Q}_p$ replacing $\mathbb{R}$.

Definition 9.

- A *naive n-dimensional p-adic analytic manifold* M is a Hausdorff space with an open cover $\{U_\alpha\}$ and homeomorphisms onto their images $\varphi_\alpha : U_\alpha \to \varphi(U_\alpha) \subset (\mathbb{Q}_p)^n$ such that the transition functions $\varphi_\alpha \varphi_\beta^{-1}$ are bianalytic.
- A map $f : M \to N$ is *analytic* if it is analytic in charts. The set of analytic maps $M \to \mathbb{Q}_p$ is denoted $\Omega^0(M)$.
- A *tangent vector* at $q \in M$ is a linear map $\Omega^0(M) \to \mathbb{Q}_p$ satisfying the Leibniz rule $v(fg) = v(f)g(q) + f(q)v(g)$. The set of all tangent vectors at $q \in M$ forms the *tangent space* T_qM. The union of all tangent spaces forms the *tangent bundle* TM.
- A *vector field* is an analytic map $X : M \to TM$ such that $X(q) \in T_qM$. The set of vector fields is denoted $\mathfrak{X}(M)$. A *k-form* is a multilinear antisymmetric map $\mathfrak{X}(M)^k \to \Omega^0(M)$.
- *Naive p-adic analytic symplectic manifolds*, *analytic symplectic maps*, *analytic symplectomorphisms*, and *analytic symplectic embeddings* are all defined analogously to the real case.

Examples.

- $((\mathbb{Q}_p)^{2n}, \omega_{\text{std}} := dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n)$ is a naive $2n$-dimensional p-adic analytic symplectic manifold.
- Open subsets and finite products of naive p-adic analytic (symplectic) manifolds are also naive p-adic analytic (symplectic) manifolds.

- Recall that the p-adic absolute value $|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ is defined as $|x|_p := p^{-\mathrm{ord}_p(x)}$ and the p-adic norm $\|\cdot\|_p : (\mathbb{Q}_p)^m \to \mathbb{R}_{\geq 0}$ is defined as

  $$\|(x_1, \ldots, x_m)\|_p := \max\{|x^1|_p, \ldots, |x^n|_p\}.$$

  Let r, R be p-adic absolute values (let r, R be in the codomain of $|\cdot|_p$). Define the p-adic ball and p-adic cylinder

  $$B_p^{2n}(r) := \{z \in (\mathbb{Q}_p)^{2n} : \|z\|_p < pr\}$$
  $$Z_p^{2n}(R) := B^2(R) \times (\mathbb{Q}_p)^{2n-2}$$

  and endow them with symplectic structures from their structures as open subsets of $((\mathbb{Q}_p)^{2n}, \omega_{\mathrm{std}})$.

Is the Non-Squeezing Theorem true for naive p-adic analytic symplectic manifolds? Will we have to study some p-adic moduli space of pseudo-holomorphic spheres to know? Earlier this year (2025), Crespo-Pelayo [1] built a simple analytic symplectomorphism that answers both questions negatively.

Theorem 2 (Crespo-Pelayo [1])**.** *For $n \geq 2$, there is an analytic symplectomorphism*

$$(\mathbb{Q}_p)^{2n} \cong Z_p^{2n}(1).$$

Proof. The symplectomorphism sends the element

$$(x_1, y_1, \ldots, x_n, y_n) \in (\mathbb{Q}_p)^{2n}$$

to the element

$$(x_1', y_1', x_2', y_2', x_3, y_4, \ldots, x_n, y_n) \in Z_p^{2n}(1),$$

where x_1', y_1', x_2', y_2' are as follows. Write out the p-adic expansions of x_1, y_1, x_2, y_2:

$$x_1 = a_0.a_1a_2\ldots$$
$$y_1 = b_0.b_1b_2\ldots$$
$$x_2 = c_0.c_1c_2\ldots$$
$$y_2 = d_0.d_1d_2\ldots$$

where $a_0, b_0, c_0, d_0 \in \mathbb{Z}_p$ and each of the sequences $a_i, b_i, c_i, d_i \in \{0, \ldots, p-1\}$ for $i \geq 1$ eventually 0. Then x_1', y_1', x_2', y_2' have the p-adic expansions

$$x_1' = a_0$$
$$y_1' = b_0$$
$$x_2' = c_0.a_1c_1a_2c_2a_3c_3\ldots$$
$$y_2' = d_0.b_1d_1b_2d_2b_3d_3\ldots.$$

This map is analytic and preserves the symplectic form because it is a translation on any p-adic ball of radius 1. □

References

[1] Luis Crespo and Álvaro Pelayo. Rigidity and flexibility in p-adic symplectic geometry, 2025.