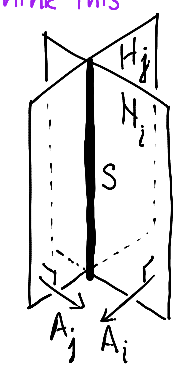
4.

$A \in \mathbb{R}^{m \times n}$ $\quad \mathbb{b} \in \mathbb{R}^m$ write this... $\quad \mathbb{R}$ could be any field $\quad$ ... think this



Q. What operations on $[A \mid \mathbb{b}]$ preserve sol set S ?

A. (i) Swap any pair of rows.

geometric meanings (ii) Multiply a row by a scalar $\neq 0$

(iii) Replace any row by its sum with a multiple of another row.

<u>Def</u> (i), (ii), and (iii) are <u>elementary row operations</u>.

<u>Theorem 4.1</u> Applying any sequence of elementary row ops to $[A \mid \mathbb{b}]$ results in a system with the same solution set.

<u>Pf</u>: (i) merely lists the equations $A_i x = b_i$ in a different order.

For (ii), note that $A_i x = b_i \overset{c \neq 0}{\Leftrightarrow} c(A_i x) = c b_i \quad$ if $c \in \mathbb{R} \setminus \{0\}$ $\quad$ set complement

$$\Leftrightarrow (cA_i)x = cb_i \, .$$

For (iii), let $[A' \mid \mathbb{b}']$ be the system obtained from $[A \mid \mathbb{b}]$ by replacing the row $[A_i \mid b_i]$ with $[A_i + cA_j \mid b_i + cb_j]$.

$[A \mid \mathbb{b}]$	$[A' \mid \mathbb{b}']$
$A_1 x = b_1$	$A_1 x = b_1$
$\vdots$	$\vdots$
$A_{i-1} x = b_{i-1}$	$A_{i-1} x = b_{i-1}$
$A_i x = b_i$	$(A_i + cA_j)x = b_i + cb_j$
$A_{i+1} x = b_{i+1}$	$A_{i+1} x = b_{i+1}$
$\vdots$	$\vdots$
$A_m x = b_m$	$A_m x = b_m$

Let $S' =$ sols of $A'x = \mathbb{b}'$. Then

"is contained in" $\quad S \subseteq S'$

because $\quad (A_i + cA_j)x = A_i x + c A_j x$

$\qquad\qquad = b_i + c b_j \quad$ when $x \in S$.

Aside: need $S = S'$, not just $S \subseteq S'$.

$$\Updownarrow$$

$S \subseteq S'$ and $S' \subseteq S$.

But $A_i = A_i' - cA_j'$ (!) so also $S' \subseteq S$.

Finally, since each of (i), (ii), (iii) preserves S, any sequence of them does, as well. $\square$

<u>E.g.</u> $A x = \mathbb{b}$ for $A = \begin{bmatrix} 3 & -2 & 2 & 9 \\ 2 & 2 & -2 & -4 \end{bmatrix}$ and $\mathbb{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$:

$$[A|\mathbb{b}] = \left[\begin{array}{cccc|c} 3 & -2 & 2 & 9 & 4 \\ 2 & 2 & -2 & -4 & 6 \end{array}\right] \underset{A_1 \rightsquigarrow A_1 - A_2}{\overset{(iii)}{\rightsquigarrow}} \left[\begin{array}{cccc|c} 1 & -4 & 4 & 13 & -2 \\ 2 & 2 & -2 & -4 & 6 \end{array}\right]$$

$$\underset{A_2 \rightsquigarrow A_2 - 2A_1}{\overset{(iii)}{\rightsquigarrow}} \left[\begin{array}{cccc|c} 1 & -4 & 4 & 13 & -2 \\ 0 & 10 & -10 & -30 & 10 \end{array}\right]$$

$$\overset{(ii)}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & -4 & 4 & 13 & -2 \\ 0 & 1 & -1 & -3 & 1 \end{array}\right]$$

$$\underset{A_1 \rightsquigarrow A_1 + 4A_2}{\overset{(iii)}{\rightsquigarrow}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 & 1 \end{array}\right] = [A'|\mathbb{b}']$$

$\Rightarrow$ $Ax = \mathbb{b}$ has same sols S as $A'x = \mathbb{b}'$.

$$\Rightarrow S = \left\{ x \in \mathbb{R}^{\cancel{?}4} \;\middle|\; \begin{array}{rl} x_1 \qquad\quad + x_4 &= 2 \\ x_2 - x_3 - 3x_4 &= 1 \end{array} \right\}$$

Whatever values we assign to x_3, x_4, we can solve for x_1, x_2 successfully

In particular,

$x_3 = x_4 = 0 \Rightarrow \begin{array}{l} x_1 = 2 \\ x_1 = 1 \end{array} \Rightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in S.$ a particular solution x_0

think of as parameters

general solution: x_3, x_4 free variables — can take on any values

$$\begin{array}{l} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \end{array} \begin{bmatrix} 2 \\ 1 \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \\ x_3 \\ x_3 \\ \\ \end{bmatrix} + \begin{bmatrix} -x_4 \\ 3x_4 \\ \\ x_4 \end{bmatrix}$$

so $x \in S \Leftrightarrow x = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_4 \\ 3x_4 \\ 0 \\ x_4 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_0 + t_1 u_1 + t_2 u_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 & 1 \\ & & -1 & 0 & 0 \\ & & 0 & -1 & 0 \end{array}\right]$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ -u_1 & -u_2 & x_0 \end{array}$$

Def: A matrix is in <u>echelon form</u> if

1. the <u>leading</u> (leftmost nonzero) entries progress to the right from each row to the next;

and 2. all 0 rows are at the bottom.

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & * & \cdots & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & * & \cdots \end{bmatrix}$$

The echelon form is <u>reduced</u> if, in addition,

3. every leading entry is 1; pivot

and 4. in each column containing a leading entry, all other entries are 0. pivot column

E.g. just did one!

$$\left[\begin{array}{cccc|c} 1 & -4 & 4 & 13 & -2 \\ 0 & 10 & -10 & -30 & 10 \end{array}\right] \quad \text{echelon form}$$

pivots

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 & 1 \end{array}\right] \quad \text{reduced echelon form}$$

pivot columns

Thm 4.3: Each matrix has unique reduced echelon form.

Pf: Exercise 16 which you aren't asked to do, but you could. □

1. Algorithm produces one.
2. Different reduced echelon forms have different sols.

How to find it?

Algorithm (Gaussian elimination)

Echelon form

Init: $i=1$

While: there is a nonzero entry in some row $\geq i$

Do: 1. pick row $\geq i$ with a leftmost such entry

2. swap that row with row i

3. add multiples of row i to rows $>i$ to cancel entry in pivot column

4. $i \to i+1$

Output: the resulting matrix

Reduced echelon form given any echelon form

Init: $i=1$

While: there is a nonzero entry in some row $\geq i$

Do: 1. rescale row i so pivot is 1

2. add multiples of row i to rows $< i$ to cancel entries in pivot column

3. $i \to i+1$

Output: the resulting matrix