Note: everything today works over any field.

5. Rows vs. columns

$$A \in \mathbb{R}^{m\times n} \Rightarrow A = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i & - \\ & \vdots & \\ - & A_m & - \end{bmatrix} \qquad A_i = [a_{i1} \cdots a_{in}] \in \mathbb{R}^n_{row}$$

i = generic row index

$$= \begin{bmatrix} | & & | & & | \\ a_1 & \cdots & a_j & \cdots & a_n \\ | & & | & & | \end{bmatrix} \qquad a_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m_{col}$$

j = generic column index

$$Ax = b \Leftrightarrow A_i x = b_i \text{ for } i = 1, \dots, m$$

but also

$$Ax = b \Leftrightarrow x_1 a_1 + \cdots + x_n a_n = b$$

$$\begin{bmatrix} x_1 a_{11} + \cdots + x_n a_{1n} \\ \vdots \\ x_1 a_{m1} + \cdots + x_n a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 a_1 + \cdots + x_n a_n = b$$

$\Leftrightarrow$ Ax is a linear combin of the a_j

$\Leftrightarrow$ b is a linear combin of the columns of A

$\Leftrightarrow$ b lies in span of cols of A

$\Leftrightarrow$ $b \in \underbrace{\mathrm{span}(a_1, \dots, a_n)}$

Def: $C(A)$ = column space of A

or image of μ_A

$\Leftrightarrow$ x lists the coefficients in a linear combination of the columns of A that equals b

why? $A \rightsquigarrow$ function taking x's to b's:

$$\mathbb{R}^m \longleftarrow \mathbb{R}^n$$
$$Ax \in \mathbb{R}^m, \quad x \in \mathbb{R}^n$$

$$\begin{bmatrix} | \\ Ax \\ | \end{bmatrix} \longleftarrow \boxed{A} \longleftarrow \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

E.g.

$$\begin{bmatrix} -3 & 6 & 4 & e \\ 1 & 2/3 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \\ \pi \end{bmatrix} = 3\begin{bmatrix} -3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ 2/3 \end{bmatrix} + 4\begin{bmatrix} 4 \\ -1 \end{bmatrix} + \pi\begin{bmatrix} e \\ 0 \end{bmatrix}$$

have class fill these in

$$\begin{bmatrix} -3 & 6 & 4 & e \\ 1 & 2/3 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 19 + \pi e \\ 1/3 \end{bmatrix} \longleftarrow\!\!| \begin{bmatrix} 3 \\ 2 \\ 4 \\ \pi \end{bmatrix}$$

"maps to"

Q. When does $Ax = b$ have a solution?

i.e., When is $Ax = b$ consistent?

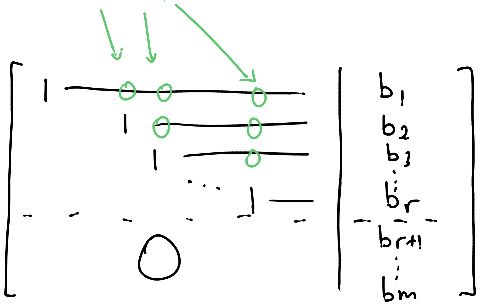
A. ✓ When there exists an x listing the coefficients in a linear combination of the columns of A that equals b.

✓ When b lies in span of columns of A

✓ When $b \in C(A)$

✓ When b lies in the image of the function "multiplication by A" μ_A

In (reduced) echelon form:



<u>Lemma 1</u>: consistent $\Leftrightarrow$ $b_{r+1}=0, \ \vdots \ , b_m = 0$

<u>Pf</u>: $\Rightarrow$: $A_i = 0$ for $i > r$ so $A x = b \Rightarrow \underset{}{\overset{0x}{\overset{\shortparallel}{A_i x}}} = b_i$ for $i > r$.

$\Leftarrow$: For any values of the free variables, the values of the pivots are determined from bottom to top.

<u>E.g.</u>

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & \pi & 4 & 1 & 7 \\ 0 & 0 & 1 & 3 & -11/5 \\ 0 & 0 & 0 & 0 & ? \end{array}\right]$$
$$\begin{array}{cccc} \text{pivot} & \text{free} & \text{pivot} & \text{free} \end{array}$$

$$x_1 + \pi x_2 + 4x_3 + x_4 = 7$$
$$x_3 + 3x_4 = -11/5$$
$$0 + 0 + 0 + 0 = ? \quad \begin{array}{l} 0 \text{ consistent} \\ \neq 0 \text{ inconsistent} \end{array}$$

make whatever you want!

<u>Def</u>: The rank of $A \in \mathbb{R}^{m\times n}$ is $\dim C(A)$ = min # columns required to span $C(A)$ (recall)

<u>E.g.</u> $\operatorname{rank}\begin{bmatrix} 1 & \pi & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \cancel{?}\ 2$ $\qquad$ $\operatorname{rank}\left[\begin{array}{cccc|c} 1 & \pi & 4 & 1 & 7 \\ 0 & 0 & 1 & 3 & -11/5 \\ 0 & 0 & 0 & 0 & ? \end{array}\right] = \cancel{?}\ \begin{array}{l} 2 \text{ if } ?=0 \\ 3 \text{ if } ? \neq 0 \end{array}$

This is the "correct" definition that I mentioned on day 1. Learn it. Use it. Reduce all questions about rank to it. You will be tested on it. It should be your go-to def.

<u>Lemma 2</u>: Fix $v_1, \dots, v_r$ cols of A with corresponding cols $v_1', \dots, v_r'$ of A', where $[A|b] \overset{\text{row op}}{\rightsquigarrow} [A'|b']$.

Then $b = c_1 v_1 + \cdots + c_r v_r \Leftrightarrow b' = c_1 v_1' + \cdots + c_r v_r'$. Note: <u>same</u> coeffs c_i.

<u>Cor</u>: $C(A) = \mathbb{R}^m \Rightarrow C(A') = \mathbb{R}^m$.

<u>Pf</u>: $b' \in \mathbb{R}^m \Rightarrow [A'|b'] \rightsquigarrow [A|b]$ for some $b \in \underset{\mathbb{R}^m}{\overset{}{C(A)}}$. $\square$

<u>Pf</u>: Check type (i), (ii), (iii) row ops directly. And observe that $[A|b] \rightsquigarrow [A'|b']$ $\Leftrightarrow [A'|b'] \rightsquigarrow [A|b]$. $\square$

<u>Prop 1</u>: $v_1, \dots, v_r$ span $C(A) \Leftrightarrow v_1', \dots, v_r'$ span $C(A')$ if $A \rightsquigarrow A'$.

<u>Pf</u>: $\Leftarrow$: $b \in C(A) \Rightarrow b' \in C(A)$ by Lemma 2

$\Rightarrow b' = c_1 v_1' + \cdots + c_r v_r'$ by hypothesis

$\Rightarrow b = c_1 v_1 + \cdots + c_r v_r$ by Lemma.

$\Rightarrow$: same (i.e., swap the primes and un-primes) $\square$

<u>Corollary</u>: rank A = rank A' if $A \rightsquigarrow A'$. Why is this useful?

This is why pivot columns are important.

<u>Prop 2</u>: rank A = # pivots in any echelon form, and pivot cols of A minimally span $C(A)$.

<u>Pf</u>: By Cor and Prop 1, need only prove for $A = U$, reduced echelon form.

Lemma 1 $\Rightarrow$ pivot cols $\begin{matrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ 0 & 0 & & 1 \\ e_1, & e_2, & \dots, & e_r \end{matrix}$ span $C(U)$, and $\mathbb{R}^r$ can't be spanned by less than r vectors: $C\left(r \overset{<r}{\boxed{}} \right) = \mathbb{R}^r \Rightarrow$ reduced echelon form has column space $\mathbb{R}^r$ by (Cor of) Lemma 2, but last row is 0, so that is impossible. $\square$

<u>Prop 3</u>: $Ax = \mathbb{b}$ consistent $\Leftrightarrow$ rank A = rank $[A \mid \mathbb{b}]$. One direction is easy; which?

<u>Pf</u>: $\Rightarrow$: consistent $\Rightarrow \mathbb{b} \in C(A) \Rightarrow C([A \mid \mathbb{b}]) = C(A)$ and same pivot cols span minimally.

$\Leftarrow$: pivot cols of A span $C([A \mid \mathbb{b}]) \Rightarrow \mathbb{b}$ lies in their span. $\square$

Parametric to implicit

implicit to parametric means: solve $Ax = \mathbb{b}$.

Given x_0 and $v_1, \dots, v_k \in \mathbb{R}^n$, find linear equations $Ax = \mathbb{b}$ so that

$$\text{sols}(Ax = \mathbb{b}) = x_0 + \text{span}(v_1, \dots, v_k).$$

<u>E.g.</u> Find implicit equation(s) for the plane

$$x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ in } \mathbb{R}^3.$$

Equivalently, which vectors x are expressible as $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \underbrace{s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}$?

Solve $x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, i.e.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s \\ t \end{bmatrix}}_{x} = \underbrace{x - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\mathbb{b}} = \begin{bmatrix} x_1 - 1 \\ x_2 - 2 \\ x_3 - 1 \end{bmatrix}$$

This is a linear system!

$$\text{cancel} \left[\begin{array}{cc|c} 1 & 2 & x_1 - 1 \\ 0 & 1 & x_2 - 2 \\ 1 & 1 & x_3 - 1 \end{array}\right] \rightsquigarrow \text{cancel} \left[\begin{array}{cc|c} 1 & 2 & x_1 - 1 \\ 0 & 1 & x_2 - 2 \\ 0 & -1 & x_3 - x_1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & x_1 - 1 \\ 0 & 1 & x_2 - 2 \\ 0 & 0 & x_3 - x_1 + x_2 - 2 \end{array}\right]$$

has solution $\Leftrightarrow$ every 0 row in echelon form has corresponding 0 on RHS,

so $\boxed{x \text{ is expressible} \Leftrightarrow x_3 - x_1 + x_2 - 2 = 0.}$