(rest of) today: $A \in \mathbb{R}^{m\times n}$

<u>Def</u>: The (system of) equation(s) $Ax = b$ is <u>inhomogeneous</u> if $b \neq 0$;
the corresponding equation(s) $Ax = 0$ is the associated <u>homogeneous</u> (system of) equation(s).

<u>Lemma</u>: For vectors $x, y \in \mathbb{R}^n$ and scalar $c \in \mathbb{R}$,

$$A(x+cy) = Ax + cAy.$$

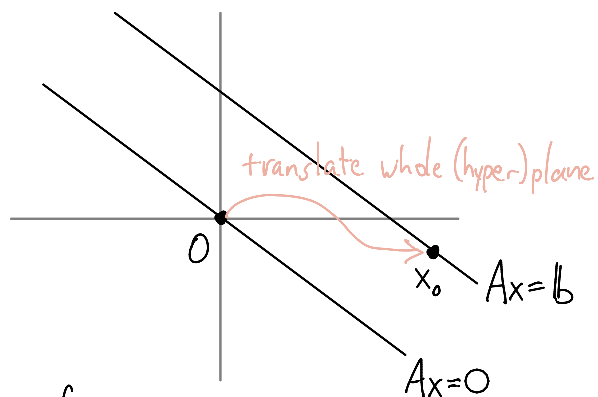
<u>Pf</u>:
$$\begin{aligned} A(x+cy) &= (x_1+cy_1)a_1 + \cdots + (x_n+cy_n)a_n \\ &= \underbrace{x_1a_1 + \cdots + x_na_n}_{} + \underbrace{cy_1a_1 + \cdots + cy_na_n}_{} \\ &= \quad Ax \quad + \quad cAy. \quad \square \end{aligned}$$

could omit this word without changing anything

<u>Thm 5.3</u>: Assume $Ax = b$ has a "particular" solution x_0.
v is a solution of $Ax = b \Leftrightarrow v$ has the form

$$v = x_0 + u$$

for some solution u of $Ax = 0$.



<u>Pf</u>: $\Leftarrow$: $v = x_0 + u \Rightarrow Av = A(x_0+u)$

$$\begin{aligned} &= Ax_0 + Au \quad \text{by Lemma} \\ &= b + 0 \\ &= b \quad \Rightarrow v \text{ is a solution of } Ax = b. \end{aligned}$$

$\Rightarrow$: Assume v solves $Ax = b$; i.e. assume $Av = b$. Then $v = x_0 + u$ for some u, namely $u = v - x_0$, and

$$\begin{aligned} A(v-x_0) &= \underset{\shortparallel}{Av} - \underset{\shortparallel}{Ax_0} \quad \text{by Lemma} \\ &= b - b \\ &= 0. \quad \square \end{aligned}$$

What happens without this hypothesis?

<u>Corollary</u>: A consistent system $Ax = b$ has a unique solution
$\Leftrightarrow$ $Ax = 0$ has only the <u>trivial solution</u> $x = 0$.

<u>Prop. 5.4</u>: $Ax = 0$ has unique solution $\Leftrightarrow$ rank $A = n$.

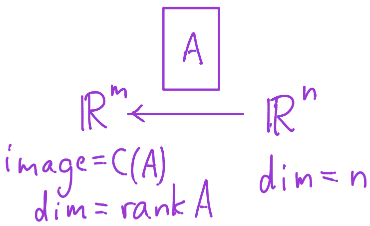
<u>Pf</u>: Equivalent: $Ux = 0$ " " " " rank $U = n$ for all U in reduced echelon form.

Why? If $A \rightsquigarrow U$ then sols A = sols U and rank A = rank U.
So let U be in r.e.f. Then

rank $U < n \Rightarrow U$ has $<n$ pivots $\Rightarrow$ some column of U has no pivot
$\Rightarrow$ some variable is free
$\Rightarrow Ux = 0$ has (at least) $\mathbb{R}$-many solutions.

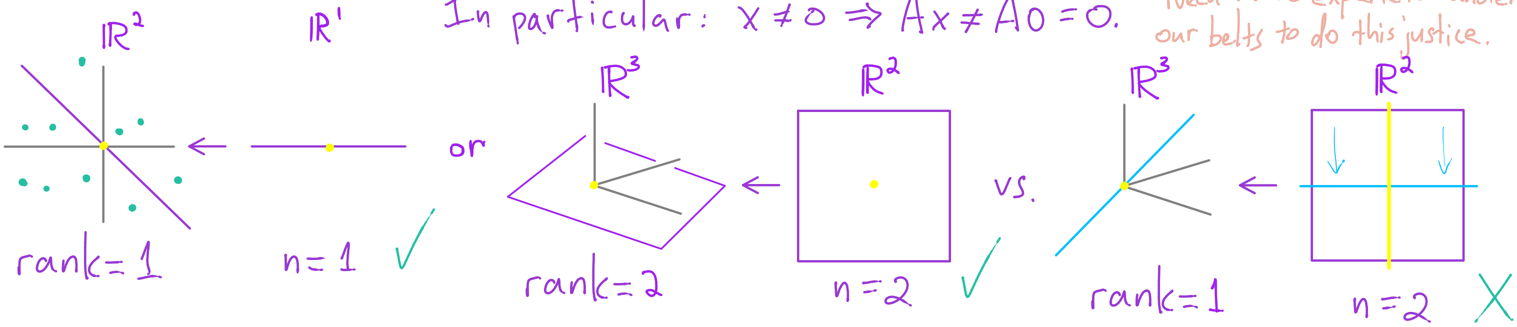
On the other hand, rank $U = n \Rightarrow U = \begin{array}{c} \quad n \\ m \left[\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline & 0 & \end{array}\right] \end{array} \Rightarrow$ the only sols have $x_1 = 0$, $x_2 = 0$, $\vdots$, $x_n = 0$. $\square$

Geometrically, why should this (Prop 5.4) be?



So rank $A = n$ means A preserves dimension of $\mathbb{R}^n$!
$\Updownarrow$
μ_A sticks $\mathbb{R}^n$ into $\mathbb{R}^m$ without compression: $x \neq y \Rightarrow Ax \neq Ay$
In particular: $x \neq 0 \Rightarrow Ax \neq A0 = 0$.

Need more experience under our belts to do this justice.



Q1. For which A does $Ax = b$ have unique solution for all $b \in \mathbb{R}^m$?
Q2. " " " is " consistent " " " ?

A. First look at the pictures: line misses . Why? rank $< m$!
general: for all $b \in \mathbb{R}^m$, $\left.\begin{array}{l} \bullet\ b \text{ has the form } Ax \\ \bullet\ b \in C(A) \\ \bullet\ b \in \text{image of } \mu_A \end{array}\right\}$ equivalent

i.e.
A2. **Prop**: $Ax = b$ consistent for all $b \in \mathbb{R}^m \Leftrightarrow \underbrace{\text{rank } A = m}_{\mathbb{R}^m = C(A)}$. $\square$

A1. $\Leftrightarrow$ $Ax = b$ • is consistent for all b and, by Cor, $\overset{A2}{\Leftrightarrow}$ rank $A = m$
• $Ax = 0$ has only the trivial solution $x = 0$. $\underset{\text{Prop. 5.4}}{\Leftrightarrow}$ rank $A = n$

$\Leftrightarrow$ rank $A = m = n$.

Def: A is <u>nonsingular</u> (~~or invertible~~) if $m = n = \text{rank } A$. defined later

A is <u>singular</u> if $m = n$ and $\text{rank } A < n$.

E.g. The $n \times n$ <u>identity matrix</u> $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ is nonsingular.

General: A nonsingular $\Leftrightarrow$ A has r.e.f. $\cancel{?}$ I_n.

(Do in class if there is time:)

Application: curve fitting

Given 3 points $\overset{v_1}{(x_1, y_1)}, \overset{v_2}{(x_2, y_2)}, \overset{v_3}{(x_3, y_3)} \in \mathbb{R}^2$ with x_1, x_2, x_3 distinct,
find a parabola $y = ax^2 + bx + c$ through them.
(HW: v_1, v_2, v_3 not collinear $\Rightarrow$ parabola exists and is unique.)

Answer:

$$
\begin{aligned}
ax_1^2 + bx_1 + c &= y_1 \\
ax_2^2 + bx_2 + c &= y_2 \\
ax_3^2 + bx_3 + c &= y_3
\end{aligned}
$$

an inhomogeneous linear system!

Solution = coeffs a, b, c on parabola through v_1, v_2, v_3

class selects points; we all solve
(Ensure one pt. has $x = 0$, for ease of row reduction.)