8.

$$\mathbb{R}^m \xleftarrow{\mu_A} \mathbb{R}^n \xleftarrow{\mu_B} \mathbb{R}^p$$

$$A(By) \longleftarrow By \longleftarrow y$$

$$By = y_1 b_1 + \cdots + y_p b_p$$

$$A(y_1 b_1 + \cdots + y_p b_p) \longleftarrow y_1 b_1 + \cdots + y_p b_p$$

$$= y_1 \underbrace{Ab_1}_{c_1} + \cdots + y_p \underbrace{Ab_p}_{c_p} \quad !$$

Lemma: $T \circ S$ is linear if T and S are.

Pf: $T\circ S(y+tz) = T(Sy + tSz)$
$= (T\circ S)y + t(T\circ S)z.$ $\square$

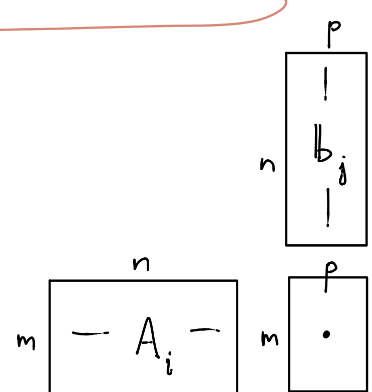
Cor: $\mu_A \circ \mu_B$ is linear, so
$\mu_A \circ \mu_B = \mu_C$ for some C !

Q. What is C? $C \in \mathbb{R}^{\cancel{?}\; m\times p}$

A. $\mu_C(y)$ = linear combination of columns $c_1, \dots, c_p$ with coeffs $y_1, \dots, y_p$. So

Q'. What are $c_1, \dots, c_p$?

Def: For an $m\times n$ matrix A and $n\times p$ matrix B, their product is the $m\times p$ matrix AB satisfying $\mu_{AB} = \mu_A \circ \mu_B$.



$$Ab_j \longleftarrow b_j \longleftarrow e_j$$

Lemma: AB has columns $Ab_1, \dots, Ab_p$.
Equivalently, $(AB)_{ij} = A_i b_j$
or $(AB)_i = A_i B$ = linear combination of rows of B with coeffs from A_i.

E.g. $A = \begin{array}{cc} & n \\ m & \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \end{array}$ $\quad B = \begin{array}{cc} & m \\ n & \begin{bmatrix} 7 \\ 11 \\ \pi \end{bmatrix} \end{array}$ $\Rightarrow AB = \cancel{?}$ $\quad 2\cdot 7 + 3\cdot 11 + 5\pi = 14 + 33 + 5\pi$
$= 47 + 5\pi$

Q. Is BA defined • here? Yes: $\overset{n\times n}{BA} = ?$ different shape than $\overset{m\times m}{AB}$
• in general? No: $C = \begin{bmatrix} 7 & 1 \\ 11 & 1 \\ \pi & 1 \end{bmatrix} \Rightarrow AC$ defined but CA not.

$$\mathbb{R}^n \xleftarrow{\overset{B}{A}} \mathbb{R}^n = \mathbb{R}^n \xleftarrow{\overset{A}{B}} \mathbb{R}^n$$

Why? $\mathbb{R}^3 \xleftarrow{C} \mathbb{R}^2 \neq \mathbb{R}^1 \xleftarrow{A} \mathbb{R}^3$

Q. If A and B square (say $n\times n$) is $AB = BA$?

A. No: $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$ — or just about any square A, B!

E.g. powers of $\underbrace{n\times n \; A}$
$\updownarrow$
directed graph with edge labels

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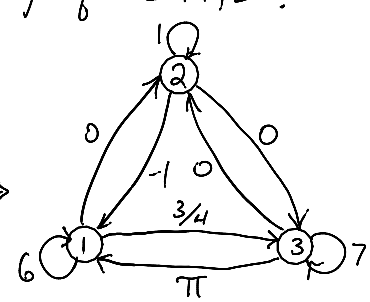
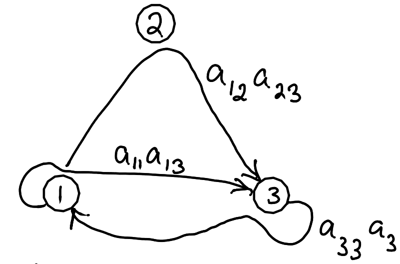
$$\begin{array}{cc} & \text{to } j \\ \text{from } i & \begin{bmatrix} 6 & 0 & 3/4 \\ -1 & 1 & 0 \\ \pi & 0 & 7 \end{bmatrix} \end{array} \longleftrightarrow$$



$(A)_{ij}$ = "distance" from i to j = "length" of edge from i to j

$(A^2)_{ij}$ = sum of "lengths" of 2-step paths from i to j

"length" = $\text{length}_1 \cdot \text{length}_2$

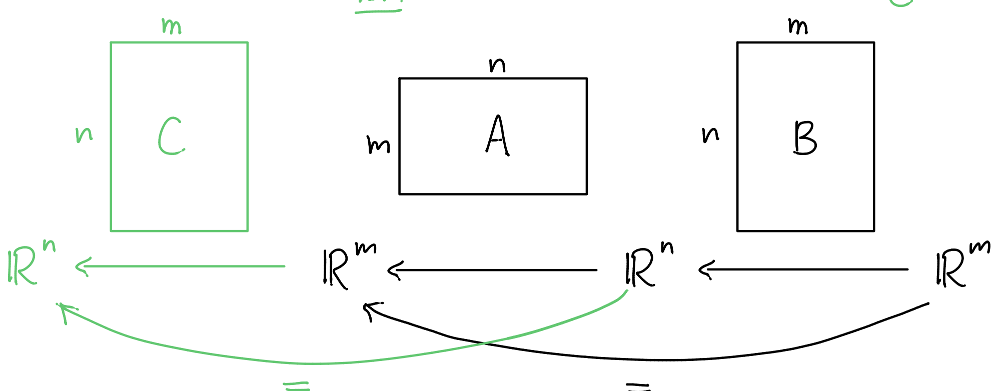


<u>Rules</u> $A, B \in \mathbb{R}^{m\times n} \Rightarrow$
- $A+B = B+A$
- $(A+B)+C = A+(B+C)$
- $0 + A = A$
- $A + (-A) = 0$
- $c(dA) = (cd)A$
- $c(A+B) = cA + cB$
- $(c+d)A = cA + dA$
- $1A = A$

(<u>Proof</u>: Matrices are just vectors drawn as rectangles. $\square$) and you already know these rules for those.

$A \in \mathbb{R}^{m\times n}$, $A' \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times q}$, $t \in \mathbb{R} \Rightarrow$
- $AI_n = A = I_m A$
- $(A+A')B = AB + A'B$
- $(tA)B = t(AB) = A(tB)$
- $(AB)C = A(BC)$ (<u>Pf</u>: Both are the matrix for $\mu_A \circ \mu_B \circ \mu_C$.)

<u>Def</u>: For $A \in \mathbb{R}^{m\times n}$, a <u>right inverse</u> is an $n\times m$ matrix B with $AB = I_{\cancel{?}\,m}$.
<u>left</u> " C $CA = I_n$.



A is <u>invertible</u> if A is square and there is a matrix B with $AB = I_n$ and $BA = I_n$. Notation: $B = A^{-1}$.

<u>E.g.</u> $\begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} -3 & 5 \\ 1 & -1 \end{bmatrix}$:
$$\begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 1 & -1 \end{bmatrix} = \left[\begin{bmatrix} -3 \\ -3 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right]$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \checkmark$$

Thm 3.2: A is invertible $\Leftrightarrow$ $\underbrace{A \text{ is nonsingular}}_{\text{reduced echelon form of } A \text{ is } I_n}$.

rank $= n = m$

Pf:

$A \rightsquigarrow I_n \Leftrightarrow$

$[A \mid I_n] \rightsquigarrow \underbrace{[I_n \mid B]}_{\text{has same sols!}}$ for some $n\times n$ B.

(i.e. solve $Ax_1 = e_1$, $Ax_2 = e_2$, $\ldots$, $Ax_n = e_n$)

$I_n x_1 = b_1$, $I_n x_2 = b_2$, $\ldots$, $I_n x_n = b_n$)

"for all"

$\Leftrightarrow Ax_j = e_j \ \forall j$ has sols $x_j = b_j \ \forall j$

$\Leftrightarrow Ab_j = e_j \ \forall j$

$\Leftrightarrow AB = I_n$.

But also $[A \mid I_n] \rightsquigarrow [I_n \mid B] \Rightarrow [I_n \mid B] \rightsquigarrow [A \mid I_n]$

$\Rightarrow [B \mid I_n] \rightsquigarrow [I_n \mid A]$ Same row operations!

$\Rightarrow BA = I_n$. $\square$

Cor 3.3: If A, B are $n\times n$ and $BA = I_n$ then $B = A^{-1}$ and $A = B^{-1}$.

Caution: $n\times n$: B right inverse of $A \Rightarrow B$ left inverse of A

$m\times n$ with $m \neq n$: FALSE! Do not make this error!

Pf: A nonsingular $\Leftrightarrow$ rank A = #cols $\Leftrightarrow Ax = 0$ has only trivial sol.

Assume $BA = I_n$. Then $Ax = 0 \Rightarrow 0 = B0 = BAx$

$= (BA)x = I_n x = x$

has only the trivial sol!

Thus, by Thm 3.2, A has an inverse A^{-1}.

But then $BA = I_n \Rightarrow \underbrace{BAA^{-1}}_{= B} = \underbrace{I_n A^{-1}}_{= A^{-1}}$. $\square$

Geometrically:

$\Leftrightarrow \mathbb{R}^n \xleftarrow{\mu_A} \mathbb{R}^n$ injective

$\Leftrightarrow \mathbb{R}^n \underset{\mu_A^{-1}}{\overset{\mu_A}{\rightleftarrows}} \mathbb{R}^n$ bijective

Def: injective and surjective