9. Review

$$\overset{A}{\begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & \pi \end{bmatrix}} \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = ? \checkmark \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = ? \checkmark A(x_1 e_1) = x_1 A e_1 = x_1 \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

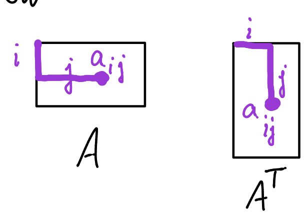
$$A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = ? \checkmark \begin{bmatrix} 3 \\ 11 \end{bmatrix} \qquad A(x_2 e_2) = x_2 \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$A e_3 = \begin{bmatrix} 5 \\ \pi \end{bmatrix} \qquad A(x_3 e_3) = x_3 \begin{bmatrix} 5 \\ \pi \end{bmatrix}$$

$$\begin{aligned} Ax = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= A(x_1 e_1 + x_2 e_2 + x_3 e_3) \\ &= x_1 A e_1 + x_2 A e_2 + x_3 A e_3 \\ &= x_1 \begin{bmatrix} 2 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 11 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ \pi \end{bmatrix} \\ &= x_1 a_1 + x_2 a_2 + x_3 a_3 = \text{linear combin of } \overset{\text{rows}}{\text{cols}} \text{ of } \overset{A^T}{A} \text{ with coeffs from } \overset{x^T}{x} \end{aligned}$$

Q. $\underbrace{[x_1 \; x_2 \; x_3]}_{x^T} \underbrace{\begin{bmatrix} 2 & 7 \\ 3 & 11 \\ 5 & \pi \end{bmatrix}}_{A^T} = ? \checkmark \; x_1[2 \;\; 7] + x_2[3 \;\; 11] + x_3[5 \;\; \pi] =$

Def: The <u>transpose</u> of a matrix A with a_{ij} in row i and col j is A^T (col, row). $\quad (A)_{ij} = (A^T)_{ji}$



E.g. A, A^T general: swap rows and cols

E.g. a is col vector $\Rightarrow a^T$ is row vector

$a^T a = ? \checkmark \; \|a\|^2$

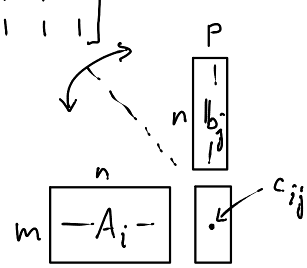
$x^T a = ? \checkmark \; x \cdot a$

$a a^T = ?$ ... some square matrix...

$$\text{proj}_a x = \underbrace{\frac{x \cdot a}{\|a\|^2}}_{\text{scalar}} \underbrace{a}_{\text{vector}} = \underbrace{a}_{n\times 1} \underbrace{\left(\frac{a \cdot x}{\|a\|^2}\right)}_{1\times 1} = a\left(\frac{a^T x}{\|a\|^2}\right) = \frac{a a^T x}{\|a\|^2} = \frac{a a^T}{a^T a} x,$$

so $\text{proj}_a = \dfrac{a a^T}{a^T a}$! $\quad$ E.g. $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{proj}_a = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Q. $(AB)^T = ? \checkmark \; B^T A^T$ $\quad$ flip page over across ⋱



E.g. $(Ax) \cdot y = y^T (Ax) = (y^T A) x = x \cdot (A^T y)$

Q. $EA =$ (A but with rows i and j swapped) for all $A \in \mathbb{R}^{m\times n}$; what's E?

A. If it's true for all A then it's true for $A = I_m$!

$$E = EI_m = \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} \quad & \quad & i & \quad & j & \quad & \quad \end{array} \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & & 1 & \\ & & & & \ddots & & \\ & & & 1 & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix} \end{array}$$

Check: $EA = \begin{bmatrix} — E_1 A — \\ \vdots \\ — E_m A — \end{bmatrix}$.

But $E_k = \begin{cases} e_k^T & \text{if } k \neq i \text{ or } j \\ e_j^T & \text{if } k = i \\ e_i^T & \text{if } k = j \end{cases}$

Thus $(EA)_k = \begin{cases} A_k & \text{if } k \neq i \text{ or } j \\ A_j & \text{if } k = i \\ A_i & \text{if } k = j \end{cases}$ ✓

Moral: what E does to I_m it does to all $A \in \mathbb{R}^{m\times n}$.

E.g. $m=3, n=2$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 3 & 11 \\ 5 & \pi \end{bmatrix} = \begin{bmatrix} 5 & \pi \\ 3 & 11 \\ 2 & 7 \end{bmatrix}$$

General: can use matrices to enact row operations!

Q. $EA =$ (A but with row i multiplied by $c \in \mathbb{R}$) $\forall A \in \mathbb{R}^{m\times n}$; what's E?

A. $E = EI_m = \begin{array}{c} \\ \\ i \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$.

E.g. $\begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 3 & 11 \\ 5 & \pi \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ -3 & -11 \\ 5 & \pi \end{bmatrix}$

Q. $EA =$ (A but with row i replaced by $A_i + cA_j$) $\forall A \in \mathbb{R}^{m\times n}$; what's E?

A. $E = EI_m = \begin{array}{c} \\ \\ i \\ j \\ \\ \end{array} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & c \\ & & & \ddots \\ & & & & 1 \end{bmatrix}$

What did we do to I_m?

$(I_m)_j = e_j^T \qquad e_i^T \rightsquigarrow e_i^T + c e_j^T$

Note: $j < i$ also possible

Moral: Elementary row operations are enacted by left multiplication

$A \rightsquigarrow EA$ for the relevant elementary matrix E.

So $[U \mid c] =$ echelon form of $[A \mid b]$

$$\Rightarrow [A \mid b] \rightsquigarrow [E_1 A \mid E_1 b]$$
$$\rightsquigarrow [E_2 E_1 A \mid E_2 E_1 b]$$
$$\vdots$$
$$\rightsquigarrow [\underbrace{E_\ell E_{\ell-1} \cdots E_2 E_1}_{E} A \mid \underbrace{E_\ell E_{\ell-1} \cdots E_2 E_1}_{E} b] \Rightarrow [U \mid c] = [EA \mid Eb].$$

Important: E is invertible!

- E elementary $\Rightarrow E^{-1}$ elementary of same type.

  Pf: (i) $E^{-1} = E$

  (ii) replace c by $\frac{1}{c}$

  (iii) $\qquad -c$

- $E^{-1} = (E_\ell E_{\ell-1} \cdots E_1)^{-1} = E_1^{-1} \cdots E_{\ell-1}^{-1} E_\ell^{-1}$.

E.g. If only type (iii) occurs, then E looks like $L = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}$,

lower-triangular with 1's on the diagonal.

In this case $A = LU$ is the LU-decomposition of A.

"lower" "upper" (-triangular)

"Most" but not all matrices A have LU decompositions

"open Schubert cell" — don't actually say this

Finding constraint equations

For which $\mathbb{b} \in \mathbb{R}^4$ does $A\mathbb{x} = \mathbb{b}$ have a solution

($\mathbb{b} \in$ image of μ_A)

when

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & 1 & -2 \\ 0 & 1 & 2 & 2 & -1 \\ 2 & -1 & 0 & 1 & -6 \end{bmatrix} ?$$

Answer: $$[A \mid \mathbb{b}] \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 1 & -3 & -1 & 0 & b_1 \\ 0 & 2 & 4 & 0 & 2 & b_1 + b_2 \\ 0 & 0 & 0 & 2 & -2 & -\frac{1}{2}b_1 - \frac{1}{2}b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & b_1 + 9b_2 - 6b_3 + 4b_4 \end{array}\right] = [U \mid \mathbb{c}]$$

constraint eqn.

Q. $U = EA$; what's E?