10. # Chapter 3: Vector spaces

$\mathbb{R}^n$ is a (real) vector space (over $\mathbb{R}$) there are others; general def after midterm

$\mathbb{C}^n$ (complex) or: $\mathbb{C}$

$\mathbb{Q}^n$ (rational) $\mathbb{Q}$

$\mathbb{F}_2$ (binary) $\mathbb{F}_2$

These are the "flat things" we've been talking about all along, though only the ones through 0.

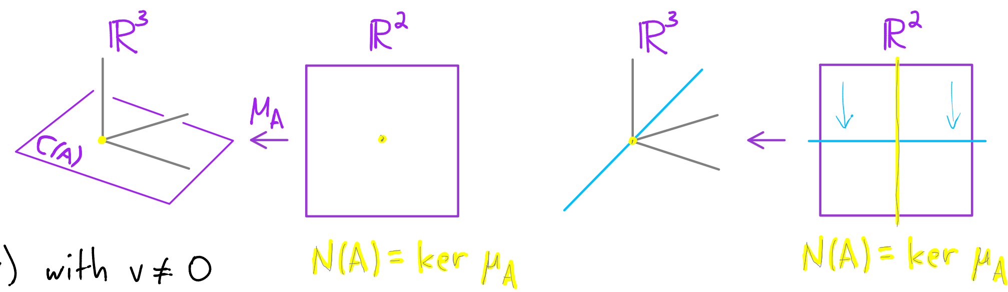
Def: A subset $V \subseteq \mathbb{R}^n$ is a <u>subspace</u> if 1. $V \neq \emptyset$ and

2. $v, w \in V$ and $c \in \mathbb{R} \Rightarrow v + cw \in V$.

Remark 1. can use instead: $0 \in V$ $(v = w;\ c = -1)$ but it's not easier to check

2. $\Rightarrow$ V closed under arbitrary linear combinations (Pf: induction.)

Examples



1. $V = \{0\}$
2. $V = \mathbb{R}^n$
3. line $\ell = \mathrm{span}(v)$ with $v \neq 0$
4. hyperplane $H = \{x \in \mathbb{R}^n \mid a x = 0\}$ for some fixed $a \in \mathbb{R}^n_{\text{row}}$
5. the <u>nullspace</u> of any matrix $A \in \mathbb{R}^{m\times n}$ Pf: $A0 = 0$ and μ_A is linear!

   $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$ (= <u>kernel</u> of μ_A) = sols $[A \mid 0]$
6. $\mathrm{span}(v_1, \dots, v_k)$ for any $\begin{bmatrix} | & & | \\ v_1, & \dots, & v_k \\ | & & | \end{bmatrix} \in \mathbb{R}^n$

   $= C(A)$ for $A = \begin{bmatrix} | & & | \\ v_1 & \dots & v_k \\ | & & | \end{bmatrix} \in \mathbb{R}^{n\times k}$

   Pf: 1. $0 = 0 v_1$.

   2. $v = c_1 v_1 + \dots + c_k v_k$ and $w = d_1 v_1 + \dots + d_k v_k$

   $\Rightarrow v + cw = (c_1 + c d_1) v_1 + \dots + (c_k + c d_k) v_k \in V$.

The point is that none of this is new; you already know what subspaces look like and how to describe all of them — yes, all:

Thm: Every subspace of $\mathbb{R}^n$ has the form of #5.
" " " " " " " " #6.

Pf: For #6, keep finding "independent" vectors that increase dimension until you can't anymore: if $\exists\, v_1 \neq 0$ in V, then $\mathrm{span}(v_1) \subseteq V$.

$v_2 \notin \mathrm{span}(v_1)$ $\qquad (v_1, v_2) \subseteq V$

$\vdots \qquad\qquad \vdots$

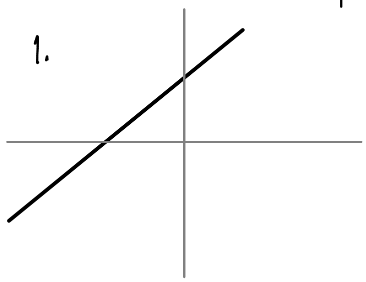
stops at v_k for some $k \leq n$

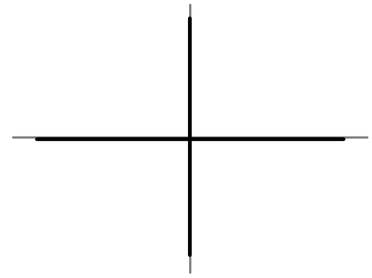
because $\mathrm{span}(v_1, \dots, v_k) = C(A)$ for $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$ is spanned by r vectors, where $r = \mathrm{rank}\, A \leq n$.

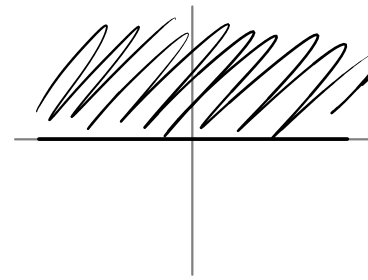
For #5: find constraint equations, given $v_1, \dots, v_k$. $\square$

$\square$

Q. Is this a subspace?

1. 

2. 

3. 

4. $x_0 + \operatorname{span}(v_1, \dots, v_k)$ yes, if $x_0 \in$ span; no if not

5. $U+V = \{u+v \mid u \in U \text{ and } v \in V\}$ for subspaces U and V

Yes: 1. $0+0 \in U+V$; 2. $\left.\begin{matrix} u+v \in U+V \\ u'+v' \in U+V \\ c \in \mathbb{R} \end{matrix}\right\} \Rightarrow u+v+c(u'+v') = (u+cu') + (v+cv') \in U+V$.

Note: $U+V$ is the smallest subspace containing U and V.

E.g. x-axis + y-axis = xy-plane.

Def: For any subset $C \subseteq \mathbb{R}^n_{col}$ set $C^\perp = \{u \in \mathbb{R}^n_{row} \mid ux = 0 \ \forall x \in C\}$

$R \subseteq \mathbb{R}^n_{row}$ set $R^\perp = \{x \in \mathbb{R}^n_{col} \mid ux = 0 \ \forall u \in R\}$

WARNING: The book calls $C^\perp$ what I call $(C^\perp)^T$; book thinks every vector is a column! $C^\perp$ in book: collection of columns

will make even more sense when we get to row space $R(A)$ | $C^\perp$ for us: collection of rows

Prop: $C^\perp$ is a subspace of $\mathbb{R}^n_{row}$

$R^\perp$ " " " " $\mathbb{R}^n_{col}$

Pf: 1. $0x = 0$. ✓

2. $vx = 0$ and $wx = 0 \Rightarrow (v+cw)x = vx + cwx = 0 + c0 = 0 \ \forall c \in \mathbb{R}$.

For $R^\perp$, transpose the argument. □

Lemma: $V^T \subseteq \mathbb{R}^n_{row}$ is a subspace $\Leftrightarrow V \subseteq \mathbb{R}^n$ is a subspace.

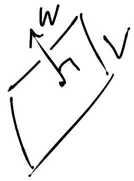
Pf: Transpose is linear. □ check! $(A+cB)^T = A^T + cB^T \ \forall A, B \in \mathbb{R}^{m \times n}, c \in \mathbb{R}$

Def: $V \subseteq \mathbb{R}^n_{col}$ and $W \subseteq \mathbb{R}^n_{row}$ are orthogonal, written $V \perp W$, if $WV = 0$. Equivalently:

- $wv = 0 \ \forall w \in W$ and $v \in V$
- $W \subseteq V^\perp$
- $V \subseteq W^\perp$

Also, $V \subseteq \mathbb{R}^n_{col}$ and $W \subseteq \mathbb{R}^n_{col}$: $V \perp W$ if $\cdot\ W^T V = 0$ $\quad W^T \perp V$

row $\qquad$ row

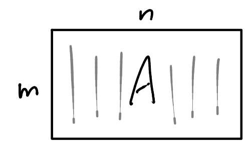


$\cdot\ V^T W = 0 \quad V^T \perp W$

$\cdot\ V \cdot W = 0$

E.g. $C(A)$ = column space of A

= image of μ_A $\qquad$ subspace as in #6.



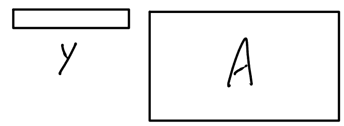
$L(A)$ = left nullspace of $A = \{ y \in \mathbb{R}^m_{row} \mid yA = 0 \}$

= kernel of $\rho_A : \mathbb{R}^m_{row} \to \mathbb{R}^n_{row}$ $\qquad$ y times any column of A is 0.

$y \mapsto yA$

right multiplication by A

book omits this $\qquad = N(A^T)^T$



WARNING: In the book's notation, $L(A) = N(A^T)^T$

Prop: $L(A) = C(A)^\perp$.

CAN SKIP: $N(A) = R(A)^\perp$ PROVED IN NEXT LECTURE

Pf: $y \in L(A) \Rightarrow yA = 0 \Rightarrow y a_j = 0 \ \forall j = 1, \dots, ?$

$\Rightarrow c_1 y a_1 + \cdots + c_n y a_n = 0 \ \forall c_1, \dots, c_n \in \mathbb{R}$ $\quad$ so

$\Rightarrow y(c_1 a_1 + \cdots + c_n a_n) = 0$ $\quad$ "

$\Rightarrow y \in C(A)^\perp$. $\qquad$ Hence $L(A) \subseteq C(A)^\perp$.

But $y \in C(A)^\perp \Rightarrow y a_j = 0 \ \forall j$ by def. (since $a_j \in C(A)$)

so $yA = 0$; hence $C(A)^\perp \subseteq L(A)$. $\square$

Similarly:

Def: $R(A)$ = row space of A = span$(A_1, \dots, A_m) \subseteq \mathbb{R}^n_{row}$ $\qquad$ the rows of A

Cor: $N(A) = R(A)^\perp$.

Pf: Transpose previous Prop. $\square$ $\qquad$ $\underset{L(A^T)^T}{\overset{N(A)}{\parallel}} = \underset{(C(A^T)^T)^\perp}{\overset{R(A)^\perp}{\parallel}}$

WARNING: The book calls $R(A)^T$ what I call $R(A)$; book thinks every vector is a column!

$\square$