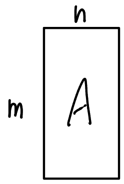
12. Q. $A \in \mathbb{R}^{m\times n} \Rightarrow \dim C(A) = ?$

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\parallel}{\text{rank } A}$

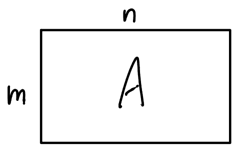
no coincidence

m fixed

Ans:

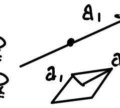


n, unless there's some coincidence



m, " " " "

	n	$\dim C(A)$		
injective	1	n	$\mathbb{R}^m \supsetneq$	$\leftarrow \mathbb{R}^1$
injective	2	n	$\mathbb{R}^m \supsetneq$	$\leftarrow \mathbb{R}^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
both (injective, surjective)	m	$n=m$	$\mathbb{R}^m = \mathbb{R}^n$	$\leftarrow \mathbb{R}^n$
surjective	$m+1$	m	$\mathbb{R}^m$	$\leftarrow \mathbb{R}^n$
surjective	$m+2$	m	$\mathbb{R}^m$	$\leftarrow \mathbb{R}^n$
	$\vdots$	$\vdots$		

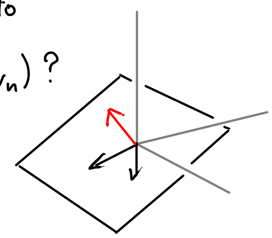


injective means no collapsing: $T(x) = T(y) \Rightarrow x = y$ "into", "one-to-one"

surjective means target covered: $\mathbb{b} \in$ target $\Rightarrow T(x) = \mathbb{b}$ for some $x \in$ source "onto"

Q $\Leftrightarrow$ Q′: Given n vectors $v_1, \dots, v_n \in \mathbb{R}^m$, what is $\dim \operatorname{span}(v_1, \dots, v_n)$?

coincidence: span has dim less than possible



Def: vectors $v_1, \dots, v_k$ are <u>linearly dependent</u> if $c_1 v_1 + \dots + c_k v_k = 0$

for some $c_1, \dots, c_k \in \mathbb{R}$ not all 0. "some nontrivial linear combination vanishes"

Prop: $v_1, \dots, v_k$ linearly dependent $\Leftrightarrow \underset{\uparrow \text{ "coincidence"}}{\dim \operatorname{span}(v_1, \dots, v_k) < k} \Leftrightarrow v_i \in \underbrace{\operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k)}_{V_i}$ for some i.

Pf: $\Leftrightarrow$: by def!

$\Rightarrow$: $c_1 v_1 + \dots + c_k v_k = 0$ with $c_i \neq 0 \Rightarrow v_i = -\frac{1}{c_i} \sum_{j\neq i} c_j v_j \in V_i$.

$\Leftarrow$: $v_i = x_1 v_1 + \dots + x_{i-1} v_{i-1} + x_{i+1} v_{i+1} + \dots + x_k v_k = \sum_{j \neq i} x_j v_j \in V_i$

$\Rightarrow \sum_{j=1}^{k} x_j v_j = 0$, where $x_i = -1$. $\square$

E.g. Are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix}$ linearly dependent?

$\Leftrightarrow N(A) \neq 0$ for $A = \begin{array}{c} \begin{bmatrix} 3 & -1 & 8 \\ 0 & 1 & 1 \\ 1 & -4 & -1 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \end{array}$!

Sol:

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0 \Leftrightarrow x \in N(A)$

dependence relation $\Leftrightarrow$ also $x \neq 0$

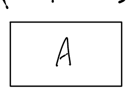
"set minus"

$$\boxed{\text{Moral: } N(A) \setminus \{0\} = \text{linear dependence relations on columns of } A}$$

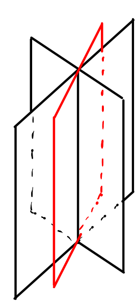
$\begin{bmatrix} 3 & -1 & 8 \\ 0 & 1 & 1 \\ 1 & -4 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \in N(A) \setminus \{0\}$, so: yes.

Q. Given rows $A_1, \dots, A_m$ of $A \in \mathbb{R}^{m\times n}$, what is $\dim \Big(\overbrace{\{A_1 x = 0\} \cap \dots \cap \{A_m x = 0\}}^{N(A)}\Big)$?

A. $n - m$ if $n \geq m$, unless there is some coincidence!



0 if $n \leq m$, " " " " "



$a_1, \dots, a_n$ linearly independent in $\mathbb{R}^m$

$\Leftrightarrow$ if $x_1 a_1 + \cdots + x_n a_n = 0$ then $x_i = 0 \ \forall i$

$\Leftrightarrow$ no column of A lies in the span of the others for $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$

$\Leftrightarrow$ $N(A) = 0$

$\Leftrightarrow$ 0 can be expressed uniquely as a linear combination of $a_1, \dots, a_n$

$\Leftrightarrow$ $b \in \operatorname{span}(a_1, \dots, a_n) = C(A)$ is uniquely a linear combination of $a_1, \dots, a_n$

$\Leftrightarrow$ $Ax = 0$ has only one solution

$[A \mid b]$ consistent

$\Leftrightarrow$ $Ax = b$ has only one solution when $b \in C(A)$

$\Leftrightarrow$ $Ax = b$ has at most one solution $\forall\, b \in \mathbb{R}^m$

$\Leftrightarrow$ M_A is injective

$\Leftrightarrow$ M_A does not decrease dimension

$\Leftrightarrow$ rank $A = n$

$\Leftrightarrow$ A has a left inverse

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\mathbb{R}^m \xleftarrow{M_A} \mathbb{R}^n$$
$$Ax \longleftarrow x$$

<u>Prop 3.2:</u> Assume $v_1, \dots, v_k$ are linearly independent.
Then $v_1, \dots, v_k, v$ is linearly independent $\Leftrightarrow$ $v \notin \operatorname{span}(v_1, \dots, v_k)$.
dependent $\Leftrightarrow$ $v \in \operatorname{span}(v_1, \dots, v_k)$

<u>Pf:</u> $\Rightarrow$: Suppose $c_1 v_1 + \cdots + c_k v_k + cv = 0$. Then $c \neq 0$ since $c = 0 \Rightarrow$ $c_1 v_1 + \cdots + c_k v_k = 0 \Rightarrow c_1 = \cdots = c_k = 0$ because $v_1, \dots, v_k$ are linearly independent.
So $v = -\frac{1}{c}(c_1 v_1 + \cdots + c_k v_k)$.

$\Leftarrow$: Previous prop. (Doesn't need $v_1, \dots, v_k$ linearly independent.) $\square$

<u>E.g.</u> Is 0 linearly independent? No: $1 \cdot 0 = 0$

<u>E.g.</u> Is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ linearly independent? No: $1 \cdot v - 1 \cdot v = 0$. $\Rightarrow$ need multisets technically

$\{\dots\}$ = set, so I list the vectors with no brackets

<u>E.g.</u> Is $\underset{v_1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}}, \underset{v}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$ $v \notin \operatorname{span}(v_1, v_2)$ but v_1, v_2, v not linearly independent.
Why doesn't this contradict Prop.?

<u>E.g.</u> Prove that $Av_1, \dots, Av_k$ are linearly independent if $v_1, \dots, v_k$ are linearly independent and $A \in \mathbb{R}^{m \times n}$ has rank n.

Sol. Suppose $c_1 A v_1 + \cdots + c_k A v_k = 0$. Then $A(c_1 v_1 + \cdots + c_k v_k) = 0$, so $c_1 v_1 + \cdots + c_k v_k = 0$ because M_A is injective. Thus $c_1 = \cdots = c_k = 0$ since $v_1, \dots, v_k$ are linearly independent.

$\square$