13. <u>Def</u>: $v_1, \dots, v_k$ is a basis for a subspace $V \subseteq \mathbb{R}^n$ if $v_1, \dots, v_k$

(i) span V and — not too few

(ii) are linearly independent. — not too many

$\Big\}$ Goldilocks

Q. Name a basis for $\mathbb{R}^n$.

A. <u>standard basis</u> $e_1, \dots, e_n$: (i) $x = x_1 e_1 + \cdots + x_n e_n \quad \forall x \in \mathbb{R}^n$.

(ii) $x = x_1 e_1 + \cdots + x_n e_n = 0 \Rightarrow x = 0$. $\left(\text{or: rank} \underbrace{\begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix}}_{= I_n} = n\right)$

<u>E.g.</u> Which of these are bases?

1. $\underset{v_1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$ in $V = \mathbb{R}^3$

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 5 & 3 \end{bmatrix}} \xrightarrow[A_2' = A_2 - 2A_1]{A_3' = A_3 - A_1} \underset{EA}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix}} \Rightarrow \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \in N(A) \setminus \{0\}$$

$\updownarrow$ dependence relation

$v_1 - 2v_2 + 3v_3 = 0$

no: fails (ii) $\quad (\Rightarrow$ fails (i)$)$

2. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ in $V = \mathbb{R}^3$ — no: too many vectors

<u>Lemma</u>: $a_1, \dots, a_n$ linearly dependent in $\mathbb{R}^m$ if $n > m$.

equivalent conditions super useful! <u>Pf</u>: rank $A \le m < n$. $\square$ $\quad$ rank $< n \Leftrightarrow$ dep.

4. $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \end{bmatrix}$ in $V = \mathbb{R}^4$

Yes: $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 2 & 1 & 4 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 1 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \Rightarrow \text{rank} = 4$

No: they don't span by Prop. 3.2.

Note: If #vectors $= \dim V$ then (i) $\Leftrightarrow$ (ii)

$v_1, \dots, v_k$ linearly independent $\Leftrightarrow$ $v_1, \dots, v_k$ is a basis for span$(v_1, \dots, v_k)$

All conditions equivalent to "linearly independent" work for "basis for their span".

<u>Prop 3.4</u>: $A \in \mathbb{R}^{n \times n}$ nonsingular $\Leftrightarrow$ cols of A form a basis of $\mathbb{R}^n$.

<u>Pf</u>: A nonsingular $\overset{m=n}{\Leftrightarrow} N(A) = 0 \overset{m=n}{\Leftrightarrow} \mu_A$ is bijective $\Leftrightarrow$ rank $A = n$. $\square$

<u>Thm 3.5</u>: Every subspace $V \subseteq \mathbb{R}^n$ has a basis.

<u>Pf</u>: $V = \text{span}(v_1, \dots, v_k)$ for some $v_1, \dots, v_k \in V$ proved in Lec. 10.

Assume k is minimal. Then $v_i \notin \text{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$ by assumption (!),

so $v_1, \dots, v_k$ is independent by $\square$. $\square$

Thm 4.2: $v_1,\dots,v_k$ and $w_1,\dots,w_\ell$ are two bases for a subspace $V \subseteq \mathbb{R}^n \Rightarrow k=\ell$.

Pf: $w_1 \in \operatorname{span}(v_1,\dots,v_k) \Rightarrow w_1 = Ax_1$ for some $x_1 \in \mathbb{R}^k$, where $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} \in \mathbb{R}^{n\times k}$

$$\vdots$$

$w_\ell = Ax_\ell$ " " x_ℓ "

$$\begin{array}{c} \ell \\ n\begin{bmatrix} | & & | \\ w_1 & \cdots & w_\ell \\ | & & | \end{bmatrix} \end{array} = \begin{array}{c} k \\ n\begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} \end{array} \begin{array}{c} \ell \\ \begin{bmatrix} | & & | \\ x_1 & \cdots & x_\ell \\ | & & | \end{bmatrix}k \end{array}$$

$$W = A \quad X$$ Want: X is square.

$\ell > k \Rightarrow N(X) \neq 0 \Rightarrow \exists\, y \in N(X)\setminus\{0\}$

$$\Rightarrow Wy = (AX)y = A(Xy) = A0 = 0$$

so $y \in N(W)\setminus\{0\}$. Thus $w_1,\dots,w_\ell$ are linearly dependent.

Therefore $\ell \le k$. By symmetry, $k \le \ell$. Thus $k=\ell$. $\square$

Cor: $\dim V$ = size of any basis of V.

E.g. $\dim C(A) = \operatorname{rank} A$: $C(A) = \operatorname{span}(v_1,\dots,v_r)$ for cols $v_1,\dots,v_r$ of A with r minimal $\Rightarrow v_1,\dots,v_r$ linearly independent.

$\dim C(A) \le n$. Why? $\operatorname{image}(\mu_A)$ spanned by $\underset{a_1}{\mu_A(e_1)},\dots,\underset{a_n}{\mu_A(e_n)}$

$\mu_A(e_1),\dots,\mu_A(e_n)$ might be linearly dependent, but they do still span $\operatorname{image}(\mu_A)$.

Grammar: $v_1,\dots,v_k$ is a basis $\mathcal{B}$

$v_i \in \mathcal{B}$ is a basis element (or basis vector)

$v_1,\dots,v_k$ and $w_1,\dots,w_k$ are bases