14. Bases for $R(A)$, $C(A)$, $N(A)$, $L(A)$

Thm 4.5: Fix $A \in \mathbb{R}^{m \times n}$ and $U = EA$ the reduced echelon form of A, with E invertible. e.g. $E \neq 0$ and doesn't even have any 0 rows

E.g.

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

$R(A)$: The nonzero rows of U form a basis.

E.g. basis for $R(A)$ is
$[1\ 0\ -1\ 0\ 1]$ U_1
$[0\ 1\ 1\ 0\ 2]$ U_2
$[0\ 0\ 0\ 1\ 1]$ U_3

Pf: The rows of U are linear combinations of the rows of A with coefficients from (rows of) E. Thus rows$(U) \subseteq R(A)$, so $R(U) \subseteq R(A)$.

But $EA = U \Rightarrow A = E^{-1}U \Rightarrow R(A) \subseteq R(U)$, so $R(U) = R(A)$. Now: are they a basis?

The pivot rows $U_1, \dots, U_r$ of U are independent because

(*) $c_1U_1 + \cdots + c_rU_r$ has entries $c_1, \dots, c_r$ in the pivot columns

$$c_1U_1 + c_2U_2 + c_3U_3 = [\underset{\uparrow}{c_1}\ \ \underset{\uparrow}{c_2}\ \ c_2 - c_1\ \ \underset{\uparrow}{c_3}\ \ c_1 + 2c_2 + c_3]$$

so if it equals $0 \Rightarrow c_1 = \cdots = c_r = 0$. □

$L(A)$: The rows of E corresponding to zero-rows of U form a basis

E.g. basis for $L(A)$ is $[-1\ -1\ 1\ 1] = E_4$

Pf: First compute $L(U) = \text{span}(e_{r+1}^T, \dots, e_m^T)$, where $r = \text{rank}\, A$, which holds because $c_1U_1 + \cdots + c_mU_m = 0 \Leftrightarrow c_1U_1 + \cdots + c_rU_r = 0$ (since $\boxed{U_{r+1}, \dots, U_m = 0}$)

last $m-r$ standard basis row vectors

$\Rightarrow c_1 = \cdots = c_r = 0$ (by (*) or better, by basis for $R(A)$).

Compare $L(U)$ to $L(A)$:
$$\begin{aligned} y \in L(U) &\Leftrightarrow yU = 0 \\ &\Leftrightarrow yEA = 0 \\ &\Leftrightarrow yE \in L(A), \end{aligned}$$

so $L(A) = L(U)E$

$= \operatorname{span}(e^T_{r+1}, \dots, e^T_m)\, E$

(i) $= \operatorname{span}(E_{r+1}, \dots, E_m)$ (ii) independent because E is invertible. □

N(A): Make U into an $n \times n$ matrix U' as follows.

1. Move rows down so all pivots sit on diagonal.
2. Add or delete 0's to ensure $n \times n$.
3. Set diag $= -1$. Note: suffices to change all 0's on diag to -1

The free-variable columns of U' are a basis for $N(A)$.

E.g.
$$U = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix} \Rightarrow U' = \begin{bmatrix} -1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 2 \\ & & -1 & & \\ 0 & 0 & 0 & -1 & 1 \\ & & & & -1 \end{bmatrix}$$

Pf: Solve equation $Ux = 0$ $(\Leftrightarrow Ax = 0)$:

$(\text{pivot var})_1 + \text{later terms} = 0 \Leftrightarrow \text{later terms} = -(\text{pivot var})_1$

$\vdots \qquad \ddots$

$(\text{pivot var})_r + \text{later terms} = 0 \Leftrightarrow \underbrace{\text{later terms}}_{\text{only involve free vars!}} = -(\text{pivot var})_r.$

So insert rows $\quad -\text{free var} = -\text{free var}.$ □

$$x_1 - x_3 + x_5 = 0 \Leftrightarrow 0 + \underbrace{\begin{bmatrix}-1\\1\\-1\\0\\0\end{bmatrix}}_{v_1} x_3 + \underbrace{\begin{bmatrix}1\\2\\0\\1\\-1\end{bmatrix}}_{v_2} x_5 = \begin{bmatrix}-x_1\\-x_2\\-x_3\\-x_4\\-x_5\end{bmatrix}$$

v_1, v_2 is a basis for $N(A)$.

pivot cols don't matter! Hence -1 vs. 0 okay.

C(A): The pivot columns of A form a basis.

E.g.
$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix} \Rightarrow C(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right).$$

v_1, v_2 say that the free var cols of A lie in span(pivot var cols).

Pf: Our basis of $N(A)$ says that $a_j \in \operatorname{span}(\text{pivot cols})$ if j is a free var col.

(E.g. v_2 says $a_1 + 2a_2 + a_4 - a_5 = 0$)

Thus the pivot cols span $C(A)$. But # pivot cols $= \operatorname{rank} A = \dim C(A)$. □

<u>Cor 4.6</u>: 1. $\dim R(A) = \dim C(A) = \operatorname{rank} A$.

2. $\dim N(A) = \#\text{cols} - \operatorname{rank} A$:

rank - nullity theorem

any field

$$\boxed{A \in \mathbb{R}^{m\times n} \Rightarrow \dim C(A) + \dim N(A) = n}$$

$$\boxed{T \text{ linear} \Rightarrow \dim(\ker T) + \dim(\operatorname{im} T) = \dim(\text{source } T)}$$

3. $\dim L(A) = m - \operatorname{rank} A$.

This is how rank-nullity is usually used.

<u>Pf</u>: $\operatorname{rank} A = \#\text{pivots}$. □

<u>Prop 4.8</u>: $V \subseteq \mathbb{R}^n$ subspace of dim $k \Rightarrow \dim V^\perp = n-k$.

$k \quad + \quad n-k \quad = \quad n$

<u>Pf</u>: V has basis $v_1, \dots, v_k$. Let $A \in \mathbb{R}^{n\times k}$ have these cols. $V = C(A)$ and $V^\perp = L(A)$. □

<u>Summary</u> $A \in \mathbb{R}^{m\times n}$ $\quad r = \operatorname{rank} A$

$r = n \Leftrightarrow A = \square$ has
- all columns independent
- n rows independent ← somewhere in A

$\Leftrightarrow N(A) = 0$

$\Leftrightarrow \mu_A$ injective

$\Leftrightarrow R(A) = \mathbb{R}^n_{\text{row}}$

$\Leftrightarrow \rho_A$ surjective

T

$r = m \Leftrightarrow A = \square$ has
- all rows independent
- m columns independent

$\Leftrightarrow C(A) = \mathbb{R}^m$

$\Leftrightarrow \mu_A$ surjective

$\Leftrightarrow L(A) = 0$

$\Leftrightarrow \rho_A$ injective

$m = r = n \Leftrightarrow A = \square$ has all $m = n$ rows and columns independent

$\Leftrightarrow A$ nonsingular $\Leftrightarrow$ invertible

$\overset{m=n}{\Leftrightarrow} N(A) = 0$

$\overset{m=n}{\Leftrightarrow} C(A) = \mathbb{R}^n$

$\Leftrightarrow \mu_A$ bijective $\overset{m=n}{\Leftrightarrow} \mu_A$ injective $\overset{m=n}{\Leftrightarrow} \mu_A$ surjective

$\Leftrightarrow \rho_A$ bijective $\overset{m=n}{\Leftrightarrow} \rho_A$ surjective $\overset{m=n}{\Leftrightarrow} \rho_A$ injective