16. Inner products and projections

can be made to work $/\mathbb{C}$ or $\mathbb{F} \subseteq \mathbb{R}$

Def: Let V be a vector space $/\mathbb{R}$. An inner product on V assigns to each pair $u, v \in V$ a number $\langle u, v\rangle \in \mathbb{R}$ such that $\forall\, u, v, w \in V$ and scalars c,

crucial!

symmetric 1. $\langle u, v\rangle = \langle v, u\rangle$

bilinear $\begin{cases} 2.\ \langle cu, v\rangle = c\langle u, v\rangle \\ 3.\ \langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \end{cases}$

needs $\mathbb{R} \rightarrow$ 4. $\langle v, v\rangle \geq 0$ and $= 0 \Leftrightarrow v = 0$.

positive ——— definite

$\Rightarrow$ lengths: $\|v\|^2 = \langle v, v\rangle$

angles: $\cos\theta = \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|}$

E.g. (a) $V = \mathbb{R}^n \quad \langle x, y\rangle = y^T x = x \cdot y$

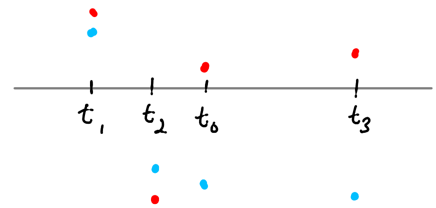
(b) $V = \mathcal{P}_k \quad t_0, \dots, t_k \in \mathbb{R} \qquad \langle p, q\rangle = \sum_{i=0}^{k} p(t_i)\, q(t_i)$

1. ✓ 2. ✓ 3. ✓ $\Longleftarrow$ $= \begin{bmatrix} p(t_0) \\ \vdots \\ p(t_k) \end{bmatrix} \cdot \begin{bmatrix} q(t_0) \\ \vdots \\ q(t_k) \end{bmatrix}$

4. ≥ 0 ✓; $\langle p, p\rangle = 0 \Rightarrow \sum_{i=0}^{k} p(t_i)^2 = 0$

$\Rightarrow p(t_i) = 0 \ \forall i$

$\Rightarrow p \equiv 0$ (remember last lecture?)



(c) $V = C^0(I)$ for $I = [a, b] \qquad \langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt$ same picture but "using all $t \in [a, b]$"

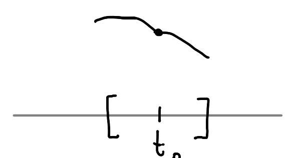
1. ✓

2. $\int c \cdots dt = c \int \cdots dt$

3. $\int (f+g)h\, dt = \int (fh + gh)\, dt = \int fh\, dt + \int gh\, dt$ $\Big\}$ $\int_a^b$ is linear!

4. $\int_a^b \underbrace{f(t)^2}_{\geq 0} dt \geq 0 \qquad f(t_0)^2 \neq 0 \Rightarrow f(t)^2 > \frac{1}{2} f(t_0)^2$ on $[t_0 - \varepsilon, t_0 + \varepsilon]$

needs $f \in C^0$ $\qquad \Rightarrow \int_a^b f(t)^2 dt > 2\varepsilon \cdot \frac{1}{2} f(t_0)^2 = \varepsilon f(t_0)^2 > 0$.



Lemma: $v_1, \dots, v_d \in V$ mutually orthogonal of length 1 under any inner product

$\langle v_i, v_j\rangle = 0$ for $j \neq i$, $\langle v_i, v_i\rangle = 1 \ \forall i$ orthonormal $\Rightarrow$ linearly independent.

Pf: $v = c_1 v_1 + \cdots + c_d v_d = 0 \Rightarrow \underset{c_i}{\underset{\|}{\langle v, v_i\rangle}} = 0 \ \forall i$. □

Application:

Thm 6.4: Given $k+1$ points $(t_0, a_0), \dots, (t_k, a_k)$ in $\mathbb{R}^2$ with $t_0, \dots, t_k$ distinct,

$\exists!\ p \in \mathcal{P}_k$ whose graph passes through the points.

there exists ↗ ↖ unique

Pf: Construct orthonormal $p_0, \dots, p_k \in \mathcal{P}_k$ under inner product (b):

$p_i(t_i) = 1$, $p_i(t_j) = 0$ for $j \neq i$. Set $\Delta(t) = (t-t_0)(t-t_1)\cdots(t-t_k)$

$$\Delta_i(t) = \frac{\Delta(t)}{(t-t_i)} \quad \text{(omit the } t-t_i \text{ factor).}$$

Then $\Delta_i(t_j) = 0$ for $j \neq i$ and $\Delta_i(t_i) \neq 0$ since $t_0, \dots, t_k$ distinct, so

$p_i(t) = \dfrac{\Delta_i(t)}{\Delta_i(t_i)}$ has $p_0, \dots, p_k$ orthonormal. Lemma $\Rightarrow$ basis

$\Rightarrow$ every $f \in \mathcal{P}_k$ has unique expression $f = c_0 p_0 + \cdots + c_k p_k$.

Note that $f(t_i) = 0 + \cdots + 0 + c_i \cdot 1 + 0 + \cdots + 0 = c_i$.

Take $p = a_0 p_0 + \cdots + a_k p_k$. $\square$

Cor: $t_0, \dots, t_k$ distinct $\Rightarrow$

$$\begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^k \\ 1 & t_1 & t_1^2 & \cdots & t_1^k \\ & & \vdots & & \\ 1 & t_k & t_k^2 & \cdots & t_k^k \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix}$$

is nonsingular.

evaluations of p at $t_0, \dots, t_k$

coeffs on the polynomial $p(t)$

Pf:

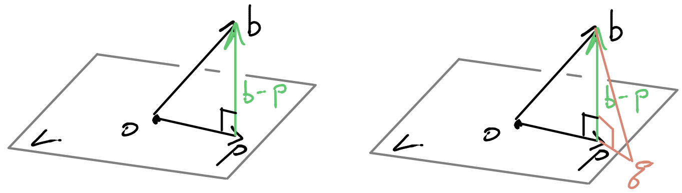
Thm 6.4 $\Rightarrow$ linear system has ! sol.

$\Rightarrow$ (square!) matrix is nonsingular. $\square$

§4.1

need $\mathbb{R}$ here: using dot product and length

<u>Def</u>: Fix a subspace $V \subseteq \mathbb{R}^m$ and $b \in \mathbb{R}^m$. The (<u>orthogonal</u>) <u>projection</u> of b onto V is the unique vector $p = \mathrm{proj}_V b \in V$ such that $b - p \in V^\perp$ under $\cdot$.



<u>Lemma 1.1</u>: Set $p = \mathrm{proj}_V b$. Then $\|b-p\| \leq \|b-q\| \ \forall q \in V$.

$\boxed{\text{p is closest to b in V.}}$

<u>Pf</u>: $\|b-q\|^2 = \|b-p\|^2 + \|p-q\|^2$. □

How to find p? Fix basis $v_1, \dots, v_n$ for V.

Need $v_1 \cdot (b-p) = 0, \dots, v_n \cdot (b-p) = 0$

$v_1^T(b-p) = 0, \dots, v_n^T(b-p) = 0$

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}, \quad \text{so} \quad \begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{bmatrix} \begin{bmatrix} | \\ b-p \\ | \end{bmatrix} = 0.$$

$\Leftrightarrow \ A^T(b-p) = 0$

$\Leftrightarrow \ A^T b = A^T p \qquad \Leftrightarrow p \in b + V^\perp.$

Q. Is that enough? No. Need also $p \in V$ — i.e. $p = Ax$ "p is a linear combination of the columns of A"

<u>Prop</u>: Given an $m \times n$ matrix of rank n, the <u>normal equation</u> $A^TAx = A^Tb$ has a unique solution $\bar{x} \in \mathbb{R}^n$, the <u>least squares solution</u> of $Ax = b$.

<u>Pf</u>: A^TA is $n \times n$.

<u>Lemma</u>: μ_{A^TA} is injective.

Lemma $\Rightarrow$ A^TA is nonsingular. □

<u>Pf of Lemma</u>: $C(A)^\perp = L(A)$

$\Rightarrow C(A) \perp \underbrace{L(A)^T}_{= N(A^T)}$ under dot product in $\mathbb{R}^m_{col}$

$\Rightarrow C(A) \cap N(A^T) = 0.$

Note: $\not\Leftarrow$

$Ax \in C(A)$

$A^TAx = 0 \Leftrightarrow A^T(Ax) = 0$

$\Leftrightarrow Ax \in \underbrace{N(A^T) \cap C(A)}_{0}$

$\Leftrightarrow Ax = 0$

$\Leftrightarrow x = 0.$ □

$x \in \ker(\mu_{A^TA}) \Leftrightarrow \mu_A(x) \in \ker(\mu_{A^T})$

$\Leftrightarrow \mu_A(x) = 0$

since $\mathrm{im}(\mu_A) \cap \ker(\mu_{A^T}) = 0$

$\Leftrightarrow x = 0.$ □

$$\begin{tikzcd} \mathbb{R}^n & \mathbb{R}^m \arrow[l, "\mu_{A^T}"'] & \mathbb{R}^n \arrow[l, "\mu_A"'] \arrow[ll, bend left, "\mu_{A^TA}"] \end{tikzcd}$$

calculate nullspace (kernel)

□