18. Recall: for vector spaces V and W, the transformation / map / function / operator $T: V \to W$ is <u>linear</u>

if $T(u+cv) = Tu + cTv \quad \forall u,v \in V$ and scalars c.

$$\Updownarrow$$

$$T\left(\sum_{i=1}^{k} c_i v_i\right) = \sum_{i=1}^{k} c_i T v_i$$

<u>E.g.</u> (i) $D: C^1(I) \to C^0(I)$ $\qquad D(f+cg) = (f+cg)' = f' + cg' = Df + cDg$

$$f \mapsto f'$$

$D: \mathcal{P}_k \to \mathcal{P}_{k-1}$ or $D: \mathcal{P}_k \to \mathcal{P}_k$

	$D: \mathcal{P}_k \to \mathcal{P}_{k-1}$	$D: \mathcal{P}_k \to \mathcal{P}_k$
rank?	k	k
image?	$\mathcal{P}_{k-1}$	$\mathcal{P}_{k-1}$
kernel?	{constants}	{constants}
injective?	no	no
surjective?	yes	no

nullity? 1: rank + nullity $= k+1$ in both cases

Why do these examples? Demonstrate that abstract vector space constructions have concrete interpretations in concrete (!) vector spaces.

(ii) $C^0(I) \xrightarrow{M_t} C^0(I)$ "multiplication by t"

$$f(t) \mapsto tf(t)$$

injective? yes: $tf(t) = 0 \Rightarrow f \equiv 0$ on I

surjective? need g/t continuous $\forall g \in C^0(I) \Rightarrow \begin{cases} \text{yes if } 0 \notin I \\ \text{no if } 0 \in I \end{cases}$

(iii) $C^0([0,1]) \to C^0([0,1])$

$$f(t) \mapsto \int_0^t f(s)\,ds$$

Fundamental Thm of Calculus

i.e. if $F' = f$ then $f \mapsto F - F(0)$

injective? Can $\int$(nonzero function) be the zero function? No: $f \not\equiv 0 \Rightarrow F \not\equiv 0$, so yes injective.

surjective? No image is $C^1 \subsetneq C^0$ $\qquad$ Pf: $F' = f$!

(iv) $ev_{0,1,3}: C^0([0,4]) \to \mathbb{R}^3$ "evaluation"

$$f \mapsto \begin{bmatrix} f(0) \\ f(1) \\ f(3) \end{bmatrix} \qquad f + cg \mapsto \begin{bmatrix} f(0) \\ f(1) \\ f(3) \end{bmatrix} + c \begin{bmatrix} g(0) \\ g(1) \\ g(3) \end{bmatrix}$$

Q. Is $ev_{0,1,3}: \mathcal{P}_4 \to \mathbb{R}^3$ injective? surjective? Find kernel and image.

A. dim: 5 $\quad$ 3 $\Rightarrow$ rank ≤ 3. rank-nullity thm $\Rightarrow$ dim ker ≥ 2.

$\Rightarrow$ not injective.

Thm 3.6.4 $\Rightarrow$ surjective! image $= \mathbb{R}^3$ $\Rightarrow$ rank $= 3$

$\Rightarrow$ dim ker $= 2$.

0, 1, 3 roots $\Rightarrow$ $t(t-1)(t-3)$, $t^2(t-1)(t-3) \in$ ker

independent because different degrees $\Rightarrow$ basis for ker.

(v) $\mathbb{R}^{2\times 2} \xrightarrow{\mu_{\left[\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right]}} \mathbb{R}^{2\times 2}$ any $B \in \mathbb{R}^{2\times 2}$ would do

$A \longmapsto \begin{bmatrix}1&2\\1&1\end{bmatrix} A$ $\qquad B(A+cA') = BA + cBA' \Rightarrow \mu_B$ is linear

<u>Def</u>: $T: V \to W$ is an <u>isomorphism</u> if T is bijective. injective and surjective

<u>Lemma 4.1</u>: T is an isomorphism $\Leftrightarrow \exists\, T^{-1}: W \to V$ with

$$T \circ T^{-1} = id_W : w \mapsto w \quad \forall\, w \in W$$
$$T^{-1} \circ T = id_V : v \mapsto v \quad \forall\, v \in V$$

<u>Pf</u>: Exercise. ($\Rightarrow$: HW, #13) content: bijective $\Leftrightarrow$ has inverse as map of sets; need linearity.

<u>Prop</u>: $T: V \to W$ isomorphism $\Leftrightarrow$ T(basis for V) = basis for W.

<u>Pf</u>: Exercise. (not assigned)

<u>Cor</u>: $\dim V = n \Leftrightarrow V \cong \mathbb{R}^n$.

<u>Def</u>: Let $T: V \to W$ be linear. If

$\mathcal{V} = (v_1, \dots, v_n)$ is an ordered basis of V

$\mathcal{W} = (w_1, \dots, w_m)$ is an ordered basis of W

then $A = [T]_{\mathcal{V},\mathcal{W}}$ is the <u>matrix of T</u> <u>with respect to $\mathcal{V}$ and $\mathcal{W}$</u> if the j^{th} column of A lists the coefficients on $w_1, \dots, w_m$ in Tv_j.

$$Tv_1 = a_{11} w_1 + \cdots + a_{m1} w_m = [w_1 \cdots w_m] \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$$

1×1 symbol

$$\vdots$$

$$Tv_n = a_{1n} w_1 + \cdots + a_{mn} w_m = [w_1 \cdots w_m] \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

a list of symbols

$$[Tv_1 \cdots Tv_n] = [w_1 \cdots w_m] \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$$

Key point: Tv_i is a linear combination of the w's; what are the coefficients? Listed in a_i.

<u>E.g.</u> $D: \mathcal{P}_3 \to \mathcal{P}_2$ $\qquad \mathcal{V} = (1, t, t^2, t^3)$ and $\mathcal{W} = (1, t, t^2) \Rightarrow [D]_{\mathcal{V},\mathcal{W}} = ?$

$$||$$

$$[D1\ Dt\ Dt^2\ Dt^3] = [0\ \ 1\ \ 2t\ \ 3t^2] = [1\ \ t\ \ t^2] \begin{bmatrix} 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{bmatrix}$$