20.

$$T[v_1 \cdots v_n] = [w_1 \cdots w_m]\,[T]_{\mathcal{V},\mathcal{W}}$$

$$\overset{\mathcal{V}'}{[v_1' \cdots v_n']} = \overset{\mathcal{V}}{[v_1 \cdots v_n]}\,P \implies [v_1' \cdots v_n']x = [v_1 \cdots v_n]Px$$

$$\overset{\mathcal{W}}{[w_1' \cdots w_m']} = \overset{\mathcal{W}'}{[w_1 \cdots w_m]}\,Q$$

$$\begin{array}{l} A = [T]_{\mathcal{V},\mathcal{W}} \\ A' = [T]_{\mathcal{V}',\mathcal{W}'} \end{array} \implies A' = Q^{-1}AP$$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ lists coefficients of $v \in V$ on basis $v_1', \ldots, v_n'$

$\Rightarrow Px$ lists coefficients of $v \in V$ on basis $v_1, \ldots, v_n$

Def: For a basis $\mathcal{B}$ of V and $T: V \to V$ linear, set $[T]_{\mathcal{B}} = [T]_{\mathcal{B},\mathcal{B}}$. Reiterate (∗)

E.g. $V = \mathbb{R}^n \Rightarrow [\mu_A]_{\mathcal{E}_n} = A$.

Cor: Bases $\mathcal{B}, \mathcal{B}'$ for V with $\overset{\mathcal{B}'}{[v_1' \cdots v_n']} = \overset{\mathcal{B}}{[v_1 \cdots v_n]}P \Rightarrow [T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P$. conjugate of $[T]_{\mathcal{B}}$ by P

In particular, $P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \Rightarrow [T]_{\mathcal{B}} = P^{-1}[T]_{\mathcal{E}_n}P$.

$[T]_{\mathcal{B}'}$ and $[T]_{\mathcal{B}}$ are similar

Pf: $V = W$, $Q = P$. □

E.g. Fix orthonormal v_1, v_2, v_3 in $\mathbb{R}^3$. Describe the linear map $L: \mathbb{R}^3 \to \mathbb{R}^3$ that

1. sends v_1, v_2, v_3 to e_1, e_2, e_3
2. rotates by $\pi/3$ around z-axis
3. sends e_1, e_2, e_3 to v_1, v_2, v_3.

Answer: $[L]_{\mathcal{E}} = \underset{3}{P}\,\underset{2}{R}\,\underset{1}{P^{-1}} \Rightarrow P^{-1}[L]_{\mathcal{E}}P = R \Rightarrow \underbrace{R = [L]_{\mathcal{B}} \Rightarrow L \text{ rotates around } v_3 \text{ by } \pi/3!}$

where $P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$ and $R = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$v_1 \mapsto -\frac{1}{2}v_1 + \frac{\sqrt{3}}{2}v_2 + 0\,v_3 = [v_1\ v_2\ v_3]\begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$

$v_2 \mapsto -\frac{\sqrt{3}}{2}v_1 + -\frac{1}{2}v_2 + 0\,v_3 = [v_1\ v_2\ v_3]\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$

$v_3 \mapsto 0\,v_1 + 0\,v_2 + 1\,v_3 = [v_1\ v_2\ v_3]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Crucial note $P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$ has two tellingly different interpretations:

multiplication by P

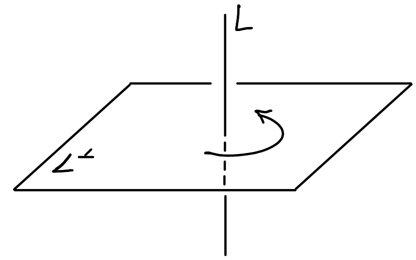
- is a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ that takes e_1, e_2, e_3 to v_1, v_2, v_3
- takes the coefficients $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of a fixed $v \in \mathbb{R}^3$ on v_1, v_2, v_3 to the coefficients Px of the same $v \in \mathbb{R}^3$ on e_1, e_2, e_3

$[v_1\ v_2\ v_3]x = [e_1\ e_2\ e_3]\underbrace{Px}_{=v!} \Leftarrow$

□

<u>Def</u>: Fix subspace $L \subseteq \mathbb{R}^n$ with $\dim L = n-2$. The <u>rotation</u> by angle α around L is the linear map rot_α^L determined by

(∗)
- $\mathrm{rot}_\alpha^L(L) = L$
- $\mathrm{rot}_\alpha^L(L^\perp)$ is usual rotation of $\mathbb{R}^2$ by α.



Q. Why set $\dim L = n-2$?

A. $\dim L^\perp = \cancel{?}\ 2$

Q. $L \cap L^\perp = \cancel{?}\ 0$ What kind of "0" is this?

Q. How is rot_α^L "determined by" (∗)?

A. Choose
- basis $v_3, \dots, v_n$ for L
- orthonormal basis x, y for $L^\perp$.

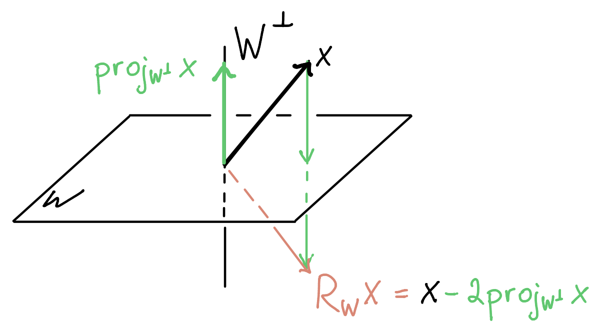
rotation of x into y by α

$$\mathcal{B} = x, y, v_3, \dots, v_n \Rightarrow [\mathrm{rot}_\alpha^L]_{\mathcal{B}} = \left[\begin{array}{c|c} \begin{matrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{matrix} & 0 \\ \hline 0 & \begin{matrix}1 & & 0 \\ & \ddots & \\ 0 & & 1\end{matrix} \end{array}\right]$$

no mixing of L with $L^\perp$

$v_3 \mapsto v_3, \dots, v_n \mapsto v_n$

<u>Def</u>: Fix subspace $W \subseteq V$ (inner product space). The <u>reflection</u> across W is $R_W = \mathrm{id}_V - 2\,\mathrm{proj}_{W^\perp}$



<u>E.g.</u> $V = \mathbb{R}^3$, $w_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $w_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix}$,

$W = \mathrm{span}(w_1, w_2)$. Find $[R_W]_{\mathcal{E}}$.

Need $\mathrm{proj}_{W^\perp}$. $\dim W = 2$ (proof?) $\Rightarrow \dim W^\perp = 1$. $v_3 = \begin{bmatrix}-1\\2\\1\end{bmatrix} \in W^\perp$

$\Rightarrow W^\perp = \mathrm{span}(v_3)$.

$2\,\mathrm{proj}_{W^\perp}$ has matrix $2\dfrac{v_3 v_3^T}{v_3^T v_3} = 2\dfrac{1}{6}\begin{bmatrix}-1\\2\\1\end{bmatrix}\begin{bmatrix}-1 & 2 & 1\end{bmatrix}$

$$= 2\frac{1}{6}\begin{bmatrix}1 & -2 & -1\\ -2 & 4 & 2\\ -1 & 2 & 1\end{bmatrix}$$

$$\Rightarrow \mathrm{id}_{\mathbb{R}^3} - 2\,\mathrm{proj}_{W^\perp} \text{ has matrix } \frac{1}{3}\begin{bmatrix}3 & & \\ & 3 & \\ & & 3\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1 & -2 & -1\\ -2 & 4 & 2\\ -1 & 2 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 2 & 1\\ 2 & -1 & -2\\ 1 & -2 & 2\end{bmatrix} = [R_W]_{\mathcal{E}}.$$

E.g. $V=\mathbb{R}^3$, $\mathcal{B}=(w_1, w_2, v_3)$. Find $[R_W]_\mathcal{B}$

Go back to def: $R_W[w_1\ w_2\ v_3] = [w_1\ w_2\ v_3][R_W]_\mathcal{B}$ W is fixed

But $[R_W w_1\ R_W w_2\ R_W v_3] = [w_1\ \ w_2\ \ -v_3]$ $W^\perp \to -W^\perp$

$$= [w_1\ \ w_2\ \ v_3]\underbrace{\begin{bmatrix}1 & & \\ & 1 & \\ & & -1\end{bmatrix}}_{[R_W]_\mathcal{B}.}$$

E.g. Use $[R_W]_\mathcal{B}$ to compute $[R_W]_\mathcal{E}$.

$$P = \begin{bmatrix} | & | & | \\ w_1 & w_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow [R_W]_\mathcal{B} = P^{-1}[R_W]_\mathcal{E}P$$

$$\Rightarrow P[R_W]_\mathcal{B}P^{-1} = [R_W]_\mathcal{E}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & -1\end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}^{-1}$$

columns of P are orthogonal!

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & -1\end{bmatrix}\begin{bmatrix}1/2 & & \\ & 1/3 & \\ & & 1/6\end{bmatrix}\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}}_{P^T}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & -1/3 \\ -1/6 & 1/3 & 1/6 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ -2/3 & -2/3 & 2/3 \end{bmatrix} \quad \checkmark$$

Summary:

abstract $\quad T[v_1 \cdots v_n] = [v_1 \cdots v_n][T]_\mathcal{B}$ (def) entries v_j are 1×1 symbols

$[T]_\mathcal{E}P = P[T]_\mathcal{B}$ columns $\begin{bmatrix} | \\ v_j \\ | \end{bmatrix}$ are $n\times 1$

in coordinates $\quad [T]_\mathcal{E}\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}[T]_\mathcal{B}$

$$\Leftrightarrow [T]_\mathcal{B} = P^{-1}[T]_\mathcal{E}P$$

$$\Leftrightarrow P[T]_\mathcal{B}P^{-1} = [T]_\mathcal{E}$$