21.

Chapter 5: Determinants

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

?

can be any $\mathbb{F}$

Thm: $\exists!$ function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying

alternating 1. $\det A = 0$ if A has two equal adjacent rows

$$\det\begin{bmatrix} \sim\sim \\ - R - \\ - R - \\ \sim\sim \end{bmatrix} = 0$$

multilinear $\left\{ \begin{array}{l} \text{2. } \det A' = c \det A \text{ if } A' \text{ is obtained by multiplying a row of } A \text{ by } c. \\ \text{3. } \det A = \det A' + \det A'' \text{ if } A, A', A'' \text{ agree in all} \\ \quad \text{rows except row } i, \text{ where } A_i = A'_i + A''_i. \end{array}\right.$

$$\det\begin{bmatrix} \sim\sim \\ - cR - \\ \sim\sim \end{bmatrix} = c \det\begin{bmatrix} \sim\sim \\ - R - \\ \sim\sim \end{bmatrix}$$

4. $\det I_n = 1$.

We've seen bilinear: $\langle \cdot , \cdot \rangle$ linear in each variable

$$\underset{A}{\det\begin{bmatrix} \sim\sim \\ \text{—} + \text{—} \\ \sim\sim \end{bmatrix}} = \underset{A'}{\det\begin{bmatrix} \sim\sim \\ \text{———} \\ \sim\sim \end{bmatrix}} + \underset{A''}{\det\begin{bmatrix} \sim\sim \\ \text{———} \\ \sim\sim \end{bmatrix}}$$

Def: $\det A$ is the <u>determinant</u> of A.

Pf: $\exists$ uses cofactors — next class.

$!$: assume $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is some function satisfying #1 – #4.

Lemma: $\det A' = -\det A$ if A' has two rows swapped from A.

Pf:

$$0 \overset{\#1}{=} \det\begin{bmatrix} \vdots \\ -x+y- \\ -x+y- \\ \vdots \end{bmatrix} \overset{\#3}{=} \det\begin{bmatrix} \vdots \\ -x- \\ -x+y- \\ \vdots \end{bmatrix} + \det\begin{bmatrix} \vdots \\ -y- \\ -x+y- \\ \vdots \end{bmatrix}$$

$$\overset{\#3}{=} \cancel{\det\begin{bmatrix} \vdots \\ -x- \\ -x- \\ \vdots \end{bmatrix}}^{\#1} + \det\begin{bmatrix} \vdots \\ -x- \\ -y- \\ \vdots \end{bmatrix} + \det\begin{bmatrix} \vdots \\ -y- \\ -x- \\ \vdots \end{bmatrix} + \cancel{\det\begin{bmatrix} \vdots \\ -y- \\ -y- \\ \vdots \end{bmatrix}}^{\#1},$$

so done if swapped rows adjacent. If not adjacent, then

$$\begin{array}{c} i \\ \\ j \end{array}\begin{bmatrix} \vdots \\ -A_i- \\ \vdots \\ -A_j- \\ \vdots \end{bmatrix} \underset{j-i \text{ steps}}{\rightsquigarrow} \begin{array}{c} i \\ i+1 \end{array}\begin{bmatrix} \vdots \\ -A_i- \\ -A_j- \\ \vdots \end{bmatrix} \underset{1 \text{ step}}{\rightsquigarrow} \begin{array}{c} i \\ i+1 \end{array}\begin{bmatrix} \vdots \\ -A_j- \\ -A_i- \\ \vdots \end{bmatrix} \underset{j-i \text{ steps}}{\rightsquigarrow} \begin{array}{c} i \\ \\ j \end{array}\begin{bmatrix} \vdots \\ -A_j- \\ \vdots \\ -A_i- \\ \vdots \end{bmatrix}$$

and $(-1)^{2(j-i)+1} = -1$. □

Cor: $\det A = 0$ if any two rows are equal.

Pf: Swap 'til you drop. □

Prop 1.4: $\det(EA) = \det E \det A$ if E is elementary.

Pf: E swaps rows: $\det(EA) \underset{\text{Lemma}}{=} -\det A \underset{\#4+\text{Lemma}}{=} \det E \det A$

E multiplies row by scalar c: $\det(EA) \underset{\#2}{=} c\det A \underset{\#2+\#4}{=} \det E \det A$

E replaces A_i with $A_i + cA_j$:

$$\det\begin{bmatrix} \vdots \\ A_i + cA_j \\ \vdots \\ -A_j- \\ \vdots \end{bmatrix} \underset{\#3}{=} \det\begin{bmatrix} \vdots \\ -A_i- \\ \vdots \\ -A_j- \\ \vdots \end{bmatrix} + \det\begin{bmatrix} \vdots \\ cA_j \\ \vdots \\ -A_j- \\ \vdots \end{bmatrix}$$

$$\overset{\#2}{=} \det A + c\,\cancel{\det\begin{bmatrix} \vdots \\ -A_j- \\ \vdots \\ -A_j- \\ \vdots \end{bmatrix}}$$ 0 by Cor.

$A = I_n \Rightarrow \det E = 1 \ (!) \quad \Rightarrow \quad = \det E \det A. \ \square$

Thm 1.2: $A \in \mathbb{R}^{n\times n}$ singular $\Leftrightarrow \det A = 0$.

If one row is a linear combination of the others, expanding by multilinearity yields 0 in every summand by alternation (#1).

any function $\mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying #1 – #4

Pf: Write $U = E_k \cdots E_1 A$ reduced echelon form.

$$\begin{aligned} \det U &= \det E_k \det(E_{k-1}\cdots E_1 A) \text{ by Prop 1.4.} \\ &= \det E_k \det E_{k-1} \det(E_{k-2}\cdots E_1 A) \\ &\ \vdots \\ &= \det E_k \det E_{k-1} \det E_{k-2} \cdots \det E_1 \det A. \end{aligned}$$

A singular $\Rightarrow U$ has zero-row U_n (bottom row)

$\Rightarrow \det U = 0 \det U = 0$ by #2: $U_n = 0U_n$. But

$$\boxed{\underbrace{\det E = \begin{cases} -1 & \text{if } E \text{ has type (i)} \\ c & \text{(ii)} \\ 1 & \text{(iii)} \end{cases}}_{\neq 0}} \Rightarrow \det A = 0.$$

A nonsingular $\overset{\#4}{\Rightarrow} 1 = \det U = \det E_k \det E_{k-1} \det E_{k-2} \cdots \det E_1 \det A$

$\Rightarrow \det A \neq 0. \ \square$

To finish proof of !, $\det A = 0$ if A singular

$$\det A = \frac{\det I_n}{\det E_k \cdots \det E_1} \text{ if } A \text{ nonsingular. } \square$$

At most one function can do this. Given that one exists, there only be one.

E.g.

$$\det\begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \overset{(i)}{=} -\det\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 4 & 6 \end{bmatrix} \overset{(iii)}{=} -\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 4 & 4 \end{bmatrix} \overset{(iii)}{=} -\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 12 \end{bmatrix} \overset{(ii)}{=} -12\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{(iii)}{=} -12 \det I_3 = -12.$$

Consequences (of $\exists!$, etc.)

Thm 1.5: $A, B \in \mathbb{R}^{n\times n} \Rightarrow \det(AB) = \det A \det B$.

Pf: A singular $\Rightarrow \det A = 0 \Rightarrow \det A \det B = 0$.

$\Downarrow$

$L(A) \neq 0 \Rightarrow L(AB) \neq 0$ since $L(AB) \supseteq L(A)$: $\boxed{}\,\boxed{A} = 0 \Rightarrow \boxed{}\,\boxed{A}\,\boxed{B} = 0$

$\Rightarrow AB$ singular $\Rightarrow \det(AB) = 0$. ✓

A nonsingular $\Rightarrow \det(AB) = \det(E_1' \cdots E_k' B)$

$= \det E_1' \cdots \det E_k' \det B$

$= \det A \det B$. □

Cor 1.6: $\det(A^{-1}) = \dfrac{1}{\det A}$.

Pf: $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det I = 1$. □

Cor: A similar to $A' \Rightarrow \det A = \det A'$.

Pf: $\det(PAP^{-1}) = \cancel{\det P} \det A\, \cancel{\det P^{-1}}$

$= \det A$. □

Note: not $\Leftrightarrow$! e.g. $A \sim I \Rightarrow A = ?$

but $\det E = 1$ for type (iii)

Prop 1.7: $\det A^T = \det A$.

Pf: Check for elementary matrices when A is nonsingular: E has same type as E^T.

Both $= 0$ if A is singular. □

Prop 1.3: A upper-triangular (or lower-) $\Rightarrow \det A = a_{11} \cdots a_{nn}$ = product of main diagonal entries.

$$\det \begin{bmatrix} * & & & \\ & * & & \\ & & \ddots & \\ \text{O} & & & * \end{bmatrix} = ** \cdots **$$

Pf: A nonsingular $\Rightarrow A \rightsquigarrow I_n$ by pulling out factors $a_{11}, \dots, a_{nn}$ and then type (iii) operations, which have det 1.

A singular $\Leftrightarrow$ $< n$ pivots $\Leftrightarrow a_{ii} = 0$ for some i. □