22\. Today: $\exists$ det satisfying #1–#4

<u>Def</u>: $\det : \mathbb{R}^{1\times 1} \to \mathbb{R}$

$$[a] \mapsto a$$

A_{11}: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & [a_{22}] \end{bmatrix}$ $\qquad$ A_{21}: $\begin{bmatrix} a_{11} & [a_{12}] \\ a_{21} & a_{22} \end{bmatrix}$

$\det : \mathbb{R}^{2\times 2} \to \mathbb{R}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto a_{11}\det A_{11} - a_{21}\det A_{21}$$
$$= a_{11}\det[a_{22}] - a_{21}\det[a_{12}]$$
$$= a_{11}a_{22} - a_{21}a_{12}$$

$\vdots$

$\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ $\qquad$ recursive definition: def for n in terms of def for n-1

$$A \mapsto a_{11}\det A_{11} - a_{21}\det A_{21} + \cdots + (-1)^{n+1}a_{n1}\det A_{n1}$$

<u>E.g.</u> $\det\begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix} = 2\det\begin{bmatrix} -2 & 3 \\ 2 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

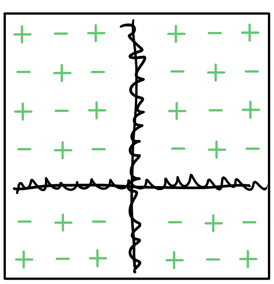
$$= 2(-8) - 2(-5) + 0$$
$$= -11.$$

1\. Don't use to compute $\geq 3\times 3$

2\. Use for $\det(\nabla) = \prod \text{diag}$

<u>Def</u>: For $A \in \mathbb{R}^{n\times n}$ with $n \geq 2$, get $(n-1)\times(n-1)$ matrix A_{ij} by deleting row i and column j.

The ij^{th} <u>cofactor</u> of A is $C_{ij} = (-1)^{i+j}\det A_{ij}$.



i

j

<u>E.g.</u> $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow$ $\quad A_{13} = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}$ $\quad A_{22} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ $\quad A_{32} = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$

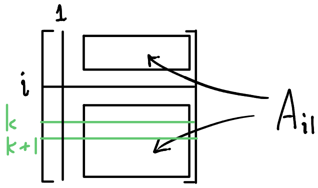
$C_{13} = +2 \qquad C_{22} = +2 \qquad C_{32} = -(2\cdot 3 - 3\cdot 1) = -3$

<u>Thm</u>: $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies #1–#4.

<u>Pf</u>: $n=2$: do it yourself.

Assume $n \geq 3$ and prove by induction on n.

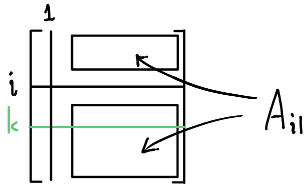
$$\left[\begin{array}{c|c} a_{11} & \\ a_{21} & \\ \vdots & A_{11} \\ a_{n1} & \end{array}\right]$$



#1: rows k and $k+1$ equal $\overset{\text{induction}}{\Rightarrow}$ $\det A_{i1} = 0$ for $i \neq k, k+1$, and

$a_{k1} \det A_{k1} = a_{(k+1)1} A_{(k+1)1}$ but $-(-1)^{k+1} = (-1)^{k+1+1}$

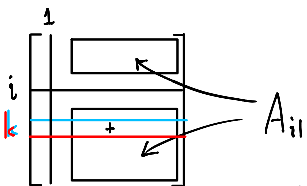
so these terms cancel.



#2: $A \overset{A_k \rightsquigarrow cA_k}{\rightsquigarrow} A'$ $\overset{\text{induction}}{\Rightarrow}$ $\det A'_{i1} = c \det A_{i1}$ but $a'_{i1} = a_{i1}$ if $i \neq k$

$\det A'_{k1} = \det A_{k1}$ but $a'_{i1} = c a_{i1}$, so

$a'_{i1} \det A'_{i1} = c\, a_{i1} \det A_{i1}$ either way.



#3: A, A', A'' agree in all rows $\neq k$, where $A_k = A'_k + A''_k$

$i \neq k$: $\det A_{i1} = \det A'_{i1} + \det A''_{i1}$
$\quad a_{i1} = a'_{i1} = a''_{i1}$

$$\Rightarrow \quad \begin{aligned} a_{i1} \det A_{i1} &= a_{i1}(\det A'_{i1} + \det A''_{i1}) \\ &= a_{i1} \det A'_{i1} + a_{i1} \det A''_{i1} \\ &= a'_{i1} \det A'_{i1} + a''_{i1} \det A''_{i1} \end{aligned}$$

$i = k$: $\det A_{k1} = \det A'_{k1} = \det A''_{k1}$
$\quad a_{k1} = a'_{k1} + a''_{k1}$

$$\Rightarrow \quad \begin{aligned} a_{k1} \det A_{k1} &= (a'_{k1} + a''_{k1}) \det A_{k1} \\ &= a'_{k1} \det A_{k1} + a''_{k1} \det A_{k1} \\ &= a'_{k1} \det A'_{k1} + a''_{k1} \det A''_{k1} \end{aligned}$$

$\Rightarrow \det A = \det A' + \det A''$ term by term.

#4: $\left[\begin{array}{c|c} a_{11} & \\ a_{21} & \\ \vdots & A_{11} \\ a_{n1} & \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ 0 & \\ \vdots & I_{n-1} \\ 0 & \end{array}\right]$ $\Rightarrow$

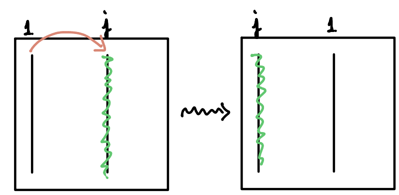
$$\begin{aligned} \det I_n &= 1 \det I_{n-1} - 0 + 0 - 0 + \cdots \pm 0 \\ &= 1(1) \quad \text{by induction} \\ &= 1. \ \square \end{aligned}$$

$\Rightarrow \exists!\ \det$

<u>Prop 2.2</u>: $\det A = a_{1j} C_{1j} + \cdots + a_{nj} C_{nj}$ for any fixed j. expand along any column

<u>Pf</u>: Swap columns 1 and j of A to get A'. Then $\det A' = -\det A$.

But $a'_{i1} = a_{ij}$ $\forall i$, and $A_{ij} \rightsquigarrow A'_{i1}$ by moving leftmost column across $j-2$ columns to column $j-1$. Hence



$$\begin{aligned} C'_{i1} &= (-1)^{i+1} \det A'_{i1} = (-1)^{i+1}(-1)^{j-2} \det A_{ij} \\ &= -(-1)^{i+j} \det A_{ij} = -C_{ij} \quad \forall i, \text{ so } a'_{i1} C'_{i1} = -a_{ij} C_{ij}. \ \square \end{aligned}$$

<u>Cor 2.1</u>: $\det A = a_{i1} C_{i1} + \cdots + a_{in} C_{in}$ for any fixed i. expand along any row

The rules for swapping columns are the same as those for rows.

<u>Pf</u>: $\det A = \det A^T$ + Prop 2.2. $\det A' = \det (A')^T = -\det A^T = -\det A$. $\square$

§5.3 Geometric interpretation

$\det(\;) = \text{area}(\;)$ 2D area = 2D volume

in $\mathbb{R}^n$: $\det(\;) =$ volume of parallelepiped. Why?

The axioms we use to define determinants are the same as those we use to define volume.

1. $v_1, \ldots, v_n$ dependent $\Rightarrow \dim(\text{span}) < n \Rightarrow$ flat $\Rightarrow$ vol $= 0$
2. scale edge by $c \Rightarrow$ vol $\rightsquigarrow c$ vol
3. $\text{vol}(\;) = \text{vol}(\;)$ Cavalieri's Principle

$\Big\}$ volume is multilinear!

4. vol(hypercube) $= 1$.

Thm 2.3 (Cramer's rule): $Ax = b$ with A nonsingular $\Rightarrow x_i = \dfrac{\det B_i}{\det A}$, where $A \underset{a_i \rightsquigarrow b}{\rightsquigarrow} B_i$.

Pf: $b = Ax = x_1 a_1 + \cdots + x_n a_n \Rightarrow \det B_i = \det \begin{bmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & x_1 a_1 + \cdots + x_n a_n & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{bmatrix}$

$$= \det \begin{bmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & x_i a_i & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{bmatrix} = x_i \det A. \;\square$$

E.g. $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Rightarrow B_1 = \begin{bmatrix} 3 & 3 \\ -1 & 7 \end{bmatrix}$ $\quad B_2 = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$

$$x_1 = \frac{21+3}{14-12} = 12 \qquad x_2 = \frac{-2-12}{14-12} = -7 \quad \Rightarrow \quad x = \begin{bmatrix} 12 \\ -7 \end{bmatrix}.$$

Magic!

Thm 2.3: $C = [C_{ij}] =$ cofactor matrix of $A \Rightarrow AC^T = (\det A) I_n$.

(i.e. A nonsingular $\Rightarrow A^{-1} = \dfrac{1}{\det A} C^T$)

Pf: The diagonal entries of AC^T are precisely the sums in Cor 2.1.

Define a matrix D_{ij} by copying row i of into row j, so $A \underset{A_j \rightsquigarrow A_i}{\rightsquigarrow} D_{ij}$.

The ij entry of AC^T is $\det D_{ij}$ as expanded along row j.

But $\det D_{ij} = \det A$ if $i = j$ $\quad (D_{ij} = A)$

$\qquad\qquad\qquad\; 0$ if $i \neq j$ $\quad$ (D_{ij} has A_i repeated). $\square$