24. $\boxed{\text{Course evaluations}}$

§6.2 Recall: λ eigenvalue of $T: V \to V$ or $n \times n$ $A = [T]_{\mathcal{B}}$

$\Leftrightarrow$ $Tv = \lambda v$ for some $v \neq 0$ eigenvector

$\Leftrightarrow$ λ root of characteristic polynomial $\det(A - tI) = p_T(t)$

Today: how many linearly independent eigenvectors with eigenvalue λ can A have?

Def: geometric multiplicity $g(\lambda) = \dim E(\lambda) = \dim \ker(T - \lambda I)$

Compare: algebraic multiplicity $a(\lambda) = \#$ times $t - \lambda$ divides $p_T(t)$

$p_T(\lambda) = 0 \Rightarrow p_T(t) = (t-\lambda)q(t)$, $\quad q(\lambda) = 0 \Rightarrow q(t) = (t-\lambda)r(t)$, ... until λ not a root

E.g. $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow p_A(t) = \det\begin{bmatrix} \lambda - t & 1 \\ 0 & \lambda - t \end{bmatrix} = (\lambda - t)^2 \Rightarrow a(\lambda) = 2$

$$E(\lambda) = N(A - \lambda I) = N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \text{ has dim } 1 = g(\lambda)$$

Theorem 2.1 (strengthened): Suppose B is a set of eigenvectors for a linear map $T: V \to V$. If $B \cap E(\lambda)$ is linearly independent for every (eigenvalue) λ, then B is linearly independent.

Pf: Let $c_1v_1 + \cdots + c_kv_k = 0$ with $v_1, \ldots, v_k \in B$. Need $c_i = 0 \ \forall i$.

$k = 1 \Rightarrow c_1 = 0$ since $v_1 \neq 0$. Assume, by induction on k, that all size $k-1$ subsets of B are independent.

Suppose $v_i \in E(\lambda_i) \ \forall i$. Then $0 = (T - \lambda_1 I)(c_1v_1 + \cdots + c_kv_k)$

$$= c_1(Tv_1 - \lambda_1 v_1) + \cdots + c_k(Tv_k - \lambda_1 v_k)$$
$$= c_1(\lambda_1 v_1 - \lambda_1 v_1) + \cdots + c_k(\lambda_k v_k - \lambda_1 v_k)$$
$$= c_1(\underbrace{\cancel{\lambda_1 - \lambda_1}}_{0})v_1 + \cdots + c_k(\lambda_k - \lambda_1)v_k$$
$$= c_2(\lambda_2 - \lambda_1)v_2 + \cdots + c_k(\lambda_k - \lambda_1)v_k$$

$\Rightarrow c_i(\lambda_i - \lambda_1) = 0 \ \forall i \geq 2$ by induction

$\Rightarrow c_i = 0 \ \forall i$ such that $\lambda_i \neq \lambda_1$

$\Rightarrow c_i = 0 \ \forall i$ by linear independence of $B \cap E(\lambda_1)$. $\square$

Corollary 2.2: $\dim V = n$ and $T: V \to V$ has n distinct eigenvalues in $\mathbb{F} \Rightarrow T$ is diagonalizable over $\mathbb{F}$.

Pf: $E(\lambda) \neq 0$ if $p_T(\lambda) = 0$, so T has n eigenvectors with distinct eigenvalues. These are independent by Thm 2.1 since $\#(B \cap E(\lambda)) = 1 \ \forall$ eigenvalues λ and hence form a basis because $\dim V = n$. $\square$

Q. Can hypothesis of Cor. 2.2. fail?

A. Yes 1. repeated roots, e.g. $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

2. complex roots, e.g. $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = [rot_{\pi/4}]_{\mathcal{E}}$ $p_A(t) = (t - 1/\sqrt{2})^2 + 1/2$

$= t^2 - t\sqrt{2} + 1$ has distinct roots

$$\frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2} \pm i\sqrt{2}}{2} = 1/\sqrt{2} \pm i/\sqrt{2} \notin \mathbb{R}.$$

E.g.

$$A = \begin{bmatrix} 2 & 0 & & \\ 0 & 2 & & \\ & & 3 & 1 \\ & & 0 & 3 \end{bmatrix} \quad \text{vs.} \quad B = \begin{bmatrix} 2 & 1 & & \\ 0 & 2 & & \\ & & 3 & 0 \\ & & 0 & 3 \end{bmatrix}$$

$$p_A(t) = (2-t)^2(3-t)^2 = p_B(t) \Rightarrow a(2) = 2$$

$g(2) =$ 2 1 $a(3) = 2$

$g(3) =$ 1 2

Prop. 2.3: $1 \le g(\lambda) \le a(\lambda)$.

Pf: Pick a basis $v_1, \dots, v_g$ of $E(\lambda)$, so $g = g(\lambda)$. Extend to a basis $\mathcal{B} = v_1, \dots, v_n$ of V.

Then
$$[T]_{\mathcal{B}} = \begin{array}{c c} \left[\begin{array}{c|c} \lambda I_g & B \\ \hline O & C \end{array}\right] & \begin{array}{c} g \\ n-g \end{array} \\ \begin{array}{cc} g & n-g \end{array} & \end{array}$$
so $p_A(t) = p_{\lambda I_g}(t)\, p_C(t)$

$= (\lambda - t)^g p_C(t) \Rightarrow g \le a(\lambda)$. □

Thm 2.4: $T: V \to V$ diagonalizable/$\mathbb{F}$ $\Leftrightarrow$ all eigenvalues lie in $\mathbb{F}$ and $g(\lambda) = a(\lambda)\ \forall\ \lambda$.

Pf: T diagonalizable $\Rightarrow [T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = A$, where $\mathcal{B}$ is a basis of eigenvectors

each $\lambda_i \leftrightarrow$ basis vector v_i Prop 2.3

$v_1, \dots, v_n$ with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{F}$. $a(\lambda) = \#$times λ appears in $A \le g(\lambda) \le a(\lambda)$

$\Rightarrow g(\lambda) = a(\lambda)$. On the other hand, assume $a(\lambda) = g(\lambda)\ \forall \lambda$ and that all eigenvalues $\in \mathbb{F}$.

Fix basis $\mathcal{B}_\lambda$ for $E(\lambda)$. The union $\mathcal{B} = \bigcup_\lambda \mathcal{B}_\lambda$ of these bases is linearly independent by Thm. 2.1, so $\mathcal{B}$ is a basis because $\sum_\lambda a(\lambda) = n$ by "all eigenvalues $\in \mathbb{F}$". □

E.g.
$$A = \begin{bmatrix} -1 & 4 & 2 \\ -1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix} \quad \text{vs.} \quad B = \begin{bmatrix} 0 & 3 & 1 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
have $p_A(t) = p_B(t) = (1-t)^2(2-t)$

$a(1) = 2$ $a(2) = 1$. $g(1) = ?$

$$A - I = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \qquad B - I = \begin{bmatrix} -1 & 3 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

rows not multiples of one row

can't $= 3$ since $\det(B - I) = 0$

rank $= 1 \Rightarrow g(1) = 2$ rank $> 1 \Rightarrow$ rank $= 2 \Rightarrow g(1) = 1$.

§6.4 What kinds of real matrices are symmetric? Projections!

$P_V = QQ^T$ if cols $q_1,\dots,q_n$ of Q are $\perp$normal basis for $V \subseteq \mathbb{R}^m$.

$(QQ^T)^T = (Q^T)^T Q^T = QQ^T = q_1q_1^T + \dots + q_nq_n^T = \mathrm{proj}_{q_1} + \dots + \mathrm{proj}_{q_n}$

also symmetric: $\lambda_1 q_1q_1^T + \dots + \lambda_n q_nq_n^T$ any linear combination of projections

$$Q\Lambda Q^T \qquad \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}$$

Def: The spectrum of a matrix is its (multi)set of eigenvalues. (atoms)

Thm 4.1 (spectral theorem): All real symmetric matrices arise this way:

$A \in \mathbb{R}^{n\times n}$ symmetric $\Rightarrow$ 1. A has real eigenvalues ($\Rightarrow$ diagonalizable/$\mathbb{R}$)

$Q^{-1} = Q^T$ 2. $\mathbb{R}^n$ has orthonormal basis $q_1,\dots,q_n$ of eigenvectors of A $\Rightarrow$

$1+2 \Leftrightarrow \exists$ orthogonal matrix Q with $\underbrace{Q^{-1}AQ = \Lambda \text{ diagonal}}_{A = Q\Lambda Q^T}$

Lemma: H symmetric $\Rightarrow Hy\cdot x = y\cdot Hx \ \forall x,y$.

Pf: $Hy\cdot x = (x^T Hy)^T = y^T H^T x = H^T x\cdot y$. But $H = H^T$. $\square$

complex conjugate

Pf of Thm: 1. Suppose $\lambda = a+bi$ eigenvalue of A, so $\bar\lambda = a-bi$ is, too.

$\mathbb{C}^{n\times n}$ finally! Set $S = (A-\lambda I)(A-\bar\lambda I) = A^2 - (\lambda+\bar\lambda)A + \lambda\bar\lambda I = A^2 - 2aA + (a^2+b^2)I \in \mathbb{R}^{n\times n}$ after all.

$\det S = 0$ since $A-\lambda I$ is singular. Pick $x \in N(S)\setminus 0$:

$$\begin{aligned} 0 = Sx\cdot x &= (A^2 - 2aA + a^2 I)x\cdot x + (b^2 I)x\cdot x \\ &= (A-aI)^2 x\cdot x + b^2 x\cdot x \\ &\overset{\text{Lemma}}{=} (A-aI)x\cdot(A-aI)x + bx\cdot bx \\ &= \|(A-aI)x\|^2 + \|bx\|^2 \Rightarrow (A-aI)x = 0 \text{ and } bx = 0 \end{aligned}$$

$\overset{x\neq 0}{\Rightarrow} x \in E(a)$ and $b=0$.

2. Pick unit vector $q_1 \in E(\lambda_1)$ and $v_2,\dots,v_n$ orthonormal basis for $q_1^\perp$. Set $\mathcal{B} = q_1, v_2, \dots, v_n$

who knows?

$[\mu_A]_{\mathcal{B}} = B = \begin{bmatrix} \lambda_1 & * \cdots * \\ 0 & \\ \vdots & C \\ 0 & \end{bmatrix}$ Change-of-basis formula: $B = P^{-1}AP$ where $P = \begin{bmatrix} | & | & & | \\ q_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

$= P^T A P$

$\Rightarrow B^T = P^T A^T (P^T)^T = P^T A^T P$ symmetric

$\Rightarrow * = 0 \ \forall *$ and C symmetric of size $n-1$

$\Rightarrow$ done by induction, the case $n=1$ being very easy. $\square$