25.

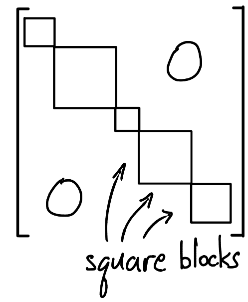
§7.1 Jordan form or "... canonical ..." or "... normal ..."

Thm (Jordan form)

Fix $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (or $\mathbb{Q}$, or $\mathbb{F}_2$, or ...)

$T: V \to V$ over $\mathbb{F}$. Assume all eigenvalues $\lambda \in \mathbb{F}$. always true if $\mathbb{F} = \mathbb{C}$!

Then V has a basis $\mathcal{B}$ in which $[T]_{\mathcal{B}} = J$ is block-diagonal



with each block being

a Jordan block of some size d for some eigenvalue λ.

$$d\left\{\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}\right. \quad (d \times d)$$

1×1: $[\lambda]$

2×2: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

The multiset of blocks — i.e., the d's and λ's — depends only on T, not on $\mathcal{B}$.

$\#$ blocks with eigenvalue λ $= g(\lambda)$

$\wedge$

$\sum$ sizes of those blocks $= \#$ diagonal entries λ $\Rightarrow g(\lambda) \le a(\lambda)$

$= a(\lambda)$

Cor: $A \in \mathbb{C}^{n\times n} \Rightarrow \exists\, P \in \mathbb{C}^{n\times n}$ with $P^{-1}AP = J$.

E.g. $A = [D]_{\mathcal{B}}$ for any basis $\mathcal{B}$ of $\mathcal{P}_k$

$D: \mathcal{P}_k \to \mathcal{P}_k$ differentiation $\Rightarrow J = ?$

1. $p_D(t) = \cancel{?}\ t^{k+1}$

Why? $Dv = \lambda v \Rightarrow D^{k+1}v = \lambda^{k+1}v$.
But $D^{k+1} = 0$! So $\lambda^{k+1}v = 0$
and hence $\lambda = 0$ if $v \neq 0$.
$\Rightarrow a(0) = k+1$

2. $\ker(D - \lambda I) = \ker(D - 0I)$
$= \ker(D)$
$= \{\text{constants}\}$
$\Rightarrow g(0) = 1$

$$\Rightarrow J = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}$$

For which basis $\mathcal{J}$ is $J = [D]_{\mathcal{J}}$?

$$0 \leftarrow 1 \leftarrow x \leftarrow x^2 \leftarrow x^3 \leftarrow \cdots \leftarrow x^k$$

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 2 & & & \\ & & 0 & 3 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & k \\ & & & & & 0 \end{bmatrix} \Rightarrow$$

$$0 \leftarrow 1 \leftarrow x \leftarrow \tfrac{x^2}{2} \leftarrow \tfrac{x^3}{3!} \leftarrow \cdots \leftarrow \tfrac{x^k}{k!}$$

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}$$

E.g. Suppose $p_A(t) = (7-t)^5$

$\dim N(A-7I) = 2$ $\quad e_1 \mapsto 0, \quad e_2 \mapsto 0$ $\qquad$ What is $J = J(A)$?

$\dim N((A-7I)^2) = 4.$ $\quad e_3 \mapsto e_1 \quad e_4 \mapsto e_2$ $\qquad \dim N((A-7I)^3)\ \cancel{?}\ 5$

Ans: $n \times n$ for $n = \cancel{?}\ 5$ $\qquad$ 2 Jordan blocks

only two choices: sizes

$$1+4: \begin{array}{ccccc} 0 & 0 & & & \\ \uparrow & \uparrow & & & \\ e_1 & e_2 \leftarrow & e_3 \leftarrow & e_4 \leftarrow & e_5 \end{array} \left[\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{array}\right] \quad \text{or} \quad 2+3: \begin{array}{ccccc} 0 & & 0 & & \\ \uparrow & & \uparrow & & \\ e_1 \leftarrow & e_2 & e_3 \leftarrow & e_4 \leftarrow & e_5 \end{array} \left[\begin{array}{cc|ccc} 0 & 1 & & & \\ & 0 & & & \\ \hline & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{array}\right] = J - 7I \Rightarrow J = \left[\begin{array}{cc|ccc} 7 & 1 & & & \\ & 7 & & & \\ \hline & & 7 & 1 & \\ & & & 7 & 1 \\ & & & & 7 \end{array}\right]$$

General meaning of a Jordan basis $\mathcal{B}$:

$v_1, \dots, v_d \leftrightarrow$ a block $\Rightarrow$

$Tv_1 = \lambda v_1 \qquad \Leftrightarrow (T-\lambda I)v_1 = 0$

$Tv_2 = \lambda v_2 + v_1 \qquad (T-\lambda I)v_2 = v_1 \neq 0$, but $(T-\lambda I)^2 v_2 = 0$

$Tv_3 = \lambda v_3 + v_2 \qquad (T-\lambda I)v_3 = v_2 \neq 0$, and $(T-\lambda I)^2 v_3 \neq 0$

$\vdots \qquad\qquad \vdots \qquad$ but $(T-\lambda I)^3 v_3 = 0$

$Tv_d = \lambda v_d + v_{d-1} \qquad (T-\lambda I)v_d = v_{d-1} \neq 0, \dots, (T-\lambda I)^{d-1} v_d \neq 0$

but $(T-\lambda I)^d v_d = 0$

$$T[v_1 \cdots v_d] = [v_1 \cdots v_d] \begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \lambda & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{bmatrix}. \quad \Leftrightarrow \dots$$

$$\dim N((A-\lambda I)^r) - \dim N((A-\lambda I)^{r-1}) = \#\text{blocks of size} \geq r.$$

Note: If T has only one Jordan block, then

v_d = almost any element of V works!

Need only $v_d \notin \ker((T-\lambda I)^{d-1})$.