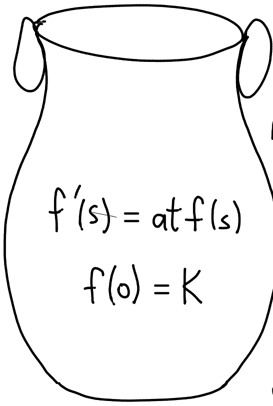
26.

§ 7.3 Systems of ODE   Ordinary Differential Equations



Riddle: what is this?

Note: the solution is a hint.

$\Rightarrow f(s) = Ke^{ats}$

Answer:

ODE on a Grecian urn!

Recall: $f = f(t)$ satisfies $f' = af \iff f = Ce^{at}$, $f(0) = C$

Q. If $x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ satisfies $x'(t) = Ax(t)$ with $A \in \mathbb{R}^{n\times n}$,

so $x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t)$

$\vdots$

$x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t)$

in what sense is "the" solution Ce^{tA} ?

Recall: $e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots + \frac{a^k}{k!} + \cdots$

Def: $A \in \mathbb{R}^{n\times n} \Rightarrow e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^k}{k!} + \cdots$ convergence issues: sequences in $\mathbb{R}^{n\times n}$

$$e^{tA} = I + tA + t^2\frac{A^2}{2!} + \cdots + t^k\frac{A^k}{k!} + \cdots$$

E.g. 1. $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Rightarrow e^{\Lambda} = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}$ and $e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{bmatrix}$

2. $P^{-1}AP = \Lambda \Rightarrow A = P\Lambda P^{-1}$

$\Rightarrow A^k = P\Lambda^k P^{-1}$

$\Rightarrow e^A = Pe^{\Lambda}P^{-1}$

$e^{tA} = Pe^{t\Lambda}P^{-1}$

3. $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 2 & \\ & -1 \end{bmatrix}}_{\Lambda}\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}}_{P^{-1}} \Rightarrow e^{t\Lambda} = \begin{bmatrix} e^{2t} & \\ & e^{-t} \end{bmatrix}$

$\Rightarrow e^{tA} = Pe^{t\Lambda}P^{-1} = \begin{bmatrix} e^{2t} & 0 \\ e^{2t} - e^{-t} & e^{-t} \end{bmatrix}$

General: $v \in \mathbb{R}^n \Rightarrow$ entries of $e^{tA}v$ are functions of t

$\cap$

$C(e^{tA})$ = analogue of Ce^{at} :   typographical pun!

vectors $e^{tA}v$ are solutions of $x'(t) = Ax(t)$.

Thm 3.3: For $A \in \mathbb{R}^{n\times n}$, solutions set of $x'(t) = Ax(t)$ is the vector space $C(e^{tA})$ of dim n.

E.g. $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$ and $x' = Ax \Rightarrow x = e^{tA}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$= v_1 \begin{bmatrix} e^{2t} \\ e^{2t} - e^{-t} \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

eigenvalues of A

$$= v_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (v_2 - v_1) e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

eigenvectors of A

Lemma: $(e^{tA})' = \left(I + tA + t^2\frac{A^2}{2!} + \cdots + t^k\frac{A^k}{k!} + \cdots\right)'$

$$= 0 + A + tA^2 + t^2\frac{A^3}{2!} + \cdots + t^k\frac{A^{k+1}}{k!} + \cdots$$

$$= A\left(I + tA + t^2\frac{A^2}{2!} + \cdots + t^k\frac{A^k}{k!} + \cdots\right)$$

$$= Ae^{tA} = e^{tA}A. \quad \square$$

Pf of Thm: $v \in \mathbb{R}^n \Rightarrow (e^{tA}v)' = (e^{tA})'v$ because $\frac{d}{dt}$ is linear

$= (Ae^{tA})v$ by Lemma

$= A(e^{tA}v)$. ✓

Need: every sol is $e^{tA}v$ for some $v \in \mathbb{R}^n$.

Assume $x'(t) = Ax(t)$. Set $y(t) = e^{-tA}x(t)$.

Amazing: same proof as sols$(f' = f) = \text{span}(e^{at})$: Divide purported sol by e^{at} and conclude by given equation that quotient is constant.

Then $y'(t) = \left(e^{-tA}x(t)\right)'$

$= \left(e^{-tA}\right)'x(t) + e^{-tA}x'(t)$

$= -Ae^{-tA}x(t) + e^{-tA}Ax(t)$ by Lemma + hypothesis

$= \left(-Ae^{-tA} + \underbrace{e^{-tA}A}_{Ae^{-tA} \text{ by Lemma}}\right)x(t)$

$= 0$

$\Rightarrow y_1'(t) = 0 \Rightarrow y(t) = v \in \mathbb{R}^n$ is constant

$\vdots$

$y_n'(t) = 0$

$\Rightarrow x(t) = e^{tA}y(t)$

$= e^{tA}v.$ ✓

$\dim C(e^{tA}) = n$:

entries of e^{tA} are functions, not scalars, so can't ask e^{tA} to be invertible.

Check that cols of e^{tA} are indep.

Need $e^{tA}v = 0 \Rightarrow v = 0$.

But $e^{tA}v = 0 \Rightarrow 0 = (e^{tA}v)\big|_{t=0}$

$= e^{0A}v$

$= Iv = v.$ $\square$

cols of $e^{tA}\big|_{t=0}$ are indep.

<u>Cor 3.4</u>: Solution set of <u>general order n ODE</u>

$$(*) \quad f^{(n)}(t) + a_{n-1}f^{(n-1)}(t) + \cdots + a_2 f''(t) + a_1 f'(t) + a_0 f(t) = 0$$

<u>with constant coeffs</u> $a_{n-1}, \dots, a_0$ is an n-dim subspace of $C^\infty(\mathbb{R})$.

<u>E.g.</u> $n = 2$: $f'' + f = 0$

$\Rightarrow$ all sols are linear combinations of sin and cos.

<u>Pf</u>: Set $x(t) = \begin{bmatrix} f(t) \\ f'(t) \\ \vdots \\ f^{(n-1)}(t) \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & \bigcirc & \\ & & 0 & 1 & & & \\ & & & \ddots & \ddots & & \\ & \bigcirc & & & 0 & 1 & \\ & & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & & -a_{n-1} \end{bmatrix}$.

Then $x'(t) = Ax(t) \Leftrightarrow (*)$.

Sols of $(*)$ are the top entries of sols of $x' = Ax$.

e.g. e^{tA} has top row $[f_1(t) \cdots f_n(t)]$

with $f_j(t)$ a sol of $(*)$ $\forall j$.

Moreover, $c_1 f_1 + \cdots + c_n f_n = 0 \Rightarrow c_1 \begin{bmatrix} f_1 \\ f_1' \\ \vdots \\ f_1^{(n-1)} \end{bmatrix} + \cdots + c_n \begin{bmatrix} f_n \\ f_n' \\ \vdots \\ f_n^{(n-1)} \end{bmatrix} = 0$

cols of e^{tA} indep.

$\Rightarrow c_1 = \cdots = c_n = 0$,

so $f_1, \dots, f_n$ are independent in $C^\infty(\mathbb{R})$. $\square$