1.

Math 403 Spring 2024

Advanced Linear Algebra

Tue / Thu 11:45 - 13:00

Physics 047

Office Hours: Tue 13:00 - 14:00 right after class
Thu 16:00 - 17:00

outside if possible
Physics 209 if not
Zoom if necessary

Safety:
- ~~masks, hand wash/sanitize~~ politely point out noncompliance — even me!
- ~~distance if possible~~
- know where the exits are from the room and the building

Policies
- covered on Tue ⇒ fair game for HW due Sat
- collaboration / academic honesty
  - Yes on HW write your solutions yourself
  - No on exams I have
    - brought numerous cases to the Office of Student Conduct
    - never lost

Index cards

1. Ezra Miller
2. he / him
3. 45^{th} grade
4. Major or potential major: Math, Music
5. What you hope to get out of this course

   students who know how to use linear algebra
6. The most important thing you've learned about how you learn

   not to take notes!
7. Hobbies: frisbee, gardening, photography, beer
8. Something unique about yourself

   hold breath for 4 minutes
   screws in right hand
   told by doctor in hospital I was going to die of rabies
   so radioactive I set off a Geiger counter from across room
   remarkable bike accident without injury

X "I'm from MA"

✓ "I'm from HI —
but I'm allergic to pineapple!"

Fields

Def: A group is a set G with an $\underbrace{\text{associative}}_{(g*g')*g'' = g*(g'*g'')}$ binary operation

$$*: G \times G \to G \qquad (g*g')*g'' = g*(g'*g'')$$
$$(g, g') \mapsto g*g'$$

that has
- an identity e with $e*g = g*e = g \ \forall g \in G$
- an inverse g^{-1} for each $g \in G$, so $g^{-1}g = e$.

G is abelian if $*$ is commutative: $g*h = h*g \ \forall g, h \in G$.

E.g. $(\mathbb{R}, +)$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}$, $\left(\begin{matrix}\text{m} \times \text{n matrices} \\ \text{with any of these,} \\ \text{coefficients}\end{matrix} +\right)$

$(\mathbb{R} \setminus \{0\}, \cdot)$, $\mathbb{C}$, $\mathbb{Q}$, $\cancel{\mathbb{Z}}$, $\cancel{\left(\begin{matrix}\text{m} \times \text{n matrices} \\ \text{with any of these,} \\ \text{coefficients}\end{matrix} \cdot\right)}$

$C^0(\mathbb{R}^n \to \mathbb{R}, +)$

$\text{Fun}(S \to A, +)$ A = any abelian group!

non-abelian: $\{A \in \mathbb{R}^{2\times 2} \mid \det A = 1\}$

Def: A field is an abelian group $(F, +)$ with additive identity $0 \in F$ such that
- $F^* = F \setminus \{0\}$ is an abelian group $(F^*, \cdot)$ and
- multiplication $\cdot$ distributes over addition $+$: $a \cdot (b+c) = a \cdot b + a \cdot c$.

E.g. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{F}_2 = \{0,1\}$, $\mathbb{F}_3 = \{-1, 0, 1\}$, $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ for $p \in \mathbb{Z}$ prime

$\mathbb{C} = \mathbb{R}(i)$, $\mathbb{Q}(i)$

Math 221 works verbatim with any F in place of $\mathbb{R}$,
except for notions of length, angle, order $(a<b)$

length $\Downarrow$ closeness (topology)

Def: A vector space over F is ... review from 221.

$(V, +)$ abelian group with an action of F:

$$F \times V \to V$$
$$(\alpha, v) \mapsto \alpha v$$

- distributes over $+$ on both sides
- associative: $\alpha(\beta v) = (\alpha\beta)v$
- $1v = v \quad \forall v \in V$.

Def: A homomorphism of vector spaces is a linear map.

E.g. $\mathcal{B}$ is a basis for V

$\Leftrightarrow \{\text{functions } \mathcal{B} \xrightarrow{f} W\} \overset{\leftrightarrow}{\cancel{\Rightarrow}} \exists! \{\text{homomorphisms } \varphi: V \to W \text{ with } \varphi|_{\mathcal{B}} = f\}$

Def: A homomorphism $\varphi: V \to W$ has

- kernel $\ker\varphi = \{v \in V \mid \varphi(v) = 0\} \subseteq V$
- image $\operatorname{im}\varphi = \{\varphi(v) \mid v \in V\} \subseteq W$.

subspaces

Thm (rank-nullity): $\dim(\ker\varphi) + \dim(\operatorname{im}\varphi) = \dim V$.

rank

nullity

Pf: Pick $\mathcal{B}' =$ basis of $\ker\varphi$

$\mathcal{B}'' \subseteq V$ with $\mathcal{B}'' \hookrightarrow \varphi(\mathcal{B}'') =$ basis of $\operatorname{im}\varphi$.

Then $\underset{\# \;=\; \#\; +\; \#}{\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''}$ is a basis of V. □

requires proof, but it's a straightforward exercise:

$$\sum_{v \in \mathcal{B}'} \alpha_v v + \sum_{w \in \mathcal{B}''} \beta_w w = 0$$

$\Rightarrow \sum_{w \in \mathcal{B}''} \beta_w \varphi(w) = 0$ since $\varphi(\mathcal{B}') = 0$

$\Rightarrow \beta_w = 0 \;\forall\, w \in \mathcal{B}''$ since $\varphi(\mathcal{B}'')$ indep.

$\Rightarrow \alpha_v = 0 \;\forall\, v \in \mathcal{B}'$ " $\mathcal{B}''$ ".

But $v \in V \Rightarrow \varphi(v) \in \operatorname{span}(\varphi(\mathcal{B}''))$, say $\varphi(v) = \varphi(b)$ with $b \in \operatorname{span}(\mathcal{B}'')$. Then $v - b \in \ker\varphi = \operatorname{span}(\mathcal{B}')$, so $v \in b + \operatorname{span}(\mathcal{B}') \subseteq \operatorname{span}(\mathcal{B}'' \cup \mathcal{B}')$.

Def: A homomorphism $\varphi: V \to W$ is an isomorphism

if φ is (injective and surjective) = bijective

$\Leftrightarrow \ker\varphi = 0$ $\quad$ $\operatorname{im}\varphi = W$

E.g. $\dim V = n \Rightarrow V \cong F^n$.

Pf: Basis $v_1, \dots, v_n$ of $V \Rightarrow v_i \mapsto e_i$ induces $\cong$: $\ker = 0$ because $e_1, \dots, e_n$ indep, and $\operatorname{im} = F^n$ because $e_1, \dots, e_n$ span F^n. □

E.g. $\operatorname{sols}_{\underset{\mathbb{C}}{\mathbb{R}}}(f'' + f = 0) \cong \mathbb{R}^2 = \operatorname{span}(\sin, \cos)$ $\updownarrow$? $\quad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, $\quad \cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

$\mathbb{C}^2 = \operatorname{span}(e^{ix}, e^{-ix})$

Def: An affine subspace is a translate of a subspace = sols (inhomogeneous linear system)

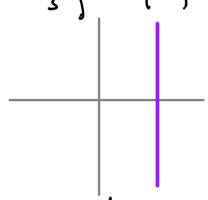
E.g. Set™ has one rule: find an affine line in $\mathbb{F}_3^4$.

card $= u \in \mathbb{F}_3^4$

$= (a, b, c, d)$. $\qquad v = (a', b', c', d') \Rightarrow$ affine line $\overline{uv}$ is

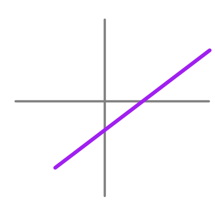
$$L = v + \{\lambda(u - v) \mid \lambda \in \mathbb{F}_3\} = \{v, u, -u-v\}.$$

Two possibilities:



(i) $L \subseteq x_1 = a$ $\qquad$ (if $a' = a$):

(ii) x_1-coordinates of all points in L are distinct (if $a' \neq a$)



Hence the rule: "If two are and one isn't then it's not a set."

Q. What makes $\mathbb{F}_3$ so special?

A. Each pair of points (cards) yields a unique third in L.