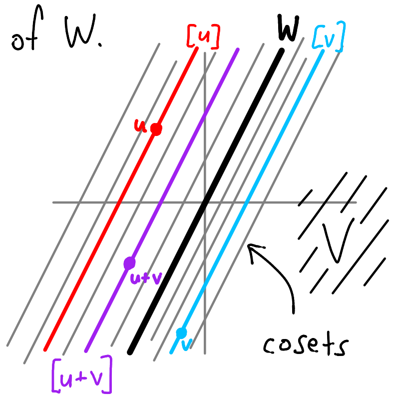
2.

Def: Fix subspace $W \subseteq V$.

1. A coset of W is an affine subspace $[v] = v + W$ for some $v \in V$.
2. The quotient V/W is the set of cosets of W.

"V mod W"



Prop: V/W is a vector space with

$$[u] + [v] = [u+v] \quad \text{and} \quad \lambda[v] = [\lambda v].$$

Pf: $(u+W) + (v+W) = (u+v) + W.$

↑ since $W + W = W$

"add ~~lines~~ affine subspaces"

$$\lambda(v+W) = \lambda v + \lambda W = \lambda v + W \text{ if } \lambda \neq 0, \text{ and}$$

$$0[v] = 0(v+W) = \{0\} \subseteq W = [0] = [0v]. \quad \square$$

Cor: $\dim V = \dim W + \dim V/W$.

Pf: $V \to V/W$ is a homomorphism (by Prop) with $\ker = W$ and $\operatorname{im} = V/W$. $\square$

Universal property of quotients

A homomorphism $\underbrace{V \xrightarrow{\varphi} U \text{ is } 0 \text{ on } W}_{W \subseteq \ker \varphi}$ $\Leftrightarrow$ φ factors through V/W:

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend left, "\varphi"] & V/W \arrow[r] & U. \end{tikzcd}$$

Pf: φ induces well defined function — forget algebraic properties, like "homomorphism"

$V/W \to U \Leftrightarrow$ each coset of W $\to$ single point in U

$\Leftrightarrow W \to$ single point in U.

φ linear

$\Leftrightarrow W \subseteq \ker \varphi$. $\square$

Def: Arbitrary homomorphism $V \to W$ has

- kernel $K \subseteq V$
- image $I \subseteq W$
- cokernel $W \twoheadrightarrow W/I$
- coimage ~~??~~ $V \twoheadrightarrow V/K$

standard abuse of notation: "$W/I = \operatorname{coker}(V \to W)$."

$$K \overset{\ker}{\subseteq} V \overset{\text{coim}}{\twoheadrightarrow} V/K \cong I \overset{\text{im}}{\subseteq} W \overset{\text{coker}}{\twoheadrightarrow} W/I$$

First Isomorphism Thm (requires proof!)

Pf: $V \twoheadrightarrow I$ by def of im, so $V \twoheadrightarrow V/K \twoheadrightarrow I$ by universal property of coker.

$\ker(V/K \twoheadrightarrow I) = \{[v] \in V/K \mid v \mapsto 0\} = [K]$ by def of ker,

$= 0 \in V/K \Rightarrow V/K \hookrightarrow I$. $\square$

Def: A sequence $V_0 \xrightarrow{\varphi_1} V_1 \longrightarrow \cdots \xrightarrow{\varphi_r} V_r$ is exact if $\ker \varphi_{i+1} = \operatorname{im} \varphi_i \ \forall i$.

E.g. $0 \to V \to V' \to 0$ exact $\Leftrightarrow \cancel{?}$ $V \cong V'$

$0 \to V \to V'$ exact $\Leftrightarrow \cancel{?}$ $V \hookrightarrow V'$

$V \to V' \to 0$ exact $\Leftrightarrow \cancel{?}$ $V \twoheadrightarrow V'$

$0 \to K \to V \to W \to W/I \to 0$ is exact

↖ exact here by FIT

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & C \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & B/A \arrow[r] & 0 \end{tikzcd}$$

$\Leftrightarrow \cancel{?}$ $A \subseteq B$ and $C \cong B/A$

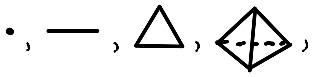
Def: $V_\bullet$: $\cdots \longrightarrow V_{i-1} \longrightarrow V_i \longrightarrow V_{i+1} \longrightarrow \cdots$ is a complex if $\underbrace{V_{i-1} \to V_i \to V_{i+1}}_{}$ is 0 $\forall i$.

$V_\bullet$ has homology $H_i V_\bullet = \ker \varphi_{i+1} / \operatorname{im} \varphi_i$.

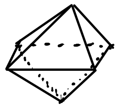
Lemma: $\Leftrightarrow$ $\operatorname{im}(V_{i-1} \xrightarrow{\varphi_i} V_i) \subseteq \ker(V_i \xrightarrow{\varphi_{i+1}} V_{i+1})$

measures how far a complex is from being exact.

Problem: you don't know many complexes yet. (Do you?)
(poll)

E.g. algebraic topology simplices  , …

of dim 0, 1, 2, 3, …

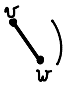
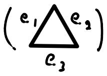
e.g. octahedron  ⇝ vector spaces $/\mathbb{F}_2$

V: basis = vertices

E: basis = edges

F: basis = faces

$C_\bullet$: $0 \xleftarrow{\partial_V} V \xleftarrow{\partial_E} E \xleftarrow{\partial_F} F \leftarrow 0$ chain complex

$\partial_E($  $) = v + w$ $\partial_F($  $) = e_1 + e_2 + e_3$

Prop: $C_\bullet$ is a complex!

Pf: 2 $(=0)$ ways to get from a simplex of dim i to a simplex of dim $i-2$. □

Compute: $H_0 C_\bullet = \ker \partial_V / \operatorname{im} \partial_E = V/B$ $B = \operatorname{span}(v+w \mid \text{vertices } v, w)$ $\dim H_0 = \cancel{?}\ 1$

$H_1 C_\bullet = \ker \partial_E / \operatorname{im} \partial_F = \cancel{?}\ 0$ exercise (not assigned)

$H_2 C_\bullet = \ker \partial_F / 0 = \cancel{?}$ $\operatorname{span}(f_1 + \cdots + f_8)$ $\dim H = \cancel{?}\ 1$

Thm (rank-nullity): $\sum_i (-1)^i \dim H_i = \sum_i (-1)^i \dim V_i$.

= Euler characteristic of $V_\bullet$

$0 \to K \to V \to I \to 0$

exact $\Rightarrow \dim K - \dim V + \dim I = 0$

$\Rightarrow \dim K + \dim I = \dim V$

Cor: exercise! : $6 - 12 + 8 = 1 - ? + 1 \Rightarrow ? = 0$

Pf of Thm: HW