4. Today: what does Jordan form mean, and how does it lead to proof?

Def: F is algebraically closed if every polynomial with coeffs in F has a root in F.

$\Rightarrow p(t) = \alpha(t-\alpha_1)^{n_1}\cdots(t-\alpha_r)^{n_r} \Rightarrow$ hypothesis of JF thm. How to guarantee all $\lambda \in F$

Fundamental Thm of Algebra: $\mathbb{C}$ is algebraically closed.

What block diagonal means:

Def (direct sum): $V = V_1 \oplus V_2$ means $V = V_1 + V_2$ and $V_1 \cap V_2 = 0$.

WARNING: does not mean $V_2 = V_1^\perp$, although that suffices with $\langle \cdot,\cdot \rangle$

Prop: $\Leftrightarrow$ V has basis $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ (partition) with $V_i = \operatorname{span} \mathcal{B}_i$ for $i = 1, 2$.

E.g. $F^n = F^m \oplus F^{n-m} = \operatorname{span}(e_1,\dots,e_m) \oplus \operatorname{span}(e_{m+1},\dots,e_n)$.

Def: $V = V_1 \oplus \cdots \oplus V_r$ if $V = V_1 + \cdots + V_r$ and $V_i \cap \sum_{j\neq i} V_j = 0 \quad \forall i$.

Prop: $\Leftrightarrow$ V has basis $\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_2$ (partition) with $V_i = \operatorname{span} \mathcal{B}_i \quad \forall i$.

E.g. V has basis $v_1,\dots,v_n \Leftrightarrow V = \operatorname{span}(v_1) \oplus \cdots \oplus \operatorname{span}(v_n)$.

Prop: φ block diagonal $\Leftrightarrow V = V_1 \oplus \cdots \oplus V_r$ with $\varphi(V_i) \subseteq V_i \quad \forall i$.

Def: V_i is φ-invariant

What Jordan blocks mean; needs:

Cayley-Hamilton Thm: $p_\varphi(\varphi) = 0$.

Pf: Fix basis $e_1,\dots,e_n$ of V, so $\varphi e_i = a_{i1}e_1 + \cdots + a_{in}e_n = A_i \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \Rightarrow \begin{bmatrix} \varphi e_1 \\ \vdots \\ \varphi e_n \end{bmatrix} = A \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$. (∗)

(entries are elements of V)

By def, $p_\varphi(t) = \det(A - tI)$. Need $p_\varphi(\varphi)e_i = 0 \quad \forall i$.

Equivalently, $\begin{bmatrix} p_\varphi(t) & & \\ & \ddots & \\ & & p_\varphi(t) \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} p_\varphi(t)e_1 \\ \vdots \\ p_\varphi(t)e_n \end{bmatrix}$ becomes $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ when evaluated at $t = \varphi$.

$\det(A-tI)I = C^T(A - tI)$, where C = cofactor matrix of $A - tI$.

But $(A - \varphi I)\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = A\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} - \begin{bmatrix} \varphi e_1 \\ \vdots \\ \varphi e_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ by (∗). Now multiply by $C^T|_{t=\varphi}$ on the left. □

Def: A minimal polynomial of φ (or A) is a monic ($m \neq 0$) polynomial $m(t)$ of minimal degree satisfying $m(\varphi) = 0$ (or $m(A) = 0$).

Prop: 1. $\exists!\ m(t)$.

2. $f(\varphi) = 0 \Rightarrow m \mid f$.

subtract monomial multiples of m from f to cancel leading terms recursively until you can't anymore

Pf: 1. follows from 2: $m \mid f$ and $\deg m = \deg f \Rightarrow f = \alpha m \overset{\text{monic}}{\Rightarrow} f = m$.

2. Assume $f(\varphi) = 0$. Write $f = qm + r$ with $\deg r < \deg m$. (∗)

Then $0 = f(\varphi) - q(\varphi)m(\varphi) = r(\varphi) \overset{(*)}{\Rightarrow} r = 0$.

Jordan block: $A \in F^{d\times d}$ or $\varphi: V \to V$ with $\dim V = d$ whose minimal polynomial is $(t-\lambda)^d$

$d=1$: $(\varphi-\lambda)v = 0 \Leftrightarrow v \in E(\lambda) \Rightarrow \mathcal{B}=\{v\}$ has $[\varphi]_{\mathcal{B}} = [\lambda]$

$d=2$: $(\varphi-\lambda)V \neq V$ or else $(\varphi-\lambda)^2 V = (\varphi-\lambda)\big((\varphi-\lambda)V\big) = (\varphi-\lambda)V = V$, but $(\varphi-\lambda)V \neq 0$ by def of minimal polynomial

$\Rightarrow \dim(\varphi-\lambda)V = 1 \Rightarrow (\varphi-\lambda)V$ is in the $d=1$ case

⋮

d arbitrary (prove by easy induction): $(\varphi-\lambda)V$ has dim $d-1$

$(\varphi-\lambda)^2 V$ $d-2$

⋮

$V_{d-k} = (\varphi-\lambda)^k V$ $d-k \Rightarrow \dim V_k = k$

⋮

Choose $v = v_d \in V \setminus V_{d-1}$. Then $(\varphi-\lambda)^d v_d = 0$ but $(\varphi-\lambda)^{d-1} v_d \neq 0$, so

$(\varphi-\lambda)v = v_{d-1} \in V_{d-1} \setminus V_{d-2}$ …

⋮

$(\varphi-\lambda)^{d-k} v = v_k \in V_k \setminus V_{k-1} \Rightarrow (\varphi-\lambda)v_k = v_{k-1} \Rightarrow \varphi v_k = \lambda v_k + v_{k-1}$ when $k \geq 2$, and $\varphi v_1 = \lambda v_1$.

⋮

So $\mathcal{B} = v_d, v_{d-1}, \dots, v_1 \Rightarrow [\varphi]_{\mathcal{B}} = d \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$ (size $d \times d$).

Pf of Jordan form thm: Need $V = V_1 \oplus \cdots \oplus V_r$ with
- V_ℓ φ-invariant $\forall \ell$
- (min. poly. of $\varphi|_{V_\ell}$) $= (t-\lambda_\ell)^{d_\ell}$ for $d_\ell = \dim V_\ell$.

Not so hard to do directly, but best done using rings and modules.

Def: A commutative ring satisfies all field axioms except
- multiplicative inverses need not exist
- multiplication need not be commutative

E.g. field, $\mathbb{Z}$, $F^{n\times n}$, $\mathbb{Z}^{n\times n}$, $R^{n\times n}$ for any commutative ring R

$F[t]$, $\mathbb{Z}[t]$, $R[t]$ for any ring R and any set t of variables

Def: A module over a ring R satisfies the same axioms as a vector space$/F$ but with scalars R.

E.g. vector space V/F with $\varphi: V \to V$ is a module $/F[t]$ with $tv = \varphi(v)$.

JF thm follows by classifying all $F[t]$-modules of $\dim_F < \infty$:   Look up "module over PID" (PID = principal ideal domain)

all are $\oplus$ invariant submodules $\langle v \rangle$ with $p^d(v) = 0$ for some irreducible p.

Note: same classification describes all finitely generated abelian groups: $\mathbb{Z}$ and $F[t]$ are both PIDs