5.

Banach spaces: complete normed vector spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

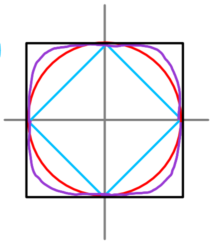
Cauchy sequences converge

Def: A **norm** on $V/\mathbb{F}$ is $\nu : V \to \mathbb{R}_+$ with (nonnegative)

- positive-definite • $\nu(x) = 0 \Leftrightarrow x = 0$
- homogeneous • $\nu(\alpha x) = |\alpha|\nu(x)$
- subadditive • $\nu(x+y) \leq \nu(x) + \nu(y)$ triangle inequality

E.g.
- Manhattan: $p=1$ • $\|x\|_1 = |x_1| + \cdots + |x_n|$
- Euclidean: $p=2$ • $\|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$
- $\lim_{p\to\infty}$ • $\|x\|_\infty = \max_{i=1}^{n} |x_i|$
- • $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$

$B_\varepsilon^{\|\cdot\|_1}(0) \subseteq B_\varepsilon^{\|\cdot\|_2}(0) \subseteq B_\varepsilon^{\|\cdot\|_\infty}(0)$



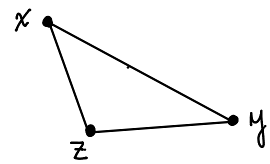
…or put any convex set here…

$B_\varepsilon^\nu(x) = \{ y \in V \mid \nu(x-y) < \varepsilon \}$ ← next lecture

default: $B_\varepsilon = B_\varepsilon^{\|\cdot\|_2}$

Def: A metric space is a set X with a distance $d : X \times X \to \mathbb{R}_+$ such that $\forall x, y \in X$

- $d(x,y) = 0 \Leftrightarrow x = y$ separates
- $d(x,y) = d(y,x)$ symmetric
- $d(x,y) \leq d(x,z) + d(z,y)$ $\forall z \in X$ triangle inequality



E.g. norm ν induces distance $d_\nu(x,y) = \nu(x-y)$, such as Euclidean metric $\|x-y\|_2$ on $\mathbb{F}^n$, Manhattan metric $\|x-y\|_1$ on $\mathbb{F}^n$

All norms "pretty much feel the same". In what sense?

Def: A topology on a set S is a collection $\mathcal{U}$ of subsets called open sets such that

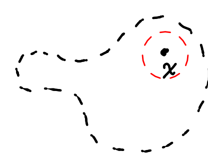
- any union of open sets is open $\quad \cup\, \mathcal{U} \in \mathcal{U}$
- any finite intersection of open sets is open $\quad \bigcap_{<\infty} \mathcal{U} \in \mathcal{U}$
- S and $\emptyset$ are open

Note: metric induces norm if homogeneous

E.g. usual topology on $\mathbb{F}^n$: $U \in \mathcal{U} \Leftrightarrow B_\varepsilon(x) \subseteq U$ $\forall x \in U$ and $\varepsilon = \varepsilon_x \ll 1$.

More generally: metric d on X ⇝ topology on X with

U open $\Leftrightarrow B_\varepsilon^d(x) \subseteq U$ $\forall x \in U$ and $\varepsilon = \varepsilon_x \ll 1$.



Def: $\mathcal{B} \subseteq \mathcal{U}$ is a base for the topology if $U \in \mathcal{U} \Rightarrow U = \bigcup_{B \in \mathcal{B}'} B$ for some $\mathcal{B}' \subseteq \mathcal{B}$.

E.g. $\{ B_\varepsilon^\nu(x) \mid x \in V \text{ and } \varepsilon \in \mathbb{R}_{>0} \}$ or $\varepsilon \ll 1$ or $\varepsilon \ll \varepsilon_0$

partially ordered (~~totally~~)

Def: $\{x_k\}_{k\in K} \to x$ if $\{x_k\}$ is eventually in U $\forall$ open $U \ni x$, meaning $\exists N_U \in K$ with $x_k \in U$ $\forall k \not\leq N_U$.

$X \subseteq S$ is closed if $x \in X$ whenever $\{x_k\} \to x$ in S with $\{x_k\} \subseteq X$. "X contains its limit points"

Prop: $X \subseteq S$ closed $\Leftrightarrow$ $S \setminus X$ open.

Pf: $S \setminus X$ open and $\{x_k\} \subseteq X \Rightarrow \lim x_k$ (if $\exists$) can't lie in $S \setminus X$, so it must lie in X. ($\{x_k\}$ is never in $S\setminus X$, eventually or otherwise)

$S \setminus X$ not open $\Rightarrow \exists y \in S \setminus X$ such that $\forall$ open $U \ni y$ $\exists x_U \in U \cap X$. Then $\{x_U\}_{y \in U}^{\text{parti} \not\leq} \to y \notin X$. $\square$

<u>Prop</u>: Any norm ν on $\mathbb{F}^n$ is continuous in the Euclidean metric.

<u>Pf</u>: Given $\varepsilon>0$, need δ so that $|\nu(x)-\nu(y)|<\varepsilon$ whenever $\|x-y\|<\delta$.

Subadditivity $\Rightarrow \nu(x) \le \nu(x-y)+\nu(y)$ and $\nu(y)\le \nu(y-x)+\nu(x)$

$\Rightarrow \nu(x)-\nu(y) \le \nu(x-y) \qquad \nu(y)-\nu(x) \le \nu(y-x)$, so

$|\nu(x)-\nu(y)| \le \nu(x-y) = \nu\left(\sum_{i=1}^n (x_i-y_i)e_i\right) \le \sum_{i=1}^n |x_i-y_i|\,\nu(e_i)$

$(*)$ $\le \|x-y\|_2 \|v\|_2$, where $v=(\nu(e_1),\dots,\nu(e_n))$.

Pick $\delta = \frac{\varepsilon}{\|v\|_2}$. $\square$ Q. why? A. Cauchy-Schwarz!

<u>Def</u>: Norms ν and μ on $V=\mathbb{F}^n$ are (<u>topologically</u>) <u>equivalent</u>, written $\nu\sim\mu$, if

$\exists\ \alpha,\beta\in\mathbb{R}_{>0}$ with $\alpha\nu(x)\le\mu(x)\le\beta\nu(x) \quad \forall x\in V$.

<u>Interpretation</u>: $\nu\sim\mu \Leftrightarrow B^\nu_{\varepsilon/\beta}(x)\subseteq B^\mu_\varepsilon(x)\subseteq B^\nu_{\varepsilon/\alpha}(x) \quad \forall x\in V$

$\alpha\nu(x-y)\le\mu(x-y)<\varepsilon \Rightarrow \nu(x-y)<\varepsilon/\alpha$

$y\in B^\mu_\varepsilon(x) \subseteq y\in B^\nu_{\varepsilon/\alpha}(x)$

$\Leftrightarrow$ every ε-ball base for the μ-topology is a base for the ν-topology

<u>E.g.</u> $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$

$\frac{1}{\sqrt{n}}\|x\|_2 \le \|x\|_\infty \le \|x\|_2$ <u>Pf</u>: exercise (not assigned)

$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

<u>Lemma</u>: $\sim$ is an equivalence relation.

<u>Pf</u>: symmetric: $\frac{1}{\beta}\mu(x)\le\nu(x)\le\frac{1}{\alpha}\mu(x)$.

transitive: exercise.

reflexive: $\alpha=\beta=1$. $\square$

<u>Thm</u>: μ,ν norms on $V=\mathbb{F}^n \Rightarrow \nu\sim\mu$.

<u>Pf</u>: By Lemma, need only check $\nu=\|\cdot\|_2$. Can assume $x\ne 0$.

$(*)$ with $y=0$ and μ instead of $\nu \Rightarrow \mu(x)\le\|x\|_2\|v\|_2$ for $v=(\mu(e_1),\dots,\mu(e_n))$

$\Rightarrow$ take $\beta=\|v\|_2$.

Set $\alpha=\min\{\mu(x)\mid \|x\|_2=1\}$, which exists by Prop because sphere S^{n-1} is closed and bounded.

and $\ne 0$ since $\mu(x)=0\Rightarrow x=0$

Then $\mu(x)=\mu\left(\|x\|_2\cdot\frac{x}{\|x\|_2}\right)=\|x\|_2\,\mu\left(\frac{x}{\|x\|_2}\right)$

$\ge \|x\|_2\,\alpha$. $\square$

CAN OMIT:

<u>Def</u>: norm on V^* <u>dual</u> to ν on V is $\nu^*(\varphi)=\max_{\nu(x)=1}|\varphi(x)|$.

well defined since $S_\nu=\{x\in V\mid \nu(x)=1\}$ is closed and bounded by Thm.