Banach spaces: complete normed vector spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Cauchy sequences converge: A norm on $V/\mathbb{F}$ is $\nu: V \rightarrow \mathbb{R}_+$ with positive-definite $\nu(x) = 0 \iff x = 0$, nonnegative $\nu(ax) = |a|\nu(x)$, homogeneous $\nu(x+y) \leq \nu(x) + \nu(y)$, subadditive, triangle inequality.

**Manhattan** $p=1 \cdot \|x\|_1 = |x_1| + \cdots + |x_n|$

**Euclidean** $p=2 \cdot \|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$

$$\lim_{p \to \infty} \|x\|_p = \max_{1 \leq i \leq n} |x_i|$$

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$

**Euclidean metric** $\|x-y\|_2$ on $\mathbb{F}^n$

**Manhattan metric** $\|x-y\|_1$ on $\mathbb{F}^n$

Def: A metric space is a set $X$ with a distance $d: X \times X \rightarrow \mathbb{R}_+$ such that $\forall x, y \in X$

- $d(x, y) = 0 \iff x = y$ separates
- $d(x, y) = d(y, x)$ symmetric
- $d(x, y) \leq d(x, z) + d(z, y)$ $\forall z \in X$ triangle inequality

Eg: norm $\nu$ induces distance $d_\nu(x, y) = \nu(x-y)$, such as Euclidean metric $\|x-y\|_2$ on $\mathbb{F}^n$

All norms "pretty much feel the same". In what sense?

Def: A topology on a set $S$ is a collection $\mathcal{U}$ of subsets called open sets such that:

- any union of open sets is open $\bigcup \mathcal{U} \in \mathcal{U}$
- any finite intersection of open sets is open $\bigcap_{\mathcal{U} \in \mathcal{U}} \in \mathcal{U}$
- $S$ and $\emptyset$ are open

Eg: usual topology on $\mathbb{F}^n$: $U \in \mathcal{U} \iff B_\varepsilon(x) \subseteq U$ $\forall x \in U$ and $\varepsilon = \varepsilon_x < 1$.

More generally: metric $d$ on $X \rightarrow$ topology on $X$ with $U$ open $\iff B_\varepsilon^d(x) \subseteq U$ $\forall x \in U$ and $\varepsilon = \varepsilon_x < 1$.

Def: $\mathcal{B} \subseteq \mathcal{U}$ is a base for the topology if $U \in \mathcal{U} \Rightarrow U = \bigcup_{B \in \mathcal{B}} B$ for some $B \subseteq B$.

Eg: $\{B_\varepsilon^d(x) | x \in V \text{ and } \varepsilon \in [0, 1] \}$ or $\varepsilon < 1$ or $\varepsilon < \varepsilon_0$.

Def: $\{x_k \}_{k \in K} \rightarrow x$ if $\{x_k\}$ is eventually in $U$ open $U \ni x$, meaning $\exists N \in \mathbb{N}$ with $x_k \in U \forall k \geq N$.

X is closed if $x \in X$ whenever $\{x_k\} \rightarrow x$ in S with $\{x_k\} \subseteq X$. "X contains its limit points"

Prop: $X \subseteq S$ closed $\iff S \setminus X$ open.

pf: $S \setminus X$ open and $\{x_k\} \subseteq X \Rightarrow \lim x_k$ (if $\exists$) can't lie in $S \setminus X$, so it must lie in $X$.

$S \setminus X$ not open $\Rightarrow \exists y \in S \setminus X$ such that $V$ open $U \ni y \exists x \in U \cap X$. Then $\{x_k\} \rightarrow y \notin X$. □
Prop: Any norm $\nu$ on $\mathbb{R}^n$ is continuous in the Euclidean metric.

Pf: Given $\varepsilon > 0$, need $\delta$ so that $|\nu(x) - \nu(y)| < \varepsilon$ whenever $\|x-y\| < \delta$.

Subadditivity $\Rightarrow$ $\nu(x) \leq \nu(x-y) + \nu(y)$ and $\nu(y) \leq \nu(y-x) + \nu(x)$

$\Rightarrow \nu(x) - \nu(y) \leq \nu(x-y)$

$\nu(y) - \nu(x) \leq \nu(y-x)$, so

$|\nu(x) - \nu(y)| \leq \nu(x-y) = \nu(\sum_{i=1}^{n} (x_i - y_i) e_i) \leq \sum_{i=1}^{n} |x_i - y_i| \nu(e_i)$. (\star) $\leq \|x-y\|_2 \|v\|_2$, where $v = (\nu(e_1), \ldots, \nu(e_n))$.

Pick $\delta = \frac{\varepsilon}{\|v\|_2}$. □

Q. why? A. Cauchy-Schwarz!

Def: Norms $\nu$ and $\mu$ on $V = \mathbb{R}^n$ are (topologically) equivalent, written $\nu \sim \mu$, if

$\exists \lambda, \beta \in \mathbb{R}_{>0}$ with $\lambda \nu(x) \leq \mu(x) \leq \beta \nu(x)$ $\forall x \in V$.

Interpretation: $\nu \sim \mu \iff B_{\beta \nu}^\mu (x) \subseteq B_{\lambda \nu}^\mu (x)$ $\forall x \in V$.

$\iff$ every $\varepsilon$-ball base for the $\mu$-topology is a base for the $\nu$-topology.

E.g., $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2$

$\frac{1}{\sqrt{n}} \|x\|_2 \leq \|x\|_{\infty} \leq \|x\|_2$  \hspace{1cm} Pf: exercise (not assigned)

Lemma: $\sim$ is an equivalence relation.

Pf: symmetric: $\frac{1}{\beta} \mu(x) \leq \nu(x) \leq \frac{1}{\alpha} \mu(x)$.

transitive: exercise.

reflexive: $\alpha = \beta = 1$. □

Thm: $\mu, \nu$ norms on $V = \mathbb{R}^n \Rightarrow \nu \sim \mu$.

Pf: By Lemma, need only check $\nu = \| \cdot \|_2$. Can assume $x \neq 0$.

(x) with $y = 0$ and $\mu$ instead of $\nu \Rightarrow \mu(x) \leq \|x\|_2 \|v\|_2$ for $v = (\mu(e_1), \ldots, \mu(e_n))$

$\Rightarrow$ take $\beta = \|v\|_2$.

Set $\alpha = \min \{ \mu(x) \mid \|x\|_2 = 1 \}$, which exists by Prop because sphere $S^{n-1}$ is closed and bounded.

Then $\mu(x) = \mu\left( \frac{x}{\|x\|_2} \right) = \|x\|_2 \mu\left( \frac{x}{\|x\|_2} \right) \\
\geq \|x\|_2 \alpha$. □

Def: norm on $V^*$ dual to $\nu$ on $V$ is $\nu^*(\varphi) = \max_{\nu(x) = 1} |\varphi(x)|$.

well defined since $S_{\nu} = \{ x \in V \mid \nu(x) = 1 \}$ is closed and bounded by Thm.