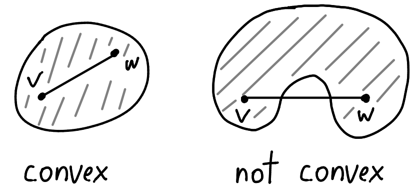
6. Convexity

Fix $\mathbb{R}$-vector space V.

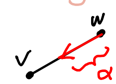
Def: $X \subseteq V$ is <u>convex</u> if $v, w \in X \Rightarrow$ line segment $\overline{vw} \subseteq X$.



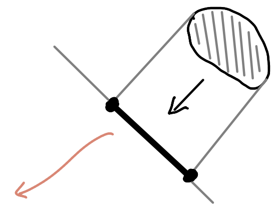
$$\overline{vw} = \{\alpha v + (1-\alpha) w \mid \alpha \in [0,1]\} = \left\{\alpha v + \beta w \,\middle|\, \begin{smallmatrix}\alpha+\beta=1\\ \alpha\geq 0,\ \beta\geq 0\end{smallmatrix}\right\}$$ = all weighted averages of v and w

$$= \{w + \alpha(v-w) \mid \alpha \in [0,1]\}$$ = parametrized line segment from w to v

E.g. any interval $\subseteq \mathbb{R}$: $\underbrace{[a,b] \quad (a,b] \quad (a,b) \quad [a,b)}$ $a=-\infty$ or $b=\infty$ allowed



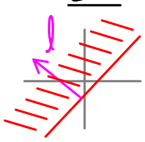
Pf: $a \overset{<}{\leq} v, w \overset{<}{\leq} b \Rightarrow [v,w] \subseteq$ " . □



Properties

1. $\varphi: V \to W$ linear $\Rightarrow \varphi(\text{convex})$ is convex. Pf: $\varphi(\overline{vw}) = \overline{\varphi(v)\varphi(w)}$.
2. " $\Rightarrow \varphi^{-1}(\text{convex})$ is convex. Pf: same! $v, w \in \varphi^{-1}(Y) \Rightarrow \overline{\varphi(v)\varphi(w)} \subseteq Y \Rightarrow \overline{vw} \subseteq \varphi^{-1}(Y)$.
3. X convex $\Rightarrow a + X$ convex $\forall a \in V$. Pf: $a+v, a+w \in a+X \Leftrightarrow v, w \in X \Leftrightarrow \overline{vw} \subseteq X \Leftrightarrow \underbrace{a + \overline{vw}}_{\overline{(a+v)(a+w)}} \subseteq a + X$.
4. $\{X_i\}_{i\in I}$ all convex $\Rightarrow \bigcap_{i\in I} X_i$ convex. Pf: $\overline{vw} \subseteq X_i\ \forall i \Leftrightarrow \overline{vw} \subseteq \bigcap_{i\in I} X_i$.

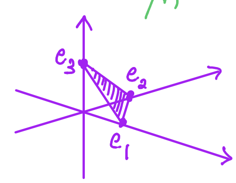
Cor: 5. Every affine subspace is convex. Pf: 1.+3.



6. Every <u>open halfspace</u> (<u>closed</u>) $H^{\pm} = \{x \in V \mid \ell(x) \overset{\leq}{\gtrless} c\}$ for some $\ell \in V^*$ and $c \in \mathbb{R}$. Pf: $H^{\pm} = \ell^{-1}(\text{ray})$; use 2.

sometimes $\mathring{H}^{\pm}$ to emphasize "open"

E.g. <u>standard $(n-1)$-simplex</u> $\sigma = \{x \in \mathbb{R}^n \mid x_i \geq 0\ \forall i \text{ and } x_1 + \cdots + x_n = 1\}$ is convex.



Pf: $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0\ \forall i\} = \bigcap_{i=1}^{n} \{e_i^*(x) \geq 0\}$ convex by 6.+4.

$\{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 1\}$ convex by 5. or 2. Now apply 4. □

Prop: $x_1, \ldots, x_n \in V \Rightarrow$ 7. $P = \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid \alpha_i \geq 0\ \forall i \text{ and } \alpha_1 + \cdots + \alpha_n = 1\}$ is convex.

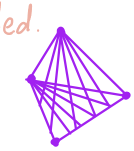
8. $X \supseteq \{x_1, \ldots, x_n\}$ convex $\Rightarrow X \supseteq P$.

<u>convex combinations</u> of $x_1, \ldots, x_n$

Def: P = <u>convex hull</u> = $\operatorname{conv}\{x_1, \ldots, x_n\}$ is a <u>polytope</u>.

Pf: 7. $P = \varphi(\sigma)$ for $\varphi: \mathbb{R}^n \to V$, $e_i \mapsto x_i$. Use 1.

8. Use induction on n. Trivial if $n=1$. And by definition if $n=2$, though that's not needed.



$n \geq 2$: need $\alpha_1 x_1 + \cdots + \alpha_n x_n \in X$. Assume $\alpha_n \neq 1$. else trivial: $x_n \in X$

Then $\alpha_1 x_1 + \cdots + \alpha_n x_n = \alpha \underbrace{(\beta_1 x_1 + \cdots + \beta_{n-1} x_{n-1})}_{\text{by induction } \in X \text{ since } \sum_{i=1}^{n-1} \beta_i = 1} + (1-\alpha) x_n$ for $\alpha = 1 - \alpha_n = \alpha_1 + \cdots + \alpha_{n-1}$

$\beta_i = \frac{\alpha_i}{\alpha}$ for $i = 1, \ldots, n-1$. □

Further examples of convex sets:

E.g. • $X = \{ f \in \mathbb{R}[t] \mid f(\alpha) > 0 \ \forall \alpha \in (0,1) \}$

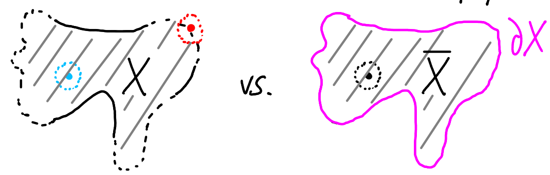
• $X = \{ A \in \mathbb{R}^{n\times n} \mid A = A^* \text{ and } \lambda > 0 \ \forall \text{ eigenvalues } \lambda \text{ of } A \}$ see Lecture 9

$\mathbb{C}$?

Def: $v \in X \subseteq V$ is an <u>interior point</u> if $v \in U \subseteq X$ for some open $U \subseteq V$.

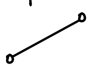
$v \in \partial X$ <u>boundary point</u> if $U \cap X$ and $U \cap (V \setminus X)$ both nonempty $\forall$ open $v \in U \subseteq V$.

$\overline{X} = X \cup \partial X$ is the <u>closure</u> of X.



Prop: $\overline{X} = \{ \lim x_k \mid \{x_k\} \subseteq X \text{ converges} \}$.

Pf: Prop, p. 11.

Q. Is  open?

A. In $\mathbb{R}^1$: yes.

In $\mathbb{R}^2$: no. <u>relative interior</u> can fix this, but let's not get into it

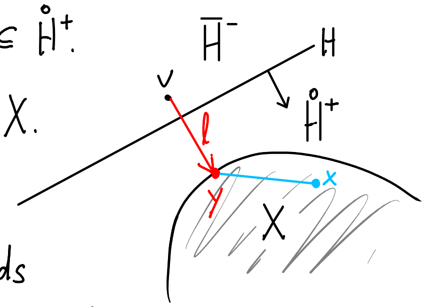
Prop: $X \subseteq V$ convex $\Rightarrow \overline{X}$ convex.

Pf: $\begin{matrix} v_k \to v \\ w_k \to w \end{matrix} \Rightarrow \alpha v_k + \beta w_k \to \alpha v + \beta w$. But $v_k \in X$ and $w_k \in X \ \forall k \overset{\alpha+\beta=1}{\Longrightarrow} \alpha v_k + \beta w_k \in X \ \forall k$ by convexity

$\Rightarrow \alpha v + \beta w \in \overline{X}$ by Prop. $\square$

Def: A hyperplane $H \subseteq V$ <u>separates</u> v from X if $v \in \overline{H}^-$ and $X \subseteq \mathring{H}^+$.

linear function $\ell \in V^*$ " " " " $\ell(v) < \ell(x) \ \forall x \in X$.



Thm: $X = \overline{X}$ convex and $v \notin X \Rightarrow \exists H$ separating v from X.

Pf: For any choice of inner product, the norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ yields

$f(x) = \|x - v\| : X \to \mathbb{R}$ continuous, bounded below (by 0), and <u>proper</u> (f^{-1}(bounded set) is bounded)

by topological equivalence: $f^{-1}([0,r]) = B_r^{\|\cdot\|}(v)$ is bounded. Since X is closed, f attains

a minimum on X, say at $y \in X$. Set $\ell = \langle y - v, \cdot \rangle$. Note $\ell \neq 0$ since $v \notin X$ (so $y \neq v$).

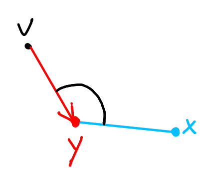
$\forall \alpha \in (0,1]$ and $x \in X$, $\cancel{\|y-v\|^2} \underset{\text{min}}{\leq} \|y + \alpha(x-y) - v\|^2 = \|y - v + \alpha(x-y)\|^2$

$= \cancel{\|y-v\|^2} + \underbrace{2\alpha \langle y-v, x-y \rangle}_{-2\alpha\langle v-y,\, x-y\rangle} + \cancel{\alpha^2} \|x-y\|^2$

law of cosines

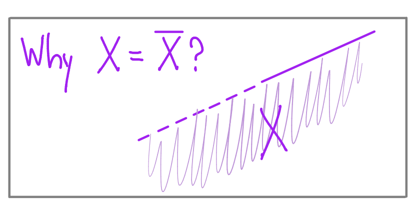
$\Rightarrow 0 \leq 2\ell(x-y) + \alpha \|x-y\|^2$

$\overset{\alpha \to 0}{\Longrightarrow} \ell(y) \leq \ell(x)$.



But $\ell(v) < \ell(y)$ because $\ell(y-v) = \|y-v\|^2 > 0$,

so $\ell(v) < \ell(x) \ \forall x \in X$. $\square$



Applications: optimization (linear programming), analysis (bounds away from 0), CS (support vector machines)