8. Grassmannians

<u>Def</u>: A <u>k-plane</u> in a vector space/F is a subspace of dimension k.

The <u>grassmannian</u> is $G_k(W) = \{k\text{-planes in } W\}$

- What does $G_k(F^n)$ "look like"?
- How close is one k-plane to another?
- Find "best" k-plane for given purpose $\longleftrightarrow$ optimization on $G_k(F^n)$

e.g. least squares $\longleftrightarrow$ on $\mathbb{P}(\mathbb{R}^n) = G_1(\mathbb{R}^n)$ for cost function $f(V) = \sum_{i=1}^{r} d(x_i, V)^2$ — data points

Q. $V \subseteq F^n$ specified by ...? ~~?~~ A. basis $v_1, \dots, v_k$

$\Rightarrow F^{k\times n} \overset{?}{\longleftrightarrow} G_k(F^n)$ no: need rank $= k$ $F_*^{k\times n}$

$F_*^{k\times n} \longrightarrow G_k(F^n)$

$A \longmapsto \text{span}(\text{rows of } A)$ $\twoheadrightarrow$ but not $\hookrightarrow$

Note: $F^n = F^n_{\text{row}}$ here; could just as easily do everything with columns

<u>E.g.</u> $A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \longmapsto V$

$\Rightarrow A' = \begin{bmatrix} 2 & 0 & 2 & 6 \\ 1 & 1 & 0 & 4 \end{bmatrix} \longmapsto V$ $\qquad A' = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} A$

<u>Prop</u>: $A, A' \in F_*^{k\times n}$ yield same $V \Leftrightarrow$ rows of A are linear combinations of rows of A'

$\Leftrightarrow A' = gA$ for some $g \in F^{k\times k}$

(*) $\Leftrightarrow$ " $g \in F_*^{k\times k} \overset{\text{Def}}{=} GL_k(F)$, the group of invertible $k\times k$ matrices

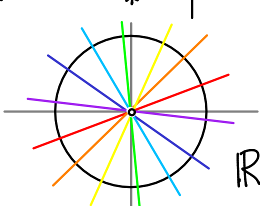
<u>Cor</u>: $G_k(F^n) = GL_k(F) \backslash F_*^{k\times n}$, the <u>quotient</u> of $F_*^{k\times n}$ modulo the action of GL_k on the left $GL_k \times F^{k\times n} \to F^{k\times n}, \dots$

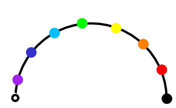
$= \{[A] \subseteq F_*^{k\times n} \mid A \in F_*^{k\times n}\}$, where $A' \in [A] \Leftrightarrow$ (*)

$[A] = GL_k(F)A$, the <u>orbit</u> of A

<u>E.g.</u> $n=2$, $k=1$

$[\alpha\ \beta] \mapsto [[\alpha\ \beta]]$





$\mathbb{R}^2 \setminus \{0\} \to G_1(\mathbb{R}^2) = \mathbb{RP}^1 = \mathbb{P}^1(\mathbb{R})$

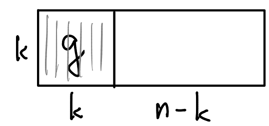
<u>Compare</u>: $V/W = \{[v] \subseteq V \mid v \in V\}$, where $v' \in [v] \Leftrightarrow v' = v + w$ for some $w \in W$

$[v] = W + v$, the coset of v — key point

<u>Note</u>: $F^n_{\text{row}} \rightsquigarrow F^n_{\text{col}} \Rightarrow G_k(F^n_{\text{col}}) = F_*^{n\times k}/GL_k(F) = \{A \cdot GL_k \mid A \in F_*^{n\times k}\}$

How does this quotient business help? What kind of structure does it induce on $G_k(F^n)$?

Suppose left k cols of $A \in F^{k\times n}$ are independent.



Let $g = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_k \\ | & & | \end{bmatrix} \in GL_k$ i.e. invertible

so $\hat{A} = g^{-1}A = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \;\ldots\text{stuff}\ldots \\ & & 1 & \end{array}\right]$ = reduced echelon form of A!

Here is a reason to be using row vectors.

Then $A \mapsto V$

$\Rightarrow \hat{A} \mapsto V$

<u>Lemma</u>: $A' \mapsto V \Rightarrow \hat{A}' = \hat{A}$. <u>Pf</u>: same row span, in echelon form, which is unique. $\square$

$\Updownarrow$

<u>Lemma'</u>: $\{A \in F^{k\times n} \mid \text{left } k \text{ cols} = I_k\} \hookrightarrow G_k(F^n)$.

Q. $=$? A. No.

rank $A = k \Rightarrow$ some set σ of k cols is independent.

Write $F_\sigma^{k\times n} = \{\hat{A} \in F^{k\times n} \mid \hat{A} \text{ has } I_k \text{ in cols from } \sigma\}$.

<u>E.g.</u> $F_{[k]}^{k\times n}$ with $[k] = \{1,\ldots,k\}$.

<u>Lemma''</u>: $F_\sigma^{k\times n} \overset{\pi_\sigma}{\hookrightarrow} G_k(F^n) \;\; \forall \; \sigma \in \binom{[n]}{k} = \binom{\{1,\ldots,n\}}{k}$. Same proof.

$\text{\rotatebox{90}{$\subseteq$}}$ affine subspace

$F^{k\times n}$ vector space

Set $G_k^\sigma = \mathrm{im}(\pi_\sigma) \cong F^{k\times(n-k)}$

$\boxed{G_k(F^n) \text{ locally looks like } F^{k\times(n-k)}}$

Summary:

<u>Prop</u>: $G_k(F^n)$ covered by $\binom{n}{k}$ affine spaces $F_\sigma^{k\times n} \overset{\pi_\sigma}{\hookrightarrow} G_k(F^n) = \bigcup_{\sigma \in \binom{[n]}{k}} G_k^\sigma$.

<u>Def</u>: A <u>manifold</u> is a topological space X with an <u>atlas</u>: a set of maps $U_\alpha \overset{\pi_\alpha}{\hookrightarrow} X$ <u>charts</u>

such that • $X = \bigcup_\alpha X_\alpha$, where $X_\alpha = \mathrm{im}(\pi_\alpha)$ is open in X

and $\forall\, \alpha$ • U_α is open in an affine space of dim d $\simeq F^d$

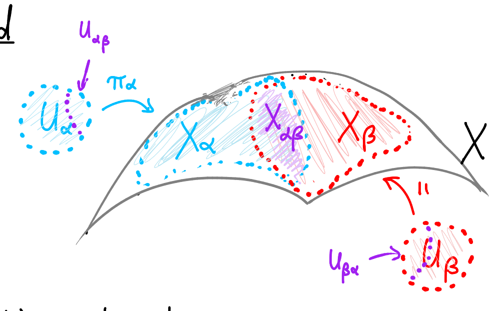
• π_α is a <u>homeomorphism</u> $U_\alpha \overset{\sim}{\to} X_\alpha$ $\quad W \subseteq U_\alpha$ open $\Leftrightarrow \pi_\alpha(W) \subseteq X_\alpha$ open

and $\forall\, \alpha, \beta$ • $\pi_\beta^{-1} \circ \pi_\alpha : U_{\alpha\beta} \overset{\sim}{\to} U_{\beta\alpha}$ is $\quad$ e.g. $U_{\alpha\beta} = \pi_\alpha^{-1}(X_{\alpha\beta}) \subseteq U_\alpha$ open

$X_{\alpha\beta} = X_\alpha \cap X_\beta$

Which one(s) is $G_k(F^n)$?

$F = \mathbb{R}$	- continuous	$\Rightarrow$	<u>topological manifold</u>
$\Uparrow$	- differentiable	$\Rightarrow$	<u>differentiable manifold</u>
$\Uparrow$	- smooth (C^∞)	$\Rightarrow$	<u>smooth manifold</u>
$F = \mathbb{C}$ or $\mathbb{R}$ $\Uparrow$	- analytic	$\Rightarrow$	<u>analytic manifold</u>
F arbitrary $\Uparrow$	- ratio of polynomials	$\Rightarrow$	<u>rational algebraic variety</u>



<u>Thm</u>: $G_k(F^n)$ is a rational algebraic variety of dim $k(n-k)$ with atlas

$\{\pi_\sigma : F_\sigma^{k\times n} \to G_k(F^n) \mid \sigma \in \binom{[n]}{k}\}$. $\square$