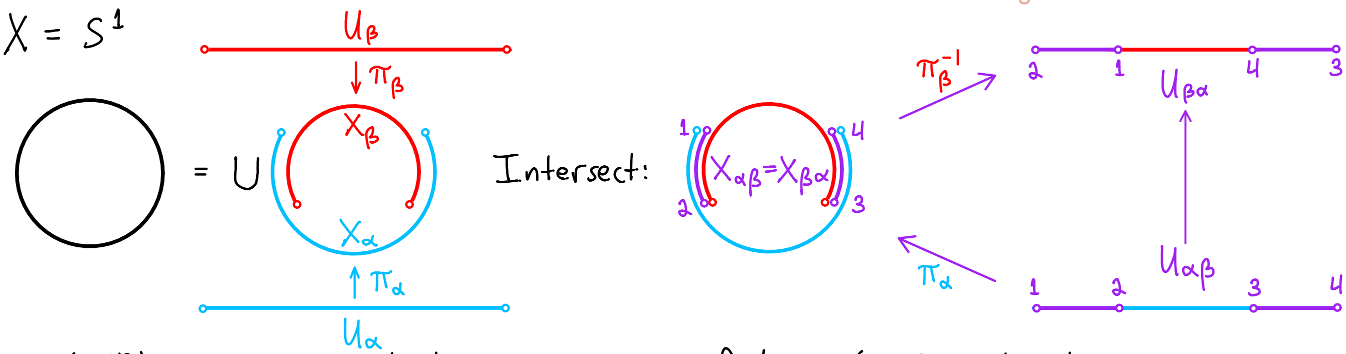
9. Review def of manifold; e.g. X = surface of Earth, atlas = actual "rectangular" maps

E.g. $X = S^1$



Thm: $G_k(F^n)$ is a rational algebraic variety of dim $k(n-k)$ with atlas

$$\left\{ \pi_\sigma : F_\sigma^{k\times n} \to G_k(F^n) \,\middle|\, \sigma \in \binom{[n]}{k} \right\}.$$

Pf: • Prop $\Rightarrow X = \bigcup_\alpha X_\alpha$: $G_k(F^n) = \bigcup_\sigma G_k^\sigma$

• and $G_k^\sigma \simeq F_\sigma^{k\times n} \simeq F^d$ for $d = k(n-k)$

• Declare $U \subseteq G_k(F^n)$ to be open $\Leftrightarrow$ $U \cap G_k^\sigma$ is open $\forall\, \sigma \in \binom{[n]}{k}$. HW 2.16: well defined

• Set $F_{\sigma,\tau}^{k\times n} = \{ A \in F_\sigma^{k\times n} \mid A_\tau = [\text{cols of } A \text{ indexed by } \tau] \text{ is invertible} \}$

$= \pi_\sigma^{-1}(G_k^\sigma \cap G_k^\tau)$. recall: [cols indexed by σ] is I_k for $A \in F_\sigma^{k\times n}$

Then $\pi_\tau^{-1} \circ \pi_\sigma : F_{\sigma,\tau}^{k\times n} \to F_{\tau,\sigma}^{k\times n}$

$A \mapsto ?$ Find matrix A' with $[A'] = [A]$ and $A'_\tau = I_k$

easy: $A' = \underbrace{A_\tau^{-1} A}$

entries are rational functions of entries of A. □

So that's how grassmannians work. Let's all do an exercise together to see these methods in action.

Def: A (complete) flag in V is a chain

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

of subspaces of V with $\dim V_i = i$ $\forall i$. Set

$$\mathcal{F}\ell_n(F) = \{\text{complete flags in } F^n\}.$$

Ex. Express $\mathcal{F}\ell_n$ as a quotient.

• of what? How do you write down a flag? Do it, then quotient modulo choices.
This a very general principle in math.

$V_1 = \langle v_1 \rangle$

$V_2 = \langle v_1, v_2 \rangle$ $\qquad A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \in GL_n(F)$ Why?

$\vdots$ (Yes we're using columns now.)

$V_i = \langle v_1, \ldots, v_i \rangle$

$\vdots$

□

- by what?

$\langle v_1\rangle = \langle \alpha v_1\rangle$ for any $\alpha \in F^*$

$\langle v_1, v_2\rangle = \langle \alpha v_1\rangle + \langle \beta_1 v_1 + \beta_2 v_2\rangle$ for any $\alpha\in F^*$, $\beta_1 \in F$, and $\beta_2 \in F^*$

$\langle v_1, v_2, v_3\rangle = \underbrace{\langle \alpha v_1\rangle + \langle \beta_1 v_1 + \beta_2 v_2\rangle} + \langle \underbrace{\text{some replacement}}_{\in \operatorname{span}(v_1,v_2,v_3)\setminus \operatorname{span}(v_1,v_2)} \text{ for } v_3\rangle$

$\vdots$

$\in \operatorname{span}(v_1,v_2,v_3)\setminus \operatorname{span}(v_1,v_2)$,

so $\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3$ with $\gamma_3 \neq 0$

$$A = AB \text{ for } B = \begin{bmatrix} F^* & F & F & \cdots & F \\ 0 & F^* & F & & \vdots \\ \vdots & 0 & F^* & & \\ \vdots & \vdots & & \ddots & F \\ 0 & 0 & \cdots & 0 & F^* \end{bmatrix} = \begin{bmatrix} \diagdown & * \\ 0 & \end{bmatrix} \subseteq GL_n(F)$$

$\Leftrightarrow$ diagonal entries all $\neq 0$, given upper-triangular

Def: $B_n^+ = \left\{ \begin{bmatrix} \diagdown\, * \\ 0 \end{bmatrix} \right\} \subseteq GL_n$ is the Borel subgroup.

Prop: $\mathcal{F}\ell_n = GL_n / B_n^+$. □

Is it a manifold? A variety? Find
- "big" subset that is an open subset of a vector space
- enough copies to cover.

Assume A "generic". What does that mean? Don't know yet. Try; see what's needed.

Use • b_{11} to make $a_{11} = 1$ needs "generic"

then • $b_{12}, \ldots, b_{1n}$ to cancel $a_{12}, \ldots, a_{1n}$

then • b_{22} to make new $a_{22} = 1$

then • $b_{23}, \ldots, b_{2n}$ to cancel $a_{23}, \ldots, a_{2n}$

$\vdots$

$$B_1 = \begin{bmatrix} a_{11}^{-1} & -a_{11}^{-1}a_{12} & \cdots & -a_{11}^{-1}a_{1n} \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{bmatrix}$$

column reduce A without swapping any columns

$$AB_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}$$

$$AB_1B_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & 0\cdots & 0 \\ \hline & & * & \end{bmatrix}$$

$$AB_1\cdots B_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ * & & \ddots \\ & & & 1 \end{bmatrix} \in U_n^-$$ unipotent subgroup

Prop: $U_n^- \hookrightarrow \mathcal{F}\ell_n = GL_n / B_n^+$.

Pf: HW.

Thm: $\mathcal{F}\ell_n$ is a rational algebraic variety with atlas $\{ wU_n^- \to GL_n/B_n^+ \mid w \text{ is a permutation matrix}\}$.

Pf: HW.