10. Orthogonal groups

intuitively: pick up S^2 and put it down on top of itself

unit sphere

Q. What is $\{\text{symmetries of } S^2 \subseteq \mathbb{R}^3\}$?

<u>Def</u>: An <u>isometry</u> of a metric space X is a bijection $\varphi : X \to X$ with

$d(\varphi x, \varphi y) = d(x,y)\ \forall\ x,y \in X.$ In S^{2}, S^{n-1}: $d(x,y) = \angle(x,y) \in [0,\pi]$

<u>Note</u>: only need $\varphi : X \twoheadrightarrow X$, since $x \neq y \Rightarrow d(\varphi x, \varphi y) = d(x,y) \neq 0 \Rightarrow \varphi x \neq \varphi y$.

<u>E.g.</u> Find (X,d) and φ satisfying everything except $\twoheadrightarrow$. e.g. $X = \mathbb{R}_+$, $\varphi : x \mapsto x+1$

Q. Is every isometry of S^2 rotation about some axis?

A. No, but if assume it preserves <u>orientation</u> then

- φ isom of $S^2 \Rightarrow$ preserves orthonormality of any basis of $\mathbb{R}^3$
- (v_1, v_2, v_3) <u>right-handed</u> if $v_1 \times v_2 = v_3$ in $\mathbb{R}^n$: <u>positively oriented</u> if $\det\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} > 0$
- <u>Def</u>: φ <u>preserves orientation</u> if φ preserves handedness

<u>E.g.</u>
- $\varphi = -I_3 \Rightarrow$ not
- $\varphi =$ reflection $\Rightarrow$ not
- $\varphi =$ rotation $\Rightarrow$ ✓

<u>Lemma</u>: $\varphi : S^{n-1} \to S^{n-1}$ isometry $\Rightarrow \{\alpha\varphi \mid \alpha \in \mathbb{R}_+\}$ is an isometry of $\mathbb{R}^n = \bigcup_{\alpha \geq 0} \alpha S^{n-1}$

i.e. φ extends to an isometry of $\mathbb{R}^n$ $\Rightarrow$ need only study isometries of $\mathbb{R}^n$.

<u>Pf</u>: $x, y \in \mathbb{R}^n \Rightarrow \varphi$ preserves
- $\|x\|$ and $\|y\|$ by construction
- $\angle(x,y) \overset{\text{def}}{=} \angle\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right)$ by isometry of S^{n-1}

$\Rightarrow$ • $\|x-y\|$ by law of cosines. $\square$

$a^2 + b^2 = c^2 + 2ab\cos C$

$\|x\|^2 \quad \|y\|^2 \quad \|x-y\|^2 \quad \|x\|\|y\| \angle(x,y)$

<u>Thm</u>: Every isometry of $\mathbb{R}^n$ has the form $A + T_v$ for some
- $A \in O_n(\mathbb{R}) = \{Q \in \mathbb{R}^{n\times n} \mid Q^{-1} = Q^T\}$ <u>orthogonal group</u>
- $T_v =$ translation by $v \in \mathbb{R}^n$.

<u>Pf</u>: Assume $\varphi \in \mathrm{isom}(\mathbb{R}^n)$ with $\varphi(0) = v$. Replace φ with $\varphi - T_v$ to assume $v = 0$.

Need $\varphi \in O_n(\mathbb{R})$; proof essentially as in Lemma + (preserves inner products $\Rightarrow$ linear). $\square$

$(Ax)_i = \langle Ax, e_i \rangle$

as a set, not pointwise

<u>Note</u>: $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ fixed by isometry φ of $\mathbb{R}^n \Leftrightarrow \varphi(0) = 0$.

<u>Cor</u>: $\mathrm{Isom}(S^{n-1}) \longleftrightarrow O_n(\mathbb{R})$.

Answer to Q.

- $A \in O_3 \Rightarrow A$ has a real eigenvalue   Why? $\deg(\text{char. poly}) = 3$ odd

  $\Rightarrow$ at least one eigenvalue $\lambda = \pm 1$, since $|\lambda| = 1$ HW

  $\Rightarrow$ eigenline = axis fixed pointwise by A or $-A$. One of these preserves orientation since 3 is odd

But • $A \in O_3 \Rightarrow A$ takes orthonormal basis of $P = \text{axis}^\perp$
to " " " "

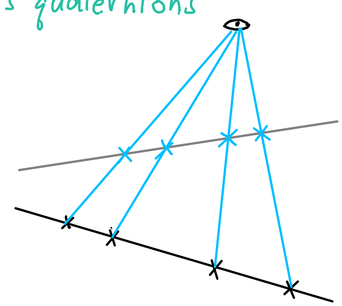
$\Rightarrow A|_P$ is rotation of P, possibly $\underbrace{\text{followed by reflection}}_{\text{but not if } A \text{ preserves orientation.}}$

Pf: $(e_1, e_2, e_3) \overset{A}{\mapsto} (v_1, v_2, v_3) \Rightarrow \det A = v_1 \cdot (v_2 \times v_3) = v_1 \cdot (\pm v_1)$.

But $\text{rot}_{\text{axis}}(\theta)$ has eigenvalues $1, \lambda, \bar\lambda$ for some $\lambda \in \mathbb{C}$ with $|\lambda|^2 = \lambda\bar\lambda = 1$,

so $\det(\ \ ) = 1 \cdot \lambda \cdot \bar\lambda = 1 \Rightarrow v_1 \cdot (\pm v_1) = 1 \Rightarrow +$. $\square$

Aside: Other symmetry groups $G \left\{ \begin{array}{l} \bullet \text{ rotation } (\mathbb{R} \text{ or } \mathbb{C} \text{ or } \mathbb{H}) \\ \bullet \text{ scaling } (\mathbb{R}, \mathbb{C}, \text{arbitrary } F) \\ \bullet \text{ translation} \\ \bullet \text{ affine or projective transformations} \end{array} \right.$ Hamilton's quaternions



Preserve: angle, distance, collinearity, ...

Applications: computer vision, rendering, 3D image reconstruction, morphometrics

E.g. face recognition $(n=2)$ $n \overset{d}{\boxed{\ \ |||| \quad A \quad |||| \ \ }} \in \mathbb{R}^{n \times d}$ $A \sim A'$ same data point if
landmarks $A' = gA \quad g \in G$

$X = G \backslash \mathbb{R}^{n \times d}$ algebraic variety, metric space: $d([A],[B])$ = something from linear algebra of A and B

Back to O_n...

Prop: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\langle \cdot, \cdot \rangle$ standard hermitian form on $\mathbb{F}^n$. TFAE. I never want to see this written in TEX!

1. $\langle xA, yA \rangle = \langle x, y \rangle \ \forall$ rows $x, y \in \mathbb{F}^n$ Def: $A \in O_n(\mathbb{F})$
2. right multiplication ρ_A preserves orthonormal bases: $v_1, \dots, v_k$ orthonormal $\Rightarrow v_1A, \dots, v_kA$ orthonormal
3. rows of A are orthonormal basis of $\mathbb{F}^n$
4. $AA^* = I_n$
5. cols of A are orthonormal basis of $\mathbb{F}^n$

Pf: Exercise (not assigned).   Discuss orally if time permits.

Def: $U_n = O_n(\mathbb{C})$ unitary group   What does it "look like"?