12. Positive (semi)definite matrices and singular values

Def: V with $\langle\cdot,\cdot\rangle$ and $\dim_{\mathbb{C}} V = n$. $\mathbb{C}$-linear $\varphi: V \to V$ is <u>self-adjoint</u> if $\varphi = \varphi^*$ and is further

- <u>positive semidefinite</u> if $\langle \varphi x, x\rangle \geq 0 \ \forall x \in V$ "$\varphi \geq 0$" e.g. diagonal and ≥ 0
- <u>positive definite</u> $\quad >$ "$\varphi > 0$" e.g. $\quad > 0$

Thm: Let $\varphi = \varphi^*$. Then $\varphi \underset{>}{\geq} 0 \Leftrightarrow$ all eigenvalues of φ are $\underset{>}{\geq} 0$.

Pf: Pick orthonormal $\mathcal{B}$ so $[\varphi]_{\mathcal{B}}$ diagonal. Now $[\varphi]_{\mathcal{B}} \underset{>}{\geq} 0 \Leftrightarrow$. $\square$

Cor: $A = A^* \geq 0 \Rightarrow \exists!\ B \geq 0$ with $B^2 = A$. **Def:** $B = \sqrt{A}$.

Pf: $\exists$: $A = UDU^* \Rightarrow \sqrt{A} = U\sqrt{D}\,U^*$.

$!$: exercise. $\square$

Prop: $A > 0 \Rightarrow \nu(x) = \sqrt{x^*Ax}$ is a norm.

Pf: $\nu(x) = \| A^{1/2} x\|_2$; apply HW2 #3: $\nu = \mu \circ \varphi$ for $\mu = \|\cdot\|_2$ and $\varphi = A^{1/2}$. $\square$

Def: $A \in \mathbb{C}^{m\times n}$ has <u>modulus</u> $|A| = \sqrt{A^*A} \geq 0$. $|A| \in \mathbb{C}^{n\times n}$

Needs: $(A^*A)^* = A^*A$ ✓

$\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0 \ \forall x \in \mathbb{C}^n$ ✓

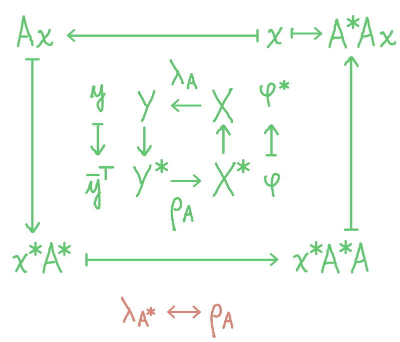
Prop: $\| |A| x\| = \|Ax\| \ \forall x \in \mathbb{F}^n$.

Pf: $\| |A|x\|^2 = \langle |A|x, |A|x\rangle = \langle |A|^* |A| x, x\rangle$

$= \langle |A|^2 x, x\rangle$ since $|A|$ hermitian

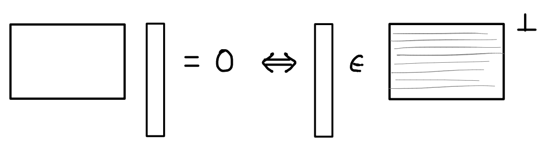
$= \langle A^*A x, x\rangle$ by def. of $|A|$

$= \langle Ax, Ax\rangle = \|Ax\|^2$. $\square$



$\lambda_{A^*} \leftrightarrow \rho_A$

Cor: $\ker A = \ker |A| = \operatorname{im}(|A|)^\perp$.



Pf: $=_1$: $\|Ax\| = 0 \Leftrightarrow \| |A| x\| = 0$.

$=_2$: $\ker T = (\operatorname{im} T^*)^\perp$, and $|A|^* = |A|$. $\square$

Note: Cor $\Rightarrow$ • $|A|$ = orthogonal projection followed by ... some $\cong$ of $\operatorname{im}|A|$ ...

• $\operatorname{im} A \overset{|A|}{\xleftarrow{\sim}} (\ker A)^\perp$. by universal property of quotients: • $|A|$ is 0 on $\ker A$. • $(\ker A)^\perp = \operatorname{im}|A|$

Goal: geometry of A in terms of $|A|$

Def: The <u>singular values</u> of A are the eigenvalues of $|A|$

$\Updownarrow$ $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$, where A^*A has eigenvalues $\lambda_1, \dots, \lambda_n$

<u>Running assumptions</u> $Y \xleftarrow{A} X$ homomorphism of hermitian v.s. $/\mathbb{F}$

- singular values $\underbrace{\sigma_1,\dots,\sigma_r}_{\neq 0}, \underbrace{\sigma_{r+1},\dots,\sigma_n}_{=0}$ of A $\quad r = \cancel{?}$ rank A
- $v_1,\dots,v_n$ orthonormal basis of X so that $|A|$ scales v_i by σ_i

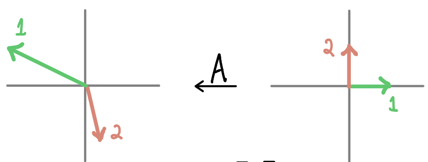
$\Rightarrow v_{r+1},\dots,v_n$ " " " $\ker A$.

Geometry of $|A|$: kill $\ker A$, rescale $v_1,\dots,v_r$ by positive $\sigma_1,\dots,\sigma_r$.

<u>Thm</u> (polar decomposition) $\quad A \in \mathbb{F}^{n\times n} \Rightarrow A = U|A|$ for some $U \in O_n(\mathbb{F})$ unique if $A \in GL_n$

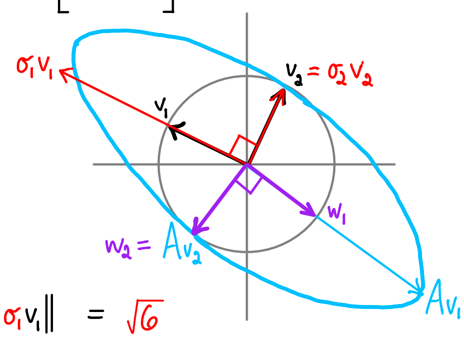
<u>Pf</u>: Follows from $\overset{\text{to come}}{\text{SVD}}$. □ $\quad z \in \mathbb{C} \Rightarrow z = e^{i\theta}|z| \quad \theta \in \mathbb{R}$ $\quad$ polar coordinates

<u>E.g.</u> $A = \begin{bmatrix} -2 & \frac{1}{5}(4-\sqrt{6}) \\ 1 & \frac{1}{5}(-2-2\sqrt{6}) \end{bmatrix} \approx \begin{bmatrix} -2 & .31 \\ 1 & -1.38 \end{bmatrix}$ $\quad A^*A = \frac{1}{5}\begin{bmatrix} -2 & 1 \\ 4-\sqrt{6} & -2-2\sqrt{6} \end{bmatrix}\frac{1}{5}\begin{bmatrix} -2 & 4-\sqrt{6} \\ 1 & -2-2\sqrt{6} \end{bmatrix} = \begin{bmatrix} 4+1 & -2 \\ \frac{-8+\cancel{2\sqrt{6}}-2-\cancel{2\sqrt{6}}}{5} & \frac{16-\cancel{8\sqrt{6}}+6+4+\cancel{8\sqrt{6}}+24}{25} \end{bmatrix}$



$$p_{A^*A}(t) = (5-t)(2-t)-4 = t^2-7t+6 = (t-6)(t-1) \quad \Leftarrow \quad = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

$\sigma_1^2 = 6$: $v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\qquad \sigma_2^2 = 1$: $v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$



$Av_1 = \frac{1}{\sqrt{5}}\left(\begin{bmatrix} 4 \\ -2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 4-\sqrt{6} \\ -2-2\sqrt{6} \end{bmatrix}\right) = \frac{1}{5\sqrt{5}}\begin{bmatrix} 24-\sqrt{6} \\ -12-2\sqrt{6} \end{bmatrix} \approx \begin{bmatrix} 1.93 \\ -1.51 \end{bmatrix}$

$Av_2 = \frac{1}{\sqrt{5}}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} 4-\sqrt{6} \\ -2-2\sqrt{6} \end{bmatrix}\right) = \frac{1}{5\sqrt{5}}\begin{bmatrix} -2-2\sqrt{6} \\ 1-4\sqrt{6} \end{bmatrix} \approx \begin{bmatrix} -.62 \\ -.79 \end{bmatrix}$

$\|Av_1\| = \|\sigma_1 v_1\| = \sqrt{6}$

$\|Av_2\| = \|\sigma_2 v_2\| = 1$

<u>Lemma</u>: $w_k = \frac{1}{\sigma_k}Av_k \Rightarrow w_1,\dots,w_r$ orthonormal in Y.

<u>Pf</u>: $\langle \sigma_i w_i, \sigma_j w_j\rangle = \langle Av_i, Av_j\rangle = \langle A^*Av_i, v_j\rangle = \langle \sigma_i^2 v_i, v_j\rangle = \sigma_i^2\langle v_i, v_j\rangle = \begin{cases} 0 & i\neq j \\ \sigma_i^2 & i=j \end{cases}$. □

<u>Thm</u>: A has <u>Schmidt decomposition</u>: $A = \sum_{k=1}^r \sigma_k w_k v_k^*$ $\quad v_k^*: x \mapsto \langle x, v_k\rangle$, so coeff. on w_k in Ax is $\sigma_k\langle x, v_k\rangle$

<u>Pf</u>: R.H.S. applied to v_i is $\sigma_i w_i = Av_i$ if $i \le r$ and 0 if $i > r$. □

<u>Lemma</u>: $A = \sum_{k=1}^r \sigma_k w_k v_k^*$ for orthonormal $v_1,\dots,v_r$ and $w_1,\dots,w_r \Rightarrow v_1,\dots,v_r$ eigenvectors of A^*A.

so this is a Schmidt decomposition

<u>Pf</u>: $A^* = \sum_{k=1}^r \sigma_k v_k w_k^*$, so in A^*A all terms vanish except $\sigma_i^2 v_i \overset{=1}{(w_i^* w_i)} v_i^*$

since $\langle w_i^*, w_j\rangle = \delta_{ij}$. But $A^*A = \sum_{i=1}^r \sigma_i^2 v_i v_i^* \Leftrightarrow A^*A v_i = \sigma_i^2 v_i \ \forall i$. □

<u>Thm</u>: A has a <u>reduced singular value decomposition</u> $A = \tilde{W}\tilde{\Sigma}\tilde{V}^*$ with

- $\tilde{V}^*$ $r\times n$ orthonormal rows; row i = (eigenvector of A^*A with eigenvalue σ_i^2)*
- $\tilde{\Sigma}$ $r\times r$ diagonal, entries $\sigma_1,\dots,\sigma_r$
- $\tilde{W}$ $m\times r$ orthonormal cols

$$\begin{bmatrix} | & & | \\ w_1 & \cdots & w_r \\ | & & | \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}\begin{bmatrix} -v_1^*- \\ \vdots \\ -v_r^*- \end{bmatrix}$$

<u>Pf</u>: Schmidt decomposition; rows of $\tilde{V}^*$ by Lemma. □