14. Perturbation theory

Vague:
- has >1 meaning
- different classes of A $\rightsquigarrow$ different behavior
- looking for continuity, diff'ability, bounds, ...

Given $A \in \mathbb{C}^{n\times n}$, how do $\Lambda(A) = \underline{\text{spectrum}}$ of $A = \{\text{roots of } p_A\}$ (multiset) and $\Lambda(\tilde{A})$ relate if $\tilde{A} = A + E$ with $\|E\| < \varepsilon$?

E.g. $A = 0 \Rightarrow |\lambda| \leq \|E\|$ for $\lambda \in \Lambda(\tilde{A}) = \Lambda(E)$.

Works for operator norm $\|\cdot\|$. What about other choices ?

Def: V: $\mathbb{C}^{m\times k}$ $\quad \mathbb{C}^{k\times n}$ $\quad \mathbb{C}^{m\times n}$

norms $\mu \quad \nu \quad \rho$ are consistent if $\rho(AB) \leq \mu(A)\nu(B) \;\; \forall A, B$

$\mu = \nu = \rho$ on $\mathbb{C}^{n\times n}$: ν is consistent

E.g. • $\|\cdot\|_2 = \sqrt{\operatorname{tr}(A^*A)}$ is consistent (HW 4)

• $\nu_\infty(A) = \max\limits_{i,j} |a_{ij}|$ norm, but $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \nu_\infty(A^2) = 2 > 1 = \nu_\infty(A)\nu_\infty(A)$ not consistent

Prop: $\|\cdot\|$ consistent on $\mathbb{C}^{n\times n} \Rightarrow \exists$ norm ν on $\mathbb{C}^n$ consistent with $\|\cdot\|$: $\nu(Ax) \leq \|A\|\nu(x)$

Pf: Fix $v \in \mathbb{C}^n \setminus \{0\}$. Set $\nu(x) = \|xv^T\|$. ($n\times n$ matrix)

• ν is a norm by HW 2.3: $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n\times n} \xrightarrow{\|\cdot\|} \mathbb{C}$ via $x \mapsto xv^T \mapsto \|xv^T\|$.

• ν consistent with $\|\cdot\|$: $\nu(Ax) = \|Axv^T\| \leq \|A\|\,\|xv^T\| = \|A\|\nu(x)$. $\square$

Def: $A \in \mathbb{C}^{n\times n}$ has spectral radius $\rho(A) = \max\{|\lambda| \mid \lambda \in \Lambda(A)\}$.

Thm: $\|\cdot\|$ consistent on $\mathbb{C}^{n\times n} \Rightarrow \rho(A) \leq \|A\| \;\; \forall A \in \mathbb{C}^{n\times n}$.

Pf: Pick ν consistent with $\|\cdot\|$ by Prop. If $\lambda \in \Lambda(A)$ and $Ax = \lambda x$ then

$|\lambda|\cancel{\nu(x)} = \nu(\lambda x) = \nu(Ax) \leq \|A\|\cancel{\nu(x)}$ $\quad x \neq 0$ $\quad$ so $|\lambda| \leq \|A\|$. $\square$

E.g. $A = 0 \Rightarrow |\lambda| \leq \|E\|$ for $\lambda \in \Lambda(\tilde{0})$

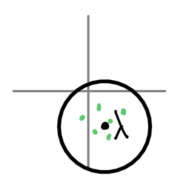
$\tilde{0} = 0 + E = E$, so $\|E\| \sim 10^{-8}$ (say) $\Rightarrow \rho(\tilde{0}) < \sim 10^{-8}$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \quad E = \begin{bmatrix} & & & \\ & & 0 & \\ & & & \\ \varepsilon & & & \end{bmatrix} \quad \text{so} \quad \tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} \Lambda(\tilde{A}) = \{\pm\varepsilon^{1/4}, \pm i\varepsilon^{1/4}\} \\ \varepsilon \sim 10^{-8} \Rightarrow \rho(\tilde{A}) \sim 10^{-2} \end{array}$$

different behavior. Nonetheless:

Thm: Locations of eigenvalues are continuous under perturbation:

if $\lambda \in \Lambda(A)$ has algebraic multiplicity $a(\lambda) = m$, $\|\cdot\|$ any norm, and $\varepsilon \ll 1$, then $\exists\, \delta > 0$ such that $\|E\| < \delta \Rightarrow B_\varepsilon(\lambda) \supseteq$ exactly m eigenvalues of $\tilde{A} = A + E$.



$a(\lambda) = 6$

Pf uses:

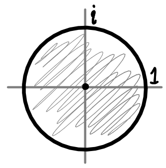
Rouché's thm: Suppose $\Omega \subseteq \mathbb{C}$ open and $\phi, f : \overline{\Omega} \to \mathbb{C}$. Assume

Math 333

- f analytic on $\overline{\Omega}$ (Taylor series at $z \to f(z)$ $\forall z \in \overline{\Omega}$) as is ϕ e.g. f, ϕ polynomials
- $\partial\Omega$ is a simple closed curve (from Math 222: $\simeq S^1$)
- $|\phi(z)| < |f(z)|$ $\forall z \in \partial\Omega$.

Then f and $f+\phi$ have the same #roots in Ω, counted with multiplicity.

E.g. $f(z) = z^n$ on $\overline{\Omega} = B_1(0)$



$\phi(z) = \varepsilon z^j$ for any $j \in \mathbb{N}$

$\Rightarrow z^n + \varepsilon z^j$ has n roots in Ω whenever $\varepsilon < 1$.

General: $z^n + \varepsilon_j z^j + \cdots + \varepsilon_1 z + \varepsilon_0$ has $\underbrace{\text{exactly } n}_{\text{all, if } j \leq n}$ roots in Ω if $\sum |\varepsilon_k| < 1$.

Can be used to prove $\mathbb{C} = \overline{\mathbb{C}}$

Lemma: $A \mapsto p_A$ (char poly) is continuous function $\mathbb{C}^{n\times n} \to P_n = \{\text{polynomials of degree} \leq n\}$.

Pf: coeffs of p_A are polynomial functions of the entries a_{ij}. □

Pf of Continuity Thm: Choose ε so $\overline{\Omega} = \overline{B_\varepsilon}(\lambda)$ has no eigenvalues of A other than λ.

Lemma $\Rightarrow$ $p_{\tilde{A}} \to p_A$ as $\tilde{A} \to A$

$\Rightarrow \underbrace{p_{\tilde{A}} - p_A}_{\phi_E(z)} \to 0$ for $z \in \overline{\Omega} \supseteq \partial\Omega$ as $E \to 0$.

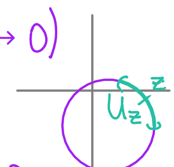
$\partial\Omega$ compact $\xRightarrow[f(z) = p_A(z) \text{ not } 0 \text{ on } \partial\Omega]{}$ $|f(z)|$ bounded away from 0: achieves min $\alpha \neq 0$ at $z_0 \in \partial\Omega$.

$\partial\Omega$ compact (closed + bounded) $\Rightarrow$ $\exists\, \delta > 0$ with $|\phi_E(z)| < \alpha$ whenever $\|E\| < \delta$.

Rouché's thm $\Rightarrow$ $f + \phi_E = p_{\tilde{A}}$ has same #roots in Ω as f does. □

Detail: compact $\Leftrightarrow$ every open cover has finite subcover [Heine–Borel]

- for each $z \in \partial\Omega$ pick δ_z with $|\phi_E(z)| < \alpha$ whenever $\|E\| < \delta_z$ (since $\phi_E(z) \to 0$)
- ϕ_E continuous $\Rightarrow$ $\phi_E(w) < \alpha$ $\forall w \in$ open nbd U_z of z whenever $\|E\| < \delta_z$
- $\partial\Omega$ compact $\Rightarrow$ finitely many U_z cover; take $\delta = \min$ of corresponding δ_z.



Maybe the δ_z accumulate around 0, but: