16. Lie algebras

<u>Def</u>: For $X \subseteq$ vector space $V/\mathbb{R}$, the <u>tangent space</u> at $p \in X$ is

$$T_pX = \{\, \gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \to X \text{ differentiable with } \gamma(0) = p \,\}.$$

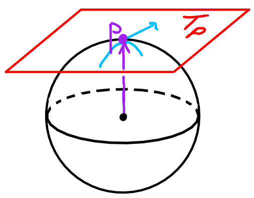
= initial velocities of differentiable paths in X through p

<u>E.g.</u> $X = S^{n-1} \subseteq \mathbb{R}^n$. $\gamma(t) \subseteq S^{n-1} \iff \gamma(t) \cdot \gamma(t) = 1$

$$\underbrace{\gamma_1(t)^2 + \cdots + \gamma_n(t)^2}_{\parallel} \Rightarrow 2\gamma_1(t)\gamma_1'(t) + \cdots + 2\gamma_n(t)\gamma_n'(t) = 0$$

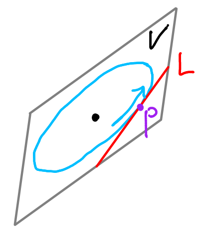
$$\iff \gamma(t) \cdot \gamma'(t) = 0$$

$$\iff \gamma'(t) \in \gamma(t)^{\perp}.$$



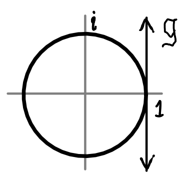
Thus $T_p S^{n-1} \subseteq p^{\perp}$. But $L \perp p \Rightarrow L = T_p(W \cap X)$ for $W = \mathrm{span}(L, p)$.

So $T_p S^{n-1} = p^{\perp}$.



<u>Def</u>: Subgroup $G \subseteq GL_n\mathbb{F}$ that is a manifold has <u>Lie algebra</u> ("Lee") $\mathfrak{g} = T_I G$. \mathfrak{g}

<u>E.g.</u> • $G = U_1 \cong O_2(\mathbb{R})$



$\Rightarrow \mathfrak{g} = \mathrm{span}_{\mathbb{R}}(i) = i\mathbb{R} = \mathfrak{u}_1$ $\quad \dfrac{d}{dt} e^{kit}\Big|_{t=0} = ki \quad k \in \mathbb{R}$

$\cong \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \mathfrak{o}_2$

• $G = GL_n(\mathbb{F}) \Rightarrow \mathfrak{g} = \mathfrak{gl}_n(\mathbb{F}) = M_n(\mathbb{F})$

<u>Pf</u>: $GL_n = M_n \setminus \underbrace{\{\det = 0\}}_{\text{closed}} \Rightarrow GL_n$ open $\Rightarrow T_p GL_n = T_p M_n$.

<u>Def</u>: $A \in M_n(\mathbb{F})$ has <u>(matrix) exponential</u> $\exp(A) = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^k}{k!} + \cdots$

convergence issues: sequences in $M_n(\mathbb{F}) \cong \mathbb{R}^m$ for some m, often using norm $\|A\|_2$

<u>Prop</u>: $\gamma : \mathbb{R} \to M_n(\mathbb{F})$ via $\gamma(t) = e^{tA}$ is differentiable with $\gamma'(t) = A\gamma(t) = \gamma(t)A$.

<u>Pf</u>: Termwise and entrywise differentiate

$$e^{tA} = I + tA + t^2\frac{A^2}{2!} + \cdots + t^k\frac{A^k}{k!} + \cdots \text{ to get}$$

$$(e^{tA})' = 0 + A + tA^2 + t^2\frac{A^3}{2!} + \cdots + t^k\frac{A^{k+1}}{k!} + \cdots = Ae^{tA} = e^{tA}A. \ \square$$

<u>Prop</u>: $AB = BA \Rightarrow e^A e^B = e^{A+B}$.

<u>Pf</u>: Binomial thm $\Rightarrow \dfrac{(A+B)^k}{k!} = \displaystyle\sum_{i+j=k} \frac{A^i}{i!}\frac{B^j}{j!}$.

These are the deg k terms in $e^A e^B$ and e^{A+B}. $\square$

<u>Cor</u>: $\exp : \mathfrak{gl}_n(\mathbb{F}) \to GL_n(\mathbb{F})$.

<u>Pf</u>: $e^A e^{-A} = e^{A-A} = e^0 = I$. $\square$

To compute more examples of $\mathfrak{g}$:

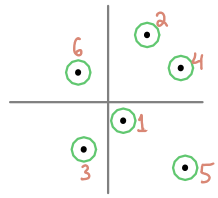
Thm: tr = det$'$: $\left.\frac{d}{dt}\right|_{t=0} \left(\det \gamma(t)\right) = \operatorname{tr}\left(\gamma'(0)\right)$ if $\gamma(0) = I$.

Pf: Suppose $\gamma(t)$ is a path in $M_n\mathbb{F}$ with distinct eigenvalues $\lambda_1(t), \dots, \lambda_n(t)$ in $\mathbb{C}$ for $t \neq 0$.

Q. How can consistent choices of numbering $\lambda_1, \dots, \lambda_n$ be made for varying t?

How do we know which eigenvalue of $\gamma(\tilde{t})$ is supposed to correspond to (say) $\lambda_1(t)$?

A. Continuity thm!



$\Lambda(t)$

$\Lambda(\tilde{t}) \subseteq \bigcup \odot$

number however you like

Then $\det \gamma(t) = \lambda_1(t) \cdots \lambda_n(t)$

$$\Rightarrow \left(\det \gamma(t)\right)' = \lambda_1'(t) \frac{\det \gamma(t)}{\lambda_1(t)} + \dots + \lambda_n'(t) \frac{\det \gamma(t)}{\lambda_n(t)}$$

$$\Rightarrow (\det{}'\gamma)(0) = \lambda_1'(0) \cdot 1 + \dots + \lambda_n'(0) \cdot 1 \qquad \text{if } \gamma(0) = I.$$

Even if $|\Lambda(\gamma(t))| < n$, same argument works by replacing $\gamma(t)$ with

$$\gamma_\varepsilon(t) = \gamma(t) + \varepsilon D \quad \text{for } D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & \ddots \\ & & & n \end{bmatrix} \text{ and taking } \lim_{\varepsilon \to 0}.$$

Uses

- $|\Lambda(A)| = n \Rightarrow |\Lambda(A+E)| = n \quad \forall \, \|E\| \ll 1$ ← why perturbation theory before Lie algebras

  holds by continuity thm $\Rightarrow |\Lambda(\gamma_\varepsilon(t))| = n \quad \forall \, \varepsilon \neq 0$ and $t \ll 1$: $A = I + \varepsilon D$, $E = \gamma(t) - I$

- $(\det \gamma_\varepsilon)'(0) \xrightarrow{\varepsilon \to 0} (\det \gamma)'(0)$

  holds because $M_n\mathbb{F} \times \mathbb{R} \to \mathbb{F}$

  $A, \varepsilon \mapsto \det(A + \varepsilon D)$ is differentiable (it is polynomial)

- $\gamma_\varepsilon'(t) = \gamma'(t)$

  holds because $\gamma_\varepsilon'(t) = \left(\gamma(t) + \varepsilon D\right)' = \gamma'(t) + \cancelto{0}{(\varepsilon D)'} \quad \forall \varepsilon, t.$ □