17. E.g. $G = SL_n(\mathbb{F}) = \{A \in GL_n(\mathbb{F}) \mid \det A = 1\}$

$\Rightarrow \mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid \operatorname{tr} A = 0\}$.

Pf: $\det \gamma(t) \equiv 1$ if $\gamma \subseteq SL_n \Rightarrow \mathfrak{g} \subseteq \mathfrak{sl}_n(\mathbb{F})$ by $\operatorname{tr} = \det'$. exercise with exp

For $\supseteq$, let $A \in \mathfrak{sl}_n(\mathbb{F})$. Then $\det e^{tA} = e^{\operatorname{tr}(tA)} = e^0 = 1 \Rightarrow e^{tA} \subseteq SL_n$.

Thus $e^{tA}$ realizes $A$ as $(e^{tA})'\big|_{t=0}$. $\square$

Lemma: $(\gamma(t)^*)' = \gamma'(t)^*$. Transpose and conjugation both commute with derivative.

Prop: $\beta, \gamma : (-\varepsilon, \varepsilon) \to M_n$ differentiable $\Rightarrow (\beta\gamma)' = \beta'\gamma + \beta\gamma'$.

Pf: Entry by entry, sum by sum, this is the usual product rule. $\square$

E.g. $\left(\underbrace{\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}}_{\beta(t)}\underbrace{\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}}_{\gamma(t)}\right)' = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}' = \begin{bmatrix} a_1'a_2 + a_1a_2' + b_1'c_2 + b_1c_2' & \cdots \\ \vdots & \ddots \end{bmatrix}$

$= \begin{bmatrix} a_1'a_2 + b_1c_2' & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} b_1'c_2 + a_1a_2' & \cdots \\ \vdots & \ddots \end{bmatrix}$

E.g. $G = O_n(\mathbb{F}) \Rightarrow \mathfrak{g} = \mathfrak{o}_n(\mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid A^* = -A\}$.

Pf: $\subseteq$: product rule + Lemma: $\gamma(t) \subseteq O_n(\mathbb{F}) \Rightarrow$

$\gamma(t)\gamma(t)^* \equiv I \Rightarrow 0 = \gamma'(t)\gamma(t)^* + \gamma(t)\gamma'(t)^* \overset{t=0}{=} \gamma'(0)I + I\gamma'(0)^*$.

$\supseteq$: $\Leftrightarrow \exists\, \gamma(t) \subseteq O_n(\mathbb{F})$ with $\gamma'(0) = A$ whenever $A^* = -A$.

Again use $\gamma(t) = e^{tA}$, which has

- $\gamma'(0) = A$ by Prop from last time
- $e^{tA}(e^{tA})^* = e^{tA}e^{tA^*} = e^{tA}e^{-tA} = e^0 = I$. $\square$

Prop: $\dim X = \dim T_pX$ for any $p$ in manifold $X$.

Pf: $\{\gamma : (-\varepsilon,\varepsilon) \to X_\alpha\} \leftrightarrow \{\gamma : (-\varepsilon,\varepsilon) \to U_\alpha\}$. Use that (diffeomorphism to image)$'$ is injective. $\square$

Cor: $\dim O_n(\mathbb{F}) = \cancel{?}$

- $\mathbb{F} = \mathbb{R}$: $A^T = -A \Rightarrow \boxed{\begin{matrix} 0 & & d \\ & \ddots & \\ 0 & & 0 \end{matrix}} \Rightarrow n^2 = 2d + n$

$\Rightarrow d = \frac{1}{2}(n^2 - n) = \binom{n}{2}$.

- $\mathbb{F} = \mathbb{C}$: $A^* = -A \Rightarrow \boxed{\begin{matrix} i\mathbb{R} & & 2d \\ & \ddots & \\ 0 & & i\mathbb{R} \end{matrix}} \Rightarrow \dim = 2d + n = n^2$.

Crucial question: Why are these subgroups of $GL_n$ manifolds in the first place?

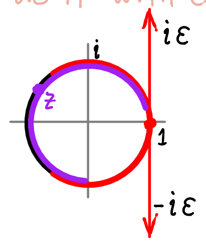
$\square$

**Thm:** Fix closed subgroup $G \subseteq GL_n(\mathbb{F})$ with Lie algebra $\mathfrak{g}$. Then

- $A \in \mathfrak{g} \Rightarrow e^A \in G$
- $B_\varepsilon = \{ A \in M_n \mid \|A\|_2 < \varepsilon \} \Rightarrow \exp_\varepsilon : \mathfrak{g} \cap B_\varepsilon \hookrightarrow G$ if $\varepsilon \ll 1$
  neighborhood of $I$ in $G$
- $G$ is a manifold with atlas $\{ g \cdot \exp_\varepsilon \mid g \in G \}$.

**Pf:** omitted, though we could do it with enough time. See Math 421, 603, 620

**E.g.** $G = U_1 = S^1$ $\quad i\varepsilon \in \mathfrak{u}_1$




$\varepsilon < \pi \Rightarrow \exp_\varepsilon : (-\varepsilon, \varepsilon) \hookrightarrow S^1$ nbd of 1
$t \mapsto e^{it}$

$|z| = 1 \Rightarrow z \cdot \exp_\varepsilon : (-\varepsilon, \varepsilon) \hookrightarrow S^1$ nbd of $z$
$t \mapsto z e^{it}$

Q. Why closed?

A. $\mathbb{Q} \subseteq \mathbb{R}$.

But that's not in $GL_1$, you complain? O.K. But $(\mathbb{R}, +) = \begin{bmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{bmatrix} \subseteq GL_2\mathbb{R}$ closed subgroup.

$x_{11} = 1$
$x_{22} = 1$
$x_{21} = 0$
intersection of polynomial level sets

To connect with previous units:

**Thm:** closed subgroup $H \subseteq G \Rightarrow G/H$ and $H \backslash G$ are manifolds.

**Pf:** omitted, and this would take more work; still doable, but not as elementary.

**E.g.**
- $\mathcal{F}\ell_n$ for $G = GL_n$ and $H = B_n^+ = $ upper-$\triangle$
- $G_k(\mathbb{F}^n)$ for $G = GL_n$ and $H = $ block upper-$\triangle$ with block sizes $k$ and $n-k$
- chains $\{V_d \subseteq V_e\}$ for $G = GL_n$ and $H = $ block upper-$\triangle$ with block sizes $d, e-d, n-e$

No need to fiddle with explicit charts.