18. Perron-Frobenius theory

Def: $A \geq B$ for real A, B of same size if $a_{ij} \geq b_{ij}$ $\forall i,j$.
$A \succ B$ ... $A \geq B$ and $a_{ij} > b_{ij}$ for some i,j.

E.g. $P \geq 0 \Leftrightarrow$ entrywise nonnegative
$>$ ... positive

Perron's Thm: $P \in \mathbb{R}^{n\times n}$ and $P > 0 \Rightarrow P$ has dominant eigenvalue $\lambda(P)$:

1. $\lambda(P) > 0$ and $Pv = \lambda(P)v$ for some $v > 0$;
2. $a(\lambda(P)) = 1$; algebraic multiplicity 1 and

for $\kappa \in \Lambda(P) \setminus \{\lambda(P)\}$:

3. $|\kappa| < \lambda(P)$ and
4. $Py = \kappa y$ and $y \neq 0 \Rightarrow y \not\geq 0$.

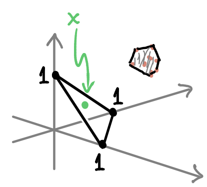
today: proof
next time: consequences

P_σ is some polytope in $\mathbb{R}^n_+$
$\lambda \gg 0 \Rightarrow P_\sigma$ entirely under $\lambda\sigma$
$\lambda \ll 1 \Rightarrow \lambda\sigma$ sufficiently under P_σ

Pf: Set $L(P) = \{\lambda \geq 0 \mid Px \geq \lambda x$ for some $x \succeq 0$ and $\mathbb{1}x = 1\}$.

Let $\mathbb{1} = [1 \cdots 1]$ so $\mathbb{1}x = \|x\|_1 = x_1 + \cdots + x_n$.

$Px \geq \lambda x \Rightarrow$ same for $\frac{x}{\|x\|_1} \Rightarrow$ assume $\mathbb{1}x = 1$.



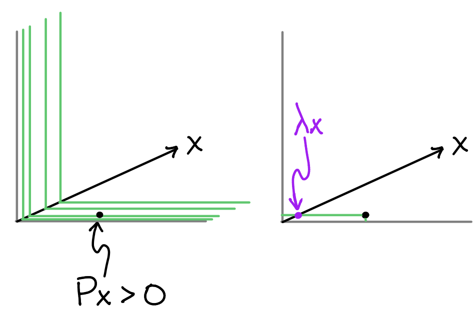
Lemma: $L(P)$ compact and has some $\lambda > 0$.

Pf: $x \in \mathbb{R}^n$ and $0 \neq x \geq 0 \Rightarrow Px > 0$.

$\lambda \to 0_+ \Rightarrow \lambda x \to 0 \Rightarrow \lambda x < \varepsilon \mathbb{1}^T$ eventually

$\Rightarrow \lambda x < Px$ "

$\Rightarrow \lambda \in L(P)$ " .



bounded: $b = b\mathbb{1}x \geq \mathbb{1}Px \geq \mathbb{1}\lambda x = \lambda\mathbb{1}x = \lambda$ as $\mathbb{1}x = 1$; $b = \|\mathbb{1}P\|_\infty \Rightarrow b = b\mathbb{1}x \geq$
$= \max$ entry of $\underbrace{\mathbb{1}P}_{\Sigma \text{ rows of } P}$

closed: $\lambda_k \to \lambda$ with $\lambda_k \in L(P)$ $\forall k \in \mathbb{N}$

$\Rightarrow \exists x_k$ with $Px_k \geq \lambda_k x_k$ $\forall k$; may as well assume $\mathbb{1}x_k = 1$.

$\{x_k\}_{k\in\mathbb{N}}$ has convergent subsequence since σ_{n-1} compact

$\sigma_{n-1} = \{x \in \mathbb{R}^n_{\geq 0} \mid \mathbb{1}x = 1\}$ simplex

$\Rightarrow$ can replace $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{x_k\}_{k\in\mathbb{N}}$ with subsequences to assume

$\lambda_k \to \lambda$ and $x_k \to x$

$\Rightarrow \lim_{k\to\infty} (Px_k \geq \lambda_k x_k)$ is $(Px \geq \lambda x) \Rightarrow \lambda \in L(P)$. $\square$

As $\lambda\sigma$ grows, its λx's lie beneath fewer Px's. $\lambda(P)$ is when the last Px works.

1. Set $\lambda(P) = \max L(P)$. Lemma $\Rightarrow \lambda(P) > 0$.

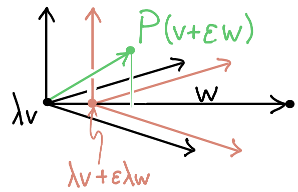
Claim: $\lambda(P) \in \Lambda(P)$. In fact, $Pv \geq \lambda(P)v$ for $v \succeq 0 \Rightarrow Pv = \lambda(P)v$.

Pf: Suppose $\lambda \in L(P)$, so $Pv \geq \lambda v$ for some $v \geq 0$. P moves $v+\varepsilon w$ to interior of $\mathbb{R}^n_{\geq \lambda v}$ (38)

Want: this $\rightsquigarrow$ $Pv \neq \lambda v \Rightarrow 0 \neq Pv - \lambda v =: w$
$\Rightarrow \lambda \neq \lambda(P)$

$\Rightarrow \varepsilon P w > 0 \; \forall \varepsilon > 0$, since $P > 0$

$\Rightarrow P(v+\varepsilon w) = Pv + \varepsilon P w > Pv = \lambda v + w$



$Pv > 0$ $\Rightarrow$ everything along $\xrightarrow{w}$ is > 0

$\geq \lambda v + \varepsilon \lambda w = \lambda(v+\varepsilon w)$ for $\varepsilon \leq \frac{1}{\lambda}$.

So $x = v + \varepsilon w \Rightarrow Px > \lambda x$

$\Rightarrow Px \geq \lambda' x$ for any $\lambda' > \lambda$ with $\lambda' - \lambda \ll 1$

$\Rightarrow \lambda \neq \lambda(P)$ by maximality. $\square$

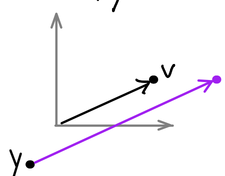
$\lambda(P) \in \Lambda(P) \Rightarrow Pv = \lambda(P)v$ for nonzero $v \geq 0 \overset{P>0}{\Rightarrow} \lambda(P)v > 0 \overset{\lambda(P)>0}{\Rightarrow} v > 0$.

2. $g(\lambda(P)) = 1$: $w \in E(\lambda(P))$ indep. of $v \Rightarrow$ line $\overleftrightarrow{vw}$ exits $\mathbb{R}^n_+ \Rightarrow E(\lambda(P)) \cap \partial \mathbb{R}^n_+ \setminus \{0\} \neq \varnothing$ $\nrightarrow\!\leftarrow$

For $a(\lambda(P)) = 1$ need: no $y \in \mathbb{R}^n$ with $Py = \lambda(P)y + \alpha v$ $(*)$ $\quad (P - \lambda(P))y \in \operatorname{span}(v)$

By $y \rightsquigarrow -y$ assume $\alpha > 0$

$y \rightsquigarrow y + \beta v$ assume $y > 0$.



$y + \beta v$ eventually in $\mathbb{R}^n_+$

$(*)$ and $v > 0 \Rightarrow Py > \lambda(P)y \Rightarrow Py > \lambda' y$ for any $\lambda' > \lambda(P)$ with $\lambda' - \lambda(P) \ll 1$ $\quad \lambda(P)$ maximal $\nrightarrow\!\leftarrow$

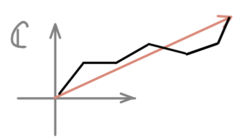
3. $\kappa \in \Lambda(P)$ with $Py = \kappa y$, both $/\mathbb{C}$

$\Rightarrow p_{i1} y_1 + \cdots + p_{in} y_n = \kappa y_i \quad \Rightarrow \quad p_{i1}|y_1| + \cdots + p_{in}|y_n| \overset{(**)}{\geq} |p_{i1} y_1 + \cdots + p_{in} y_n| = |\kappa||y_i|$

$\Rightarrow |\kappa| \in L(P) \Rightarrow |\kappa| \leq \lambda(P)$. But

$|\kappa| = \lambda(P) \Rightarrow \begin{bmatrix} |y_1| \\ \vdots \\ |y_n| \end{bmatrix} \in E(\lambda(P))$ by Claim $\Rightarrow = \alpha v$ and "=" in $(**)$

$\Rightarrow y_1, \ldots, y_n$ lie along a ray in $\mathbb{C}$



$\| \diagup \| = \Sigma \| \diagup \|$ $\Rightarrow$ $\diagup \parallel \diagup$

$y_i = \omega |y_i| \; \forall i$ for some $\omega \in U_1$

$\Rightarrow \omega \alpha v \in E(\lambda(P)) \Rightarrow \kappa = \lambda(P)$.

4. $P > 0 \Rightarrow P^T > 0 \Rightarrow \exists \varphi^T > 0$ in $E(\lambda(P^T))$ $\quad$ want this for $Py = \kappa y$

$\Rightarrow y \ngtr 0$ if $\varphi y = 0$.

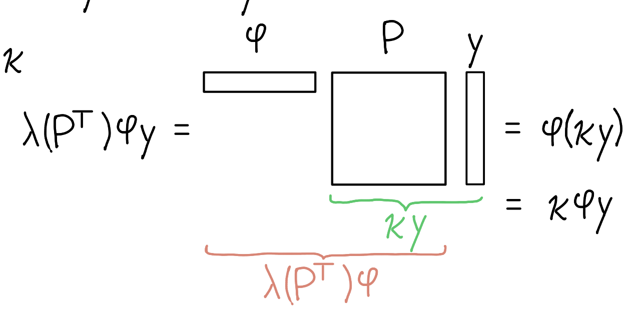
$Py = \kappa y$ and $P^T \varphi^T = \lambda \varphi^T \Rightarrow \lambda \varphi y = \varphi P y = \varphi(\kappa y) = \kappa \varphi y$

$\varphi P = \lambda \varphi$ $\qquad \Rightarrow \varphi y = 0$ if $\lambda \neq \kappa$

Take $\lambda = \lambda(P^T)$

Lemma: $= \lambda(P)$.

Pf: $(P - \lambda I)^T = P^T - \lambda I$. $\square$



$\square$