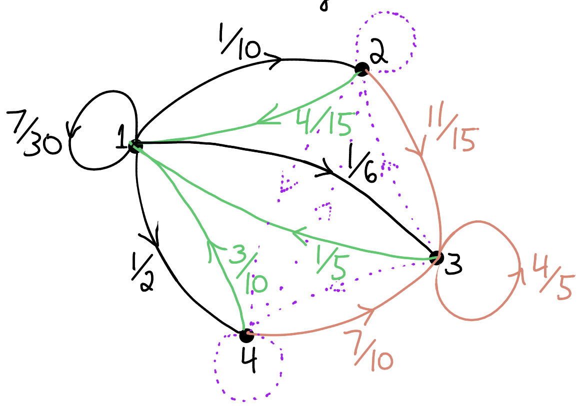
19. **General:** $A \in F^{n\times n} \longleftrightarrow$ labeled directed graph

$e_j \mapsto \sum_{i=1}^{n} a_{ij} e_i \longleftrightarrow$ labels $a_{1j}, \dots, a_{nj}$ on edges exiting vertex $j$

↖ $j^{th}$ column of $A$

E.g.




$$\begin{bmatrix} 7/30 & 4/15 & 1/5 & 3/10 \\ 1/10 & 0 & 0 & 0 \\ 1/6 & 11/15 & 4/5 & 7/10 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$$

**Def:** $P \geq 0$ stochastic if all col sums $= 1$ $(\mathbb{1}P = \mathbb{1})$ $(p_{1j} + \dots + p_{nj} = 1 \ \forall j)$ $11/15$ $4/5$ $7/10$

edge labels $\longleftrightarrow$ transition probabilities or fractions:

really the same { • How much of the stuff at $j$ moves to $i$ ?

• What chance does the thing at $j$ have of moving to $i$ ? finite Markov chain

instead of just one item, think of huge # of trials: place token and play

vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\longleftrightarrow$ where the tokens got placed in your trials $\longleftrightarrow$ stuff

Q. Iterate $x \mapsto Px \mapsto PPx \mapsto \cdots$

$x = e_j$: where does token sit after $k$ iterations? $\mathbb{P}(\text{token at } i \mid \text{started at } j)$

$x =$ arbitrary $\geq 0$ : how much stuff is at $i$ after $k$ iterations?

E.g. PageRank: $n$ webpages,

$p_{ij} = .85(\text{fraction of links } j \to i)$ $\Rightarrow$ where are you likely to be

$+15\%$ go somewhere random after $k$ iterations? Rank by $\mathbb{P}$.

$\Rightarrow P > 0$

**Thm:** $P > 0$ stochastic $\Rightarrow$ • $\lambda(P) = 1$ and

• $\forall\, 0 \neq x \geq 0$, $P^k x \to \alpha v$ for some $\alpha > 0$, where $Pv = v > 0$.

**Pf:** $P > 0 \Rightarrow P^T > 0 \Rightarrow$ same $\overset{\text{dominant}}{\text{eigenvalues}}$ by Lemma. But

$\mathbb{1}P = \mathbb{1} \Rightarrow \mathbb{1}$ is (unique!) dominant eigenvector of $P^T$ by Perron's Thm.

$\Rightarrow \lambda(P^T) = 1 = \lambda(P)$.

For $P^k$, first assume $P$ diagonalizable, so

$$x = \sum_{i=1}^{n} \alpha_i v_i \quad \text{for} \quad P v_i = \lambda_i v_i$$

$\Rightarrow P^k x = \sum_{i=1}^{n} \alpha_i \lambda_i^k v_i$. $\quad \lambda(P)$ dominant $\Rightarrow$ all other $|\lambda_i| < 1$

$\Rightarrow$ all summands $\to 0$ except $\lambda(P)$ term

$\Rightarrow P^k x \to \alpha v$ for some $\alpha$.

Why $\alpha > 0$?

$\mathbb{1}P = \mathbb{1} \Rightarrow \mathbb{1}P^k = \mathbb{1}$

$\Rightarrow \mathbb{1}x = \mathbb{1}P^k x \to \mathbb{1}\alpha v = \alpha \mathbb{1} v$

$\Rightarrow \alpha \leftarrow \dfrac{\mathbb{1}P^k x}{\mathbb{1}v} > 0$ because $0 \neq x \geqslant 0$ and $v > 0$.

For $P$ not diagonalizable: HW 5. $\square$

<u>Cor</u>: Markov chain transition probabilities all $> 0$

$\Rightarrow$ convergence to unique <u>stationary distribution</u>.

computable by iterative methods from any $x > 0$!

<u>Frobenius Thm</u>: $M \in \mathbb{R}^{n \times n}$ and $M \geqslant 0 \Rightarrow \exists\, \lambda(M) \in \Lambda(M)$ satisfying

1. $\lambda(M) \geqslant 0$, and $\exists\, 0 \neq v \geqslant 0$ $(v \succ 0)$ with $Mv = \lambda(M)v$;
2. $\kappa \in \Lambda(M)$ and $|\kappa| = \lambda(M) \Rightarrow \kappa = e^{2\pi i k/m} \lambda(M)$ for some $k, m \in \mathbb{N}$ with $m \leqslant n$;
3. $\kappa \in \Lambda(M) \Rightarrow |\kappa| \leqslant \lambda(M)$.

<u>Pf summary</u>: Express $M$ as lim of $P > 0$. $\square$