

Multilinear algebra

Def: Fix vector spaces V_1, \dots, V_r and W over F .

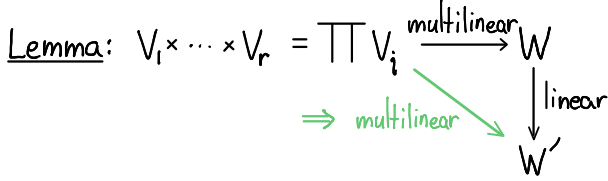
$f: V_1 \times \dots \times V_r \rightarrow W$ multilinear if

$$f(\dots, v_{i-1}, \alpha v_i + v_i', v_{i+1}, \dots) = \alpha f(\dots, v_{i-1}, v_i, v_{i+1}, \dots) + f(\dots, v_{i-1}, v_i', v_{i+1}, \dots)$$

$$\forall i \quad \forall v_i, v_i' \in V_i \quad \forall \alpha \in F \quad \text{with } v_j \text{ fixed for } j \neq i.$$

E.g. • $A \in F^{m \times n} \Rightarrow (v, w) \mapsto vAw$ for $v \in F_{row}^m$ and $w \in F_{col}^n$ bilinear

• $V_i = F^n \quad \forall i = 1, \dots, n$ and $(v_1, \dots, v_n) \mapsto \det [v_1 \dots v_n]$

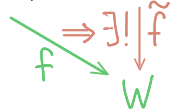


Interpretation: {vector spaces W with multilinear $\prod V_i \rightarrow W$ }

forms a category: Obj + Mor

Def: The tensor product of V_1, \dots, V_r is a universal such thing:

multilinear $t: V_1 \times \dots \times V_r \rightarrow T$ such that $\forall f \exists! \downarrow \tilde{f}$ with $f = \tilde{f} \circ t$



"the most general multilinear map from $\prod V_i$ "

Thm: T exists. Notation: $T = \bigotimes_{i=1}^r V_i = V_1 \otimes \dots \otimes V_r$

$$t(v_1, \dots, v_r) = v_1 \otimes \dots \otimes v_r$$

Q. Does every element of $V_1 \otimes \dots \otimes V_r$ have the form $v_1 \otimes \dots \otimes v_r$?

A. No. Key example: $w \in W = F_{col}^m \Rightarrow w \otimes \varphi = w\varphi$ has rank 1

very general $\varphi \in V^* = F_{row}^n$

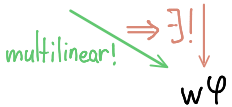
$$i \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes [0 \dots 0 \ 1 \ 0 \dots 0] = i \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & 1 \end{bmatrix}$$

φ

$$\left(x \mapsto x_i e_i = i \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \in \text{basis } B.$$

But $W \otimes V^* = \text{Hom}(V, W) = F^{m \times n}$, and lots of matrices

$(w, \varphi) \mapsto w \otimes \varphi \mapsto (x \mapsto \varphi(x)w)$ have rank > 1 .



Pf: Construct T by "freely" multiplying elements of V_1, \dots, V_r because

" " " " is multilinear.

$$\dots \otimes (v_i + v_i') \otimes \dots = (\dots \otimes v_i \otimes \dots) + (\dots \otimes v_i' \otimes \dots)$$

Want:

$$\dots \otimes (\alpha v_i) \otimes \dots = \alpha (\dots \otimes v_i \otimes \dots)$$

Set $T = M/N$ for $M = \text{span}((v_1, \dots, v_r) \mid v_i \in V_i \ \forall i)$

$$N = \text{span}((\dots, \alpha v_i + v_i', \dots) - \alpha(\dots, v_i, \dots) - (\dots, v_i', \dots))$$

$\prod V_i \hookrightarrow M$ map of sets — not multilinear, but

$\prod V_i \hookrightarrow M \rightarrow M/N$ multilinear by construction.

Suppose $\prod V_i \xrightarrow{\quad} M$ Then \downarrow because $\prod V_i$ is a basis of M (!)
 $\quad \quad \quad \searrow f \quad \downarrow$ But f multilinear $\Rightarrow N \subseteq \ker(\downarrow)$.
 $\quad \quad \quad \quad \quad W$

Universal property of quotients $\Rightarrow M \xrightarrow{\quad} W$ induces unique $M/N \xrightarrow{\quad} W = T$. \square
 How to compute?

Lemma: $B \subseteq V$ independent iff \exists linear $\{\varphi_b: V \rightarrow F\}_{b \in B}$ with $\varphi_b(b) = \delta_{b,b}$. *f true but not needed*

Pf: $\sum_{b \in B} \alpha_b b = 0 \Rightarrow 0 = \varphi_{b'}(\sum_{b \in B} \alpha_b b) = \alpha_{b'} \quad \forall b' \in B$. \square

Thm: B_i basis for $V_i \Rightarrow B_1 \times \dots \times B_r \xrightarrow{\sim}$ basis B for T .
 $(b_1, \dots, b_r) \mapsto b_1 \otimes \dots \otimes b_r$

E.g. $\mathbb{R}^2 \otimes \mathbb{R}^3$ has basis $e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_2 \otimes e_2, e_2 \otimes e_3$

Pf: Multilinear map on $\prod V_i$ determined by values on $\prod B_i$:

$v_i = \sum_j \alpha_j b_j^i \Rightarrow f(v_1, \dots, \text{stuff}) = \sum_j \alpha_j f(b_j^1, \dots, \text{stuff})$ and similarly for $i > 1$.

Thus $B = \{b_1 \otimes \dots \otimes b_r \mid b_i \in B_i \forall i\}$ spans T . *Need independence.*

Suppose $f_i: V_i \rightarrow F \quad \forall i$. Set $f = f_1 \otimes \dots \otimes f_r: \prod V_i \rightarrow F$ *product in F*

f multilinear \Rightarrow induces $\tilde{f}: T \rightarrow F$. $(v_1, \dots, v_r) \mapsto f_1(v_1) \dots f_r(v_r)$

Take $f_i = b_i^* \in B_i^*$ dual basis, so

$b_i^*(b_i) = 1$ but $b_i^*(b'_i) = 0$ when $b'_i \in B_i \setminus \{b_i\}$.

Then $b \in B \rightsquigarrow \tilde{f}_b: T \rightarrow F$ satisfying

$b \mapsto 1$

$b' \mapsto 0$ for $b' \in B \setminus \{b\}$. Use Lemma. \square

Cor: $\dim V_1 \otimes \dots \otimes V_r = (\dim V_1) \dots (\dim V_r)$. \square

E.g. $v \in \mathbb{R}^4 \quad w \in \mathbb{R}^2$ (get from class) $\Rightarrow v \otimes w =$

Universal property redux: $\{\text{multilinear } \prod_{i=1}^r V_i \rightarrow W\} \leftrightarrow \{\text{linear } \bigotimes_{i=1}^r V_i \rightarrow W\}$.

E.g. $\exists!$ isomorphism $V \otimes W \rightarrow W \otimes V$ Pf: HW5; use $V \times W \rightarrow W \otimes V$
 $v \otimes w \mapsto w \otimes v. \quad (v, w) \mapsto w \otimes v.$