20. Multilinear algebra

<u>Def</u>: Fix vector spaces $V_1, \dots, V_r$ and W over F.

$f : V_1 \times \cdots \times V_r \to W$ <u>multilinear</u> if

$$f(\dots, v_{i-1}, \alpha v_i + v_i', v_{i+1}, \dots) = \alpha f(\dots, v_{i-1}, v_i, v_{i+1}, \dots) + f(\dots, v_{i-1}, v_i', v_{i+1}, \dots)$$

$\forall i \quad \forall v_i, v_i' \in V_i \quad \forall \alpha \in F$ with v_j fixed for $j \neq i$.

<u>E.g.</u>
- $A \in F^{m \times n} \Rightarrow (v, w) \mapsto vAw$ for $v \in F^m_{\text{row}}$ and $w \in F^n_{\text{col}}$ <u>bilinear</u>
- $V_i = F^n \ \forall i = 1, \dots, n$ and $(v_1, \dots, v_n) \mapsto \det[v_1 \ \cdots \ v_n]$

<u>Lemma</u>:
$$\begin{tikzcd} V_1 \times \cdots \times V_r = \prod V_i \arrow[r, "\text{multilinear}"] \arrow[dr] & W \arrow[d, "\text{linear}"] \\ & W' \end{tikzcd}$$
$\Rightarrow$ multilinear

Interpretation: {vector spaces W with multilinear $\prod V_i \to W$} forms a <u>category</u>: Obj + Mor

<u>Def</u>: The <u>tensor product</u> of $V_1, \dots, V_r$ is a universal such thing:

multilinear $t : V_1 \times \cdots \times V_r \to T$ such that $\forall f \ \exists! \downarrow \tilde{f}$ with $f = \tilde{f} \circ t$

$$\begin{tikzcd} V_1 \times \cdots \times V_r \arrow[r, "t"] \arrow[dr, "f"'] & T \arrow[d, "\exists! \tilde{f}"] \\ & W \end{tikzcd}$$

"the most general multilinear map from $\prod V_i$"

<u>Thm</u>: T exists. Notation: $T = \bigotimes_{i=1}^{r} V_i = V_1 \otimes \cdots \otimes V_r$

$$t(v_1, \dots, v_r) = v_1 \otimes \cdots \otimes v_r$$

Q. Does every element of $V_1 \otimes \cdots \otimes V_r$ have the form $v_1 \otimes \cdots \otimes v_r$?

A. No. Key example: $w \in W = F^m_{\text{col}}$, $\varphi \in V^* = F^n_{\text{row}}$ $\Rightarrow w \otimes \varphi = w\varphi$ has rank 1

$$\underset{w}{i\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} \otimes \underset{\varphi}{\begin{array}{c} \qquad\quad j \\ \begin{bmatrix} 0 \cdots 0 & 1 & 0 \cdots 0 \end{bmatrix} \end{array}} = \begin{array}{c} \qquad j \\ i\begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{array}$$

$$\left(x \mapsto x_j e_i = i\begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \in \text{basis } B.$$

very general

But $W \otimes V^* = \mathrm{Hom}(V, W) = F^{m \times n}$, and lots of matrices have rank > 1.

$$\begin{tikzcd} (w, \varphi) \arrow[r, mapsto] \arrow[dr, "\text{multilinear!}"'] & w \otimes \varphi \arrow[r, mapsto] \arrow[d, "\exists!"] & (x \mapsto \varphi(x) w) \\ & w\varphi & \end{tikzcd}$$

<u>Pf</u>: Construct T by "freely" multiplying elements of $V_1, \dots, V_r$ because " " " " is multilinear.

Want:
$$\cdots \otimes (v_i + v_i') \otimes \cdots = (\cdots \otimes v_i \otimes \cdots) + (\cdots \otimes v_i' \otimes \cdots)$$
$$\cdots \otimes (\alpha v_i) \otimes \cdots = \alpha(\cdots \otimes v_i \otimes \cdots)$$

Set $T = M/N$ for $M = \mathrm{span}((v_1, \dots, v_r) \mid v_i \in V_i \ \forall i)$

$$N = \mathrm{span}((\dots, \alpha v_i + v_i', \dots) - \alpha(\dots, v_i, \dots) - (\dots, v_i', \dots))$$

$\prod V_i \hookrightarrow M$ map of sets — not multilinear, but

$\prod V_i \hookrightarrow M \twoheadrightarrow M/N$ multilinear by construction.

Suppose
$$\begin{tikzcd} \prod V_i \arrow[r] \arrow[dr, "f"'] & M \arrow[d] \\ & W \end{tikzcd}$$
Then $\downarrow$ because $\prod V_i$ is a basis of M (!)

But f multilinear $\Rightarrow$ $N \subseteq \ker(\downarrow)$.

Universal property of quotients $\Rightarrow$ $\begin{matrix} M \\ \downarrow \\ W \end{matrix}$ induces unique $\begin{matrix} M/N = T. \ \square \\ \downarrow \\ W \end{matrix}$

How to compute?

<u>Lemma</u>: $B \subseteq V$ independent iff $\exists$ linear $\{\varphi_b : V \to F\}_{b \in B}$ with $\varphi_b(b') = \delta_{b,b'}$. f true but not needed

<u>Pf</u>: $\sum_{b\in B} \alpha_b b = 0 \Rightarrow 0 = \varphi_{b'}\left(\sum_{b \in B} \alpha_b b\right) = \alpha_{b'} \quad \forall\, b' \in B.$ $\square$

<u>Thm</u>: B_i basis for $V_i \Rightarrow B_1 \times \cdots \times B_r \xrightarrow{\sim}$ basis B for T.

$$(b_1, \ldots, b_r) \mapsto b_1 \otimes \cdots \otimes b_r$$

<u>E.g.</u> $\mathbb{R}^2 \otimes \mathbb{R}^3$ has basis $e_1\otimes e_1$, $e_1 \otimes e_2$, $e_1 \otimes e_3$, $e_2 \otimes e_1$, $e_2 \otimes e_2$, $e_2 \otimes e_3$

<u>Pf</u>: Multilinear map on $\prod V_i$ determined by values on $\prod B_i$:

$v_1 = \sum_j \alpha_j b_1^j \Rightarrow f(v_1, \ldots\text{stuff}\ldots) = \sum_j \alpha_j f(b_1^j, \ldots\text{stuff}\ldots)$ and similarly for $i > 1$.

Thus $B = \{b_1 \otimes \cdots \otimes b_r \mid b_i \in B_i \ \forall i\}$ spans T. Need independence.

Suppose $f_i : V_i \to F \ \forall i$. Set $f = f_1 \otimes \cdots \otimes f_r : \prod V_i \to F$ product in F

f multilinear $\Rightarrow$ induces $\tilde{f} : T \to F$.
$$\begin{tikzcd} (v_1, \ldots, v_r) \arrow[r, mapsto] \arrow[d, "t"'] & f_1(v_1)\cdots f_r(v_r) \\ v_1 \otimes \cdots \otimes v_r \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Take $f_i = b_i^* \in B_i^*$ dual basis, so

$$b_i^*(b_i) = 1 \quad \text{but} \quad b_i^*(b_i') = 0 \quad \text{when } b_i' \in B_i \setminus \{b_i\}.$$

Then $b \in B \rightsquigarrow \tilde{f}_b : T \to F$ satisfying

$$b \mapsto 1$$

$$b' \mapsto 0 \quad \text{for } b' \in B \setminus \{b\}.$$ Use Lemma. $\square$

<u>Cor</u>: $\dim V_1 \otimes \cdots \otimes V_r = (\dim V_1) \cdots (\dim V_r)$. $\square$

<u>E.g.</u> $v \in \mathbb{R}^4 \quad w \in \mathbb{R}^2$ (get from class) $\Rightarrow v \otimes w =$

<u>Universal property redux</u>: $\{\text{multilinear } \prod_{i=1}^r V_i \to W\} \leftrightarrow \{\text{linear } \bigotimes_{i=1}^r V_i \to W\}$.

<u>E.g.</u> $\exists!$ isomorphism $V \otimes W \to W \otimes V$

$$v \otimes w \mapsto w \otimes v.$$

<u>Pf</u>: HW5; use $V \times W \to W \otimes V$

$$(v, w) \mapsto w \otimes v.$$ $\square$