21.

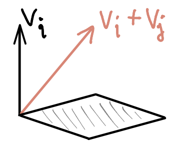
Exterior algebra

Def: An alternating operator $\varphi : \underbrace{V \times \cdots \times V}_{r} = V^{\times r} \to W$ is a multilinear map such that

$v_1, \dots, v_r$ linearly dependent $\Rightarrow \varphi(v_1, \dots, v_r) = 0$.

E.g. volume of parallelepiped on $v_1, \dots, v_n \in \mathbb{R}^n$: vol $= 0$ if $v_1, \dots, v_n$ linearly dependent

2.21 $\Rightarrow$ vol = det
- $v_i \rightsquigarrow \alpha v_i \Rightarrow \text{vol} \rightsquigarrow \alpha\, \text{vol} \quad \forall \alpha$ — including $\alpha < 0$: signed or oriented volume
- $v_i \rightsquigarrow v_i + v_j$ for $j \neq i \Rightarrow$ vol unchanged — same height $\Rightarrow$ same vol
- $\text{vol}(e_1, \dots, e_n) = 1$.



Def: r^{th} exterior power of V is a universal alternating operator:

alternating map $\Lambda^r : V^{\times r} \to U$ such that $\forall\, \varphi$ alternating $\exists!\, \tilde{\varphi}$ with $\varphi = \tilde{\varphi} \circ \Lambda^r$.

$$\begin{array}{ccc} V^{\times r} & \xrightarrow{\Lambda^r} & U \\ & \searrow^{\text{alternating } \varphi} & \downarrow \exists!\, \tilde{\varphi} \\ & & W \end{array}$$

Thm: Λ^r exists.

Pf: Set $U = \Lambda^r V = V^{\otimes r} / \underbrace{\text{span}(v_1 \otimes \cdots \otimes v_r \mid \text{two of the } v\text{'s are equal})}_{Y}$ ($V^{\otimes r} = V \otimes \cdots \otimes V$) — r-forms

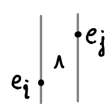
$V^{\times r} \to \Lambda^r V$

$(v_1, \dots, v_r) \mapsto v_1 \wedge \cdots \wedge v_r$ (\wedge) — multilinear because factors through $V^{\otimes r}$

alternating because $v_1 = \sum_{j>1} \alpha_j v_j \Rightarrow v_1 \wedge \cdots \wedge v_r = \sum_{j>1} \alpha_j v_j \wedge (v_2 \wedge \cdots \wedge v_r)$
$= 0$, and same for $i > 1$.

$\varphi : V^{\times r} \to W$ multilinear $\Rightarrow \varphi$ factors through $V^{\otimes r}$...

alternating $\Rightarrow$ ... and kills $Y \Rightarrow$ factors through $V^{\otimes r}/Y$. $\square$

E.g. $v, w \in \mathbb{R}^4_{\text{col}}$ (get from class) $\Rightarrow v \wedge w =$ 

Prop: $V \xrightarrow{\varphi} W$ linear induces canonical linear map $\Lambda^r V \xrightarrow{\Lambda^r \varphi} \Lambda^r V$. Λ^r is a functor.

$v_1 \wedge \cdots \wedge v_r \mapsto \varphi(v_1) \wedge \cdots \wedge \varphi(v_r)$

Pf: HW5, including entries of matrix if A is given. $\square$

Quintessential E.g. $V = W$ and $r = n = \dim V$: determinant of $\varphi : V \to V$ is $\det \varphi = \Lambda^n \varphi$.

Note: $\det \varphi = \Lambda^{\text{top}} \varphi$ since $\Lambda^r V = 0$ for $r \geq n+1$.

$\varphi(e_j) = v_j = \sum_{i=1}^n a_{ij} e_i \Rightarrow \Lambda^n \varphi(e_1 \wedge \cdots \wedge e_n) = v_1 \wedge \cdots \wedge v_n = \left(\sum_{i=1}^n a_{i1} e_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^n a_{in} e_i\right) = \underline{\quad}\, e_1 \wedge \cdots \wedge e_n$

$= \sum_{i_1, \dots, i_n} a_{i_1 1} e_{i_1} \wedge \cdots \wedge a_{i_n n} e_{i_n}$.

Terms are 0 unless $i_1, \dots, i_n$ distinct, so $i_j = \pi(j)$ for some permutation $\pi \in S_n$. Thus

$$v_1 \wedge \cdots \wedge v_n = \sum_{\pi \in S_n} a_{\pi(1)1} e_{\pi(1)} \wedge \cdots \wedge a_{\pi(n)n} e_{\pi(n)}$$

$\det A_\pi$ for permutation matrix A_π

$$= \underbrace{\sum_{\pi \in S_n} (-1)^\pi a_{\pi(1)1} \cdots a_{\pi(n)n}}_{\det A} \underbrace{e_1 \wedge \cdots \wedge e_n}_{e_{[n]}}$$

<u>Notation</u>: For $\sigma = \{\sigma_1 < \cdots < \sigma_r\} \subseteq [n]$ and $E = e_1, \dots, e_n \in V$ set

$$e_\sigma = e_{\sigma_1} \wedge \cdots \wedge e_{\sigma_r}$$

and $\Lambda^r E = \{ e_\sigma \mid \sigma \in \binom{[n]}{r} \}$.

<u>Prop</u>: E is a basis for $V \Rightarrow \Lambda^r E$ spans $\Lambda^r V$.

<u>Pf</u>: $\Lambda^r E = \{$squarefree elements of basis $E^{\otimes r}$ of $V^{\otimes r}\}$

and nonsquarefree elements $\mapsto 0$ in $\Lambda^r V$. □

Q. coeff. on e_σ in $v_1 \wedge \cdots \wedge v_r = ?$ i.e. How can $v_1 \wedge \cdots \wedge v_n$ be expressed as a linear combination of elements e_σ?

A. $\det A_\sigma$, where $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{bmatrix}$ and A_σ takes rows indexed by σ.

<u>Pf</u>: In rows from σ, get one e_i from each column j with coeff a_{ij}:

$$\sigma = \begin{Bmatrix} s_1 = s_{\pi(2)} \\ s_2 = s_{\pi(3)} \\ s_3 = s_{\pi(1)} \end{Bmatrix}$$

$$v_1 \wedge \cdots \wedge v_r = \sum_{\sigma \in \binom{[n]}{r}} \sum_{\pi \in S_r} a_{\sigma_{\pi(1)}1} e_{\sigma_{\pi(1)}} \wedge \cdots \wedge a_{\sigma_{\pi(r)}r} e_{\sigma_{\pi(r)}}$$

$$= \sum_{\sigma \in \binom{[n]}{r}} \sum_{\pi \in S_r} a_{\sigma_{\pi(1)}1} \cdots a_{\sigma_{\pi(r)}r} \, e_{\sigma_{\pi(1)}} \wedge \cdots \wedge e_{\sigma_{\pi(r)}}$$

$$= \sum_{\sigma \in \binom{[n]}{r}} \underbrace{\sum_{\pi \in S_r} a_{\sigma_{\pi(1)}1} \cdots a_{\sigma_{\pi(r)}r} (-1)^\pi}_{\det A_\sigma} e_\sigma . \quad \square$$

<u>Thm</u>: E is a basis for $V \Rightarrow \Lambda^r E$ is a basis for $\Lambda^r V$.

<u>Pf</u>: spans by Prop.

independent: existence of determinants $\Rightarrow (v_1, \dots, v_r) \mapsto \det([v_1 \cdots v_r]_\sigma)$ is alternating, so

induces $e_\sigma^* : \Lambda^r V \to F$ with $e_\sigma^*(e_\tau) = \delta_{\sigma,\tau}$; Lemma $\Rightarrow$ independent. □

<u>Cor</u>: $\dim V = n \Rightarrow \dim \Lambda^r V = \binom{n}{r}$ if $r \leqslant n$ and $\Lambda^r V = 0$ if $r > n$. □

<u>Prop</u>: $v \in V \Rightarrow v\wedge : \Lambda^r V \to \Lambda^{r+1} V$ linear

$\omega \in \Lambda^j V \Rightarrow \omega\wedge : \Lambda^r V \to \Lambda^{r+j} V$ linear

$(\omega_1 \wedge) \circ (\omega_2 \wedge) = (\omega_1 \wedge \omega_2)\wedge : \Lambda^r V \to \Lambda^{r+j+k} V$ if $\omega_1 \in \Lambda^j V$ and $\omega_2 \in \Lambda^k V$ (associativity)

<u>Remark</u>: $\Rightarrow \Lambda^* V = \bigoplus_r \Lambda^r V$ is a ring (Lec. 4).