Def: The cross product of two vectors $v, w \in \mathbb{R}^3$ is $u = v \times w$ whose entries are the coeffs on i, j, k in $\det \begin{bmatrix} i & j & k \\ - & v & - \\ - & w & - \end{bmatrix}$.

Prop: $u \cdot v = u \cdot w = 0$ and $\|u\| =$ area of parallelogram spanned by v and w. $\square$ $\quad \mathbb{R}^n$?

Def: $\mathrm{vol}(v_1, \dots, v_r) = |\mathrm{vol}(v_1, \dots, v_r, u_{r+1}, \dots, u_n)|$ for any $\perp$normal basis $u_{r+1}, \dots, u_n$ of $\{v_1, \dots, v_r\}^\perp$.

Def: cross product u of $v_2, \dots, v_n \in \mathbb{R}^n$ satisfies $v_j \cdot u = 0 \ \forall j = 2, \dots, n$ and $\|u\| = \mathrm{vol}(v_2, \dots, v_n)$.

Thm: $u = \sum_{i=1}^n u_i e_i \Rightarrow -v_2 \wedge \cdots \wedge v_n$ has coeff. $(-1)^i u_i$ on $e_{\bar{i}}$, where $\bar{i} = [n] \setminus \{i\}$.

Pf: $v_2 \wedge \cdots \wedge v_n \in \mathrm{span}(e_{\bar{1}}, \dots, e_{\bar{n}}) \Rightarrow$ it is $-\sum_{i=1}^n (-1)^i u_i e_{\bar{i}}$ for some u. For any $w \in \mathbb{R}^n$,

$$w \wedge v_2 \wedge \cdots \wedge v_n = \left(\sum_{i=1}^n w_i e_i\right) \wedge \left(-\sum_{i=1}^n (-1)^i u_i e_{\bar{i}}\right)$$

$$= \sum_{i=1}^n - w_i u_i (-1)^i e_i \wedge e_{\bar{i}}$$

$$= w \cdot u \ e_1 \wedge \cdots \wedge e_n.$$

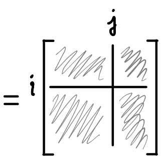
$$w = v_j \Rightarrow v_j \cdot u \ e_{[n]} = v_j \wedge v_2 \wedge \cdots \wedge v_n = 0 \overset{e_{[n]} \neq 0}{\Rightarrow} v_j \cdot u = 0 \ \forall j \geq 2. \quad (*)$$

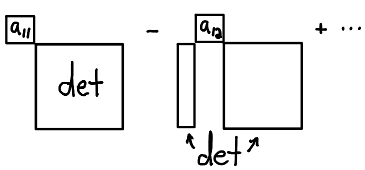
Let $w \in \{v_2, \dots, v_n\}^\perp$ and $\|w\| = 1$. Then

$\det(w, v_2, \dots, v_n)$

$$\mathrm{vol}(v_2, \dots, v_n) = |\mathrm{vol}(w, v_2, \dots, v_n)| = |\text{coeff. on } e_{[n]} \text{ in } w \wedge v_2 \wedge \cdots \wedge v_n|$$

$$= |w \cdot u| \overset{(*)}{=} \frac{u}{\|u\|} \cdot u = \frac{\|u\|^2}{\|u\|} = \|u\|. \ \square$$

Recall: Laplace expansion of det along a row: $\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{ij}$ for $A_{ij} =$ .

E.g. $i = 1$:  $+ \cdots$

Along (top) r rows: $A[\leq r] =$ top r rows $\qquad A[\leq r]^\sigma = r \times r$ submatrix with cols from $\sigma \in \binom{[n]}{r}$

$A[> r] =$ bottom $n-r$ rows $\quad A[>r]^{\bar{\sigma}} \qquad \bar{\sigma} = [n] \setminus \sigma$

Thm: $\det A = \sum_{\sigma \in \binom{[n]}{r}} (-1)^\sigma \det A[\leq r]^\sigma \det A[>r]^{\bar{\sigma}}$ $\qquad (-1)^\sigma = (-1)^{\#\text{swaps needed to put } \sigma, \bar{\sigma} \text{ in order}}$

Pf: $\det A =$ coeff. on $e_{[n]}$ in $v_1 \wedge \cdots \wedge v_n$, where $A = \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$.

Factor $v_1 \wedge \cdots \wedge v_n$: (coeff. on e_σ in $v_1 \wedge \cdots \wedge v_r$) $= \det A[\leq r]^\sigma$

$\qquad e_{\bar{\sigma}} \qquad v_{r+1} \wedge \cdots \wedge v_n = \det A[>r]^{\bar{\sigma}}$

$$\Rightarrow v_1 \wedge \cdots \wedge v_n = (v_1 \wedge \cdots \wedge v_r) \wedge (v_{r+1} \wedge \cdots \wedge v_n) = \sum_\sigma \det A[\leq r]^\sigma \det A[>r]^{\bar{\sigma}} \underbrace{e_\sigma \wedge e_{\bar{\sigma}}}_{(-1)^\sigma e_{[n]}}. \ \square$$

Pf very simple and explains exactly where the sign comes from

Thm: rank $A < r \Leftrightarrow$ all $\underbrace{\text{minors of size } r}_{\det(r\times r \text{ submatrix})}$ vanish.

entries are the r-minors of A (HW5)

Pf: $A \in F^{m\times n}$ represents $\varphi: F^n \to F^m$. rank $\varphi < r \Leftrightarrow \dim(\operatorname{im}\varphi) < r \Leftrightarrow \Lambda^r \varphi = 0$. $\square$

Intuition: r-dim vol in dim n needs to specify
- an r-dim subspace V
- a full-dim (i.e. dim r) volume in V

Thm: $v_1\wedge\cdots\wedge v_r$ identifies $V = \operatorname{span}(v_1,\dots,v_r)$ up to a scalar if $v_1,\dots,v_r$ independent:

$$w_1\wedge\cdots\wedge w_r = \alpha v_1\wedge\cdots\wedge v_r \text{ for some } \alpha\neq 0 \Leftrightarrow \underbrace{\operatorname{span}(w_1,\dots,w_r)}_{= W} = V.$$

Geometric interpretation: $F=\mathbb{R}$ and $u_1,\dots,u_r$ $\perp$normal in V

$$\Rightarrow v_1\wedge\cdots\wedge v_r = \alpha u_1\wedge\cdots\wedge u_r \text{ with } |\alpha| = \operatorname{vol}(v_1,\dots,v_r).$$

Pf of Thm: $W = V \Rightarrow \underbrace{\begin{bmatrix} - w_1 - \\ \vdots \\ - w_r - \end{bmatrix}}_{\mathcal{W}} = A \underbrace{\begin{bmatrix} - v_1 - \\ \vdots \\ - v_r - \end{bmatrix}}_{\mathcal{V}}$ for some $A \in GL_r$

multiplies all r-minors by same scalar, namely $\det A$

$\Rightarrow w_1\wedge\cdots\wedge w_r = \det A \; v_1\wedge\cdots\wedge v_r$, because coeff on e_σ in $w_1\wedge\cdots\wedge w_r$ is

$$e_\sigma^*(w_1\wedge\cdots\wedge w_r) = \det\left(A\begin{bmatrix} - v_1 - \\ \vdots \\ - v_r - \end{bmatrix}^\sigma\right) = \det A \; e_\sigma^*(v_1\wedge\cdots\wedge v_r).$$

Now suppose $w_1\wedge\cdots\wedge w_r = \alpha v_1\wedge\cdots\wedge v_r \neq 0$ for some $\alpha\in F$.

Then $w_i\wedge v_1\wedge\cdots\wedge v_r = \alpha^{-1} w_i\wedge w_1\wedge\cdots\wedge w_r = 0 \quad \forall i$.

Lemma: $w\wedge v_1\wedge\cdots\wedge v_r = 0 \Leftrightarrow w\in\operatorname{span}(v_1,\dots,v_r)$.

Pf: $\Leftarrow$: def. of alternating.

$\Rightarrow$: $w\notin\operatorname{span}(v_1,\dots,v_r) \Rightarrow$ can complete $w, v_1,\dots,v_r$ to a basis of F^n whose $\wedge$ is nonzero. $\square$

Def: $\left\{ e_\sigma^*(v_1\wedge\cdots\wedge v_r) \,\middle|\, \sigma\in\binom{[n]}{r}\right\}$ = Plücker coordinates of V.

Cor: $G_r(F^n) \longleftrightarrow \{$decomposable forms in $P\Lambda^r F^n = G_1(\Lambda^r F^n)\}$.