Math 403 Homework #1, Spring 2024

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...
(1 point for each of up to 3 collaborators who also list you) /3

Due: 11:59pm Saturday 27 January 2024

Reading assignments (item numbers are lecture numbers as in PDF lecture notes) /57

1. for Thu. 11 January
   - [Hefferon, p.153–154] field axioms
   - [Cornell, "Fields" (about 1/3 of the way through `4330-week1.pdf`)]: more on fields
   - [Climenhaga, §7.2, §9.1] isomorphism, rank-nullity
2. for Tue. 16 January
   - [Lax, Chapter 1] quotients
   - [Climenhaga, §5.1] quotients
   - [Cornell, "Quotient Spaces" (`4330-week5.pdf`)] universal properties
   - [Cornell, "Exact Sequences" (about halfway through `4330-week5.pdf`)]
3. for Thu. 18 January
   - [Lax, Chapter 2] duality
   - [Climenhaga, §5.1, §8.2] duality, transpose
   - [Treil, §5.1–5.5] Hermitian inner products, ajoints; §5.2–5.4 should be mostly review
4. for Tue. 23 January
   - [Lax, Appendix 15] Jordan canonical form (very short proof)
   - [Hefferon, §5.IV.1] characteristic polynomial and minimal polynomial
   - [Treil, §9.3 and §9.5] generalized eigenspaces and Jordan canonical form
5. for Thu. 26 January
   - [Lax, Chapter 14, p.214–221] norms, equivalence, continuity, local compactness
   - [Stewart–Sun, §II.1] norms, equivalence, Hahn–Banach theorem
6. for Tue. 30 January
   - [Lax, Chap.12, p.187–190] Convex sets
7. for Thu. 1 February
   - [Serge Lang, *Linear Algebra*, Chapter XII, §1–§2] separating hyperplanes
   - [Serge Lang, *Linear Algebra*, Chapter XII, §3–§4] support hyperplane, extreme point

Exercises

1. (Freshman's Dream): The *characteristic* of a field F is the smallest positive integer p such that the sum $1 + \cdots + 1$ of p multiplicative identities is 0 in F. Prove that p is prime if it is finite. If F has characteristic p, show that $(a+b)^p = a^p + b^p$ for $a, b \in F$. /3

2. Fix a vector space V and a subspace $W \subseteq V$ over a field F. Let $\pi : V \to V/W$ be the *projection homomorphism* given by $\pi(v) = v + W$. Write X for the set of all subspaces of V that contain W, and write Y for the set of all subspaces of V/W. Prove that π induces a bijection between these two sets, with /3

$$\begin{aligned} X &\to Y \\ L &\mapsto \pi(L) = \{\pi(v) \mid v \in L\} \end{aligned}$$

and

$$\begin{aligned} Y &\to X \\ M &\mapsto \pi^{-1}(M) = \{v \in V \mid \pi(v) \in M\}. \end{aligned}$$

3. Show that giving an exact sequence $\cdots \to V_{i-1} \to V_i \to V_{i+1} \to \cdots$ is the same as giving a collection of short exact sequences $0 \to K_i \to V_i \to K_{i+1} \to 0$, one for each i. (The long exact sequence is said to be constructed by *splicing* the short exact sequences together.) /3

4. Rank-nullity theorem for exact sequences: Given an exact sequence /3

$$0 \to V_0 \to V_1 \to \cdots \to V_r \to 0,$$

prove that $\sum_{i=0}^{r}(-1)^i \dim V_i = 0$.

5. Rank-nullity for arbitrary complexes: Given a complex $0 \to V_0 \to V_1 \to \cdots \to V_r \to 0$, write $B_i = \operatorname{im}(V_{i-1} \to V_i)$ and $Z_i = \ker(V_i \to V_{i+1})$. Prove that /3

$$\sum_{i=0}^{r}(-1)^i \dim V_i = \sum_i (-1)^i \dim H_i,$$

where $H_i = Z_i/B_i$ is the i^{th} homology of the complex. [In Exercise 4, $B_i = Z_i = K_i$, so $H_i = Z_i/B_i = 0$ for all i.]

6. Two elements u and v in a vector space V are *congruent modulo* a subspace $W \subseteq V$, written $u \equiv v \pmod{W}$, if $u + W = v + W$. Show that congruence modulo W is an *equivalence relation* on V, meaning that it is /3
   - reflexive: $v \equiv v \pmod{W}$ for all $v \in V$;
   - symmetric: if $u \equiv v \pmod{W}$, then $v \equiv u \pmod{W}$; and
   - transitive: if $u \equiv v \pmod{W}$ and $v \equiv x \pmod{W}$ then $u \equiv x \pmod{W}$.

7. Give an example of three subspaces Y_1, Y_2, and Y_3 in $\mathbb{R}^2$ such that $Y_1 + Y_2 + Y_3 = \mathbb{R}^2$ and $Y_i \cap Y_j = \{\mathbf{0}\}$ for all $i \neq j$, but $\mathbb{R}^2$ is not the direct sum of Y_1, Y_2, and Y_3. /3

8. Prove that if V is a vector space and $W \subseteq V$ is a subspace, then W has a *complement*: a subspace $U \subseteq V$ such that $V = W \oplus U$. Hint: V/W has a basis; lift it back to V. /3

9. For vectors $\mathbf{x} = (1, 2i, 1+i)$ and $\mathbf{y} = (i, 2-i, 3)$, compute
   (a) $\langle \mathbf{x}, \mathbf{y} \rangle$, $||\mathbf{x}||^2$, $||\mathbf{y}||^2$, and $||\mathbf{y}||$; /3
   (b) $\langle 3\mathbf{x}, 2i\mathbf{y} \rangle$ and $\langle 2\mathbf{x}, i\mathbf{x} + 2\mathbf{y} \rangle$; /3
   (c) $||\mathbf{x} + 2\mathbf{y}||$. [Use parts (a) and (b) for this.] /3

10. Prove that for vectors in an inner product space, $||\mathbf{x} \pm \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \pm 2\mathrm{Re}\langle \mathbf{x}, \mathbf{y} \rangle$. /3

11. For any $m \times n$ complex matrix A, prove that $\ker(A^* A) = \ker(A)$. /3

12. Prove that if P is self-ajoint (that is, $P^* = P$) and idempotent (that is, $P^2 = P$) then P is the matrix for an orthogonal projection. /3

13. If V is a vector space over $\mathbb{C}$ of dimension n, then it is also a vector space over $\mathbb{R}$ of dimension $2n$. (If this isn't clear to you, then write down a proof.) Given a Hermitian inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ on V as a complex vector space, show that the real part $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}} = \mathrm{Re}(\langle \mathbf{x}, \mathbf{y} \rangle)$ is an inner product on V as a real vector space. /3

14. Find all possible Jordan forms of linear transformations with characteristic polynomial $(t-1)^2(t+2)^2$. /3

    Solution: OK, so now what happens if I write a paragraph or two in response to one of these questions? Does the inter-paragraph spacing look okay?

    It will, of course, be important to write more than one paragraph to detect whether or not it looks okay.

15. Find all possible Jordan forms of linear transformations with characteristic polynomial $(t-2)^3(t+1)$ and minimal polynomial $(t-2)^2(t+1)$. /3

16. How many similarity classes are there for 3×3 matrices whose only eigenvalues are -3 and 4? /3

17. Prove or disprove: two $n \times n$ matrices are similar if and only if they have the same characteristic polynomial and minimal polynomial. /3