Math 403 Homework #2, Spring 2024

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...
(1 point for each of up to 3 collaborators who also list you) /3

Due: 11:59pm Saturday 10 February 2024

Reading assignments /48

7. for Thu. 1 February
   - [Serge Lang, *Linear Algebra*, Chapter XII, §1–§2] separating hyperplanes
   - [Serge Lang, *Linear Algebra*, Chapter XII, §3–§4] support hyperplane, extreme point
8. for Tue. 6 February
   - [Wikipedia, *Grassmannian*] as much as you're willing; thru §3 (*"as a set"*) at least
   - [Wikipedia, *Group action*] get a feel for the concept; no need to overdo it
9. for Thu. 8 February
   - [multivariable calculus, any source]: open & closed sets; differentiability; chain rule $D(f \circ g) = Df \circ Dg$; inverse function theorem: Df invertible $\Rightarrow f$ locally invertible
   - [Wikipedia, *Topological space*] get a feel; through §5.1 (*Metric spaces*), say
   - [Wikipedia, *Differentiable manifold*] read up through §2 (*Definition*)
10. for Tue. 13 February
    - [Lax, 7.Isometry, p.87–89] isometry, orthogonal group
    - [Lax, 7.Complex Euclidean structure, p.95–96] unitary matrices
    - [Treil, §5.6–§5.7] unitary transformations, rigid motions
11. for Thu. 15 February
    - [Treil, §6.1–§6.2] Spectral theorem, normal operators

Exercises

1. Find the Jordan form of a 5×5 matrix A whose sole eigenvalue is 3, given that $A - 3I$, $(A-3I)^2$, $(A-3I)^3$, and $(A-3I)^4$ have kernels of dimension 2, 3, 4, and 5, respectively. Do the same thing if the kernel dimensions are 2, 4, 5, and 5, respectively. /3

2. Find a Jordan form and a corresponding Jordan basis for the matrix /3

$$A = \begin{bmatrix} 7 & 1 & 2 & 2 \\ 1 & 4 & -1 & -1 \\ -2 & 1 & 5 & -1 \\ 1 & 1 & 2 & 8 \end{bmatrix}.$$

3. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ of finite dimension. Let $\varphi : V \hookrightarrow W$ be an injective homomorphism. Show that if μ is a norm on W then $\mu \circ \varphi$ is a norm on V. /3

4. Let ν be a norm on $V = \mathbb{R}^n$ or $\mathbb{C}^n$ with unit ν-ball $B_\nu = \{\mathbf{x} \in V \mid \nu(\mathbf{x}) \leq 1\}$. Prove that B_ν is closed, bounded, convex, and *equilibrated*: $\mathbf{x} \in B_\nu$ and $|\alpha| \leq 1 \Rightarrow \alpha\mathbf{x} \in B_\nu$. Also show that the origin lies interior to B_ν. Conversely, if $B \subseteq V$ is closed, bounded, convex, equilibrated subset with 0 in its interior, then a norm $\nu_B : V \to \mathbb{R}$ with unit ball B can be defined by $\nu_B(\mathbf{x}) = \inf\{\alpha > 0 \mid \frac{1}{\alpha}\mathbf{x} \in B\}$. /3

5. The Hahn–Banach theorem says that for any norm ν on $\mathbb{C}^n$ and any subspace $V \subseteq \mathbb{C}^n$, if $\varphi : V \to \mathbb{C}$ is a linear functional whose maximum magnitude $|\varphi(\mathbf{x})|$ for $\mathbf{x}$ in the unit ν-sphere of V is α, then φ can be extended to a linear functional on $\mathbb{C}^n$ whose maximum magnitude on the unit ν-sphere in $\mathbb{C}^n$ is α. Show that the Hahn–Banach theorem is equivalent to the statement that separating hyperplanes exist for points outside of any closed, bounded, convex, equilibrated set that contains 0 in its interior. /3

6. Let V be the vector space of real (or complex) absolutely convergent series. The function $x \mapsto \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$ for $p \geq 1$ is a norm on V. (You may assume this, although no complaints if you'd like to prove it.) Prove that $\|x\|_p$ is not topologically equivalent to $\|x\|_q$ if $p < q$. /3

7. Sketch the convex hull of $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. /3

8. Fix an operator $\varphi : \mathbb{R}^n \to \mathbb{R}^n$. If $X \subseteq \mathbb{R}^n$ is convex and $\mathbf{x}$ is an extreme point of X, must $\varphi(\mathbf{x})$ be an extreme point of $\varphi(X)$? What if φ is assumed invertible? /3

9. Prove that the intersection of finitely many closed halfspaces in $\mathbb{R}^n$ can have only finitely many extreme points. Must it have any extreme points at all? If it does have extreme points, must the intersection equal the convex hull of its extreme points? /3

10. Fix a column vector $\mathbf{b} \in \mathbb{R}^n$ and an $n \times n$ real matrix A. Show that the set of solutions $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ of the inequality $A\mathbf{x} \leq \mathbf{b}$ is convex, where $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all i. What kind of convex set is it? (No need to provide full justification of this last bit.) /3

11. A *cone* is a subset $C \subseteq \mathbb{R}^n$ closed under nonnegative scaling: $\mathbf{x} \in C$ implies $\alpha\mathbf{x} \in C$ for all $\alpha \in \mathbb{R}_{\geq 0}$. A ray $\mathbb{R}_{\geq 0}\mathbf{x} = \{\alpha\mathbf{x} \mid \alpha \in \mathbb{R}_{\geq 0}\} \subseteq C$ is *extreme* if $\mathbf{x} \in \mathbf{y} + C \subseteq C$ implies $\mathbf{y} \in \mathbb{R}_{\geq 0}\mathbf{x}$. Let ν be an outer support vector for a closed convex cone C, so $\langle \nu, \mathbf{x} \rangle \leq 0$ for all $\mathbf{x} \in C$. Assume ν attains a unique maximum on C. Prove that the minimum angle between ν and a unit vector $\mathbf{x} \in C$ occurs when $\mathbf{x}$ lies on an extreme ray of C. /3

12. How many $k \times n$ matrices of rank k with entries in a field $\mathbb{F}_q$ of size q are there? Hint: how many nonzero row vectors are there in $\mathbb{F}_q^n$? How many vectors are linearly independent from the first one? How many are independent from the first two? /3

13. Show that the action of $GL_k(F)$ on the set of $k \times n$ matrices of rank k is *free*: if $g \neq g'$ are invertible $k \times k$ matrices and A is a $k \times n$ matrix of rank k, then $gA \neq g'A$. Use this (and the previous exercise) to compute the cardinality of the set $G_k(\mathbb{F}_q^n)$ of k-planes in $\mathbb{F}_q^n$. Hint: use #12 twice, once with $k = n$ and once with $k < n$. /3

14. Does the first claim in #13 remain true if the rank of A is less than k? /3

15. Use #13 to count the number of Sets in the game of Set. (Note: a Set is an affine line, not a linear subspace of dimension 1, so this is not quite a special case of #13. There is more than one way around this; I'll leave it to you to find at least one.) Confirm your count by counting a different way: a line L is determined by any pair of points on L. /3