Math 403 Homework #3, Spring 2024

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...
(1 point for each of up to 3 collaborators who also list you) /3

Due: 11:59pm Saturday 2 March 2024

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Reading assignments

13. for Thu. 22 February
    - [Treil, §6.3.3–§6.3.4] singular value decomposition
    - [Lax, Chap.7: §Norm of a Linear Map, §Spectral Radius]
    - [Lax, Chap.8: §Norm and Eigenvalues]
    - [Treil, §6.4] Applications of SVD: spectral radius, operator norm, condition number
14. for Tue. 27 February
    - [Stewart–Sun, §II.2.1] matrix norms, consistency
    - [Stewart–Sun, §IV.1 through Thm 1.3] general perturbation theorems
15. for Thu. 29 February
    - [Stewart–Sun, §IV.1.2 through Thm 1.6] Bauer–Fike theorem
    - [Lax, Appendix 7] Gershgorin's Theorem
    - [Stewart–Sun, §IV.2 through §IV.2.1] Gerschgorin theory

16–17. for Tue. 5 March and Thu. 7 March
    - [Tapp, Chap.5] Lie algebras as tangent spaces to the identity
    - [Lax, Chap.9: Matrix-valued Functions, through §Matrix Exponential]
    - [Wikipedia, *Exponential map (Lie theory)*]

Exercises

1. Prove that a subset of a manifold is open if and only if its preimage under every chart is an open subset of a vector space. (Note: This can be used to specify the topology on a manifold without knowing what a topological space is, since vector spaces have the usual notion of "open": what it means for a neighborhood of a point in a manifold to be open is well defined independent of which charts are used to verify openness.) /3

2. Prove that the sphere $S^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid ||\mathbf{x}|| = 1\}$ is a smooth manifold. Is it a rational algebraic variety over the field $\mathbb{R}$? /3

3. Fix a field F. Prove that the unipotent lower-triangular matrices $U^- \subseteq GL_n(F)$ map injectively to the set $\mathcal{F}\ell_n(F)$ of complete flags in F^n expressed as the set of orbits of the group B^+ acting on the right of $GL_n(F)$; that is, $U^- \hookrightarrow \mathcal{F}\ell_n(F) = GL_n(F)/B^+$. /3

4. Let $S_n \subseteq GL_n(F)$ be the set of permutation matrices. Prove that $\mathcal{F}\ell_n(F)$ is a rational algebraic variety with atlas $\{\pi_w : wU^- \to \mathcal{F}\ell_n(F) \mid w \in S_n\}$ naturally indexed by S_n. You will need to use that any level set of a polynomial with coefficients in F is closed. /3

5. The *standard complex structure* on $\mathbb{R}^{2n}$ is the block-diagonal $2n \times 2n$ matrix J_{2n} whose diagonal blocks are all $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Prove that $A \in GL_{2n}(\mathbb{R})$ is complex-linear if and only if A commutes with the standard complex structure: $AJ_{2n} = J_{2n}A$. /3

6. Prove that if $\lambda \in \mathbb{C}$ is an eigenvalue of a unitary matrix then $|\lambda| = 1$. /3

7. Show that the standard Hermitian inner product on $\mathbb{C}^n$ defines a distance $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$ on $\mathbb{C}^n$. Give an example of an isometry φ of $\mathbb{C}^n$ such that $\varphi(\mathbf{0}) = \mathbf{0}$ but φ is not $\mathbb{C}$-linear. /3

8. Let φ be an orthogonal transformation of a real inner product space V. Assume that $\varphi^2 = -I$. Show that $\dim V$ is even, say $2n$. Moreover, prove that there exists a dimension n subspace $W \subset V$ and an isometry $\psi : W \to W^\perp$ such that, in the decomposition $V = W \oplus W^\perp$, the operator φ is given by the block matrix /3

$$\begin{bmatrix} \mathbf{0} & -\psi^* \\ \psi & \mathbf{0} \end{bmatrix}$$

(N.B. The result means that φ can be thought of as multiplication by i on a complex vector space whose real and imaginary parts are W and $W^\perp$.)

9. True or false: the sum of two normal operators is normal. Justify. /3

10. Show that the space of positive (semi)definite real symmetric matrices is convex. Is the same true with "complex Hermitian" in place of "real symmetric"? /3

11. Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. Find all square roots of A; note that they are all self-adjoint. /3

12. Find a singular decomposition of the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Use it to find $\max_{\|\mathbf{x}\| \leq 1} \|A\mathbf{x}\|$ and $\min_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$, as well as the vectors where this maximum and minimum are attained. Describe geometrically the image under A of the closed unit disk in $\mathbb{R}^2$. /3

13. Prove that the operator norm of a matrix A coincides with the Frobenius norm of A if and only if A has rank at most 1. /3